Topological fluid dynamics

Boris Khesin*

Notices of the AMS, vol. **52:1** (2005), 9–19pp.

Topological fluid dynamics is a young mathematical discipline which studies topological features of flows with complicated trajectories and their applications to fluid motions, and develops group-theoretic and geometric points of view on various problems of hydrodynamical origin. It is situated at a crossroad of several disciplines, including Lie groups, knot theory, partial differential equations, stability theory, integrable systems, geometric inequalities, and symplectic geometry. Its main ideas can be traced back to the seminal 1966 paper [1] by V. Arnold on the Euler equation for an ideal fluid as the geodesic equation on the group of volume-preserving diffeomorphisms.

One of the most intriguing observations of topological fluid dynamics is that one simple construction in Lie groups enables a unified approach to a great variety of different dynamical systems, from the simple (Euler) equation of a rotating top to the (also Euler) hydrodynamics equation, one of the most challenging equations of our time.

A curious application of this theory is an explanation of why longterm dynamical weather forecasts are not reliable: Arnold's explicit estimates related to curvatures of diffeomorphism groups show that the earth weather is essentially unpredictable after two weeks as the error in the initial condition grows by a factor of 10^5 for this period, i.e., one looses 5 digits of accuracy. (Ironically, 15 day(!) weather forecasts for any country in the world are now readily available online at www.accuweather.com .) Another application is related to the Sakharov–Zeldovich problem on whether a neutron star can extinguish by "reshaping" and turning to radiation the excessive magnetic energy.

In this introductory article we will touch on these and several other purely mathematical problems motivated by fluid mechanics, referring the interested reader to the book [4] for further details and the extensive bibliography.

1 Energy relaxation.

1.A. The minimization problems.

The first problem we are going to discuss is related to topological obstructions to energy relaxation of a magnetic field in a perfectly conducting medium. A motivation for this problem is the following model of a magnetic field of a star. Imagine that the star is

*Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3, Canada; e-mail: khesin@math.toronto.edu

filled with some perfectly conducting medium (say, plasma), which carries a "frozen in" magnetic field. Then the topology of the field's trajectories cannot change under the fluid flow, but its magnetic energy can. The conducting fluid keeps moving (due to Maxwell's equations) until the excess of magnetic energy over its possible minimum is fully dissipated (this process is called "energy relaxation"). It turns out that mutual linking of magnetic lines may prevent complete dissipation of the magnetic energy. The problem is to describe lower bounds for energy of the magnetic field in terms of topological characteristics of its trajectories.

More precisely, consider a divergence-free ("magnetic"[1]) vector field ξ in a (simply connected) bounded domain $M \subset \mathbb{R}^3$ tangent to its boundary. By the energy of the field ξ we will mean the square of its L_2-norm, i.e., the integral

$$E(\xi) = \int_M (\xi, \xi)\, d^3x\,,$$

where $(.,.)$ is the Euclidean inner product on M. *Given a divergence-free field* ξ*,* the problem is *to find the minimum energy (or, to give an appropriate lower bound for)* $\inf_h E(h_*\xi)$ *of the push-forward fields* $h_*\xi$ *under the action of all volume-preserving diffeomorphisms* h *of* M*.*

A topological obstruction to the energy relaxation can be seen in the example of a magnetic field confined to two linked solid tori. Assume that the field vanishes outside those tubes and the field trajectories are all closed and oriented along the tube axes inside. To minimize the energy of a vector field with closed orbits one has to shorten the length of most trajectories. This, however, leads to a fattening of the solid tori (since the acting diffeomorphisms are volume-preserving). For a linked configuration, as in Fig.1, the solid tori prevent each other from endless fattening and therefore from further shrinking of the orbits. Therefore, heuristically, in the volume-preserving relaxation process the magnetic energy of the field supported on a pair of linked tubes is bounded from below and cannot attain arbitrarily small values.

The topological obstruction is even more evident in the two-dimensional case of the energy minimization problem. Let M be a bounded domain in $\mathbb{R}^2$. The problem is *to describe the infimum and the minimizers of the* Dirichlet integral

$$E(u) = \int_M (\nabla u, \nabla u)\ d^2x$$

among all the smooth functions u (in the domain M) *that can be obtained from a given function* u_0 *by the action of area-preserving diffeomorphisms of* M *to itself.*

To see that this is the two-dimensional counterpart of the above 3D problem, one considers the skew gradient $J\nabla u$, the pointwise rotation by $\pi/2$ of the true gradient ∇u, on which the functional E has, of course, the same value. Then u can be regarded as a Hamiltonian (or stream) function of the field $J\nabla u$ and its definition is invariant: Any area-preserving change of coordinates for the function u implies the corresponding diffeomorphism action on the field $J\nabla u$.

[1] Note that magnetic fields are always divergence-free due to the absence of magnetic charges.

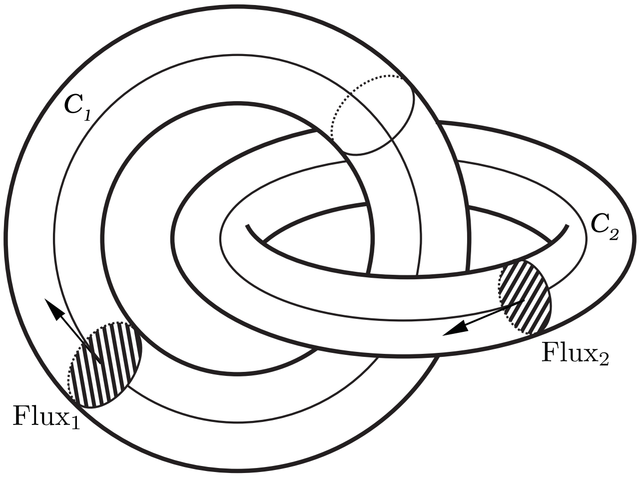

Figure 1: C_1, C_2 are axes of the tubes; Flux_1, Flux_2 are the corresponding fluxes.

For instance, consider a function u vanishing on the boundary of a 2D disk $M = \{x^2 + y^2 \leq 1\}$ and having a single critical point inside. Then the minimum of the Dirichlet functional is attained on the function u_0 that depends only on the distance to the center of the disk and whose sets $\{(x, y) \mid u_0(x, y) \leq c\}$ of smaller values have the same areas as those of the initial function u, see [2]. This can be shown by applying the Cauchy–Schwarz and isoperimetric inequalities. If the initial function has several critical points (say, two maxima and a saddle point), the situation is more subtle and far from being solved. P.Laurence and E.Stredulinsky (2000) showed the existence of weak minimizers with some topological constraints. Numerical experiments suggest various types of (non-smooth) minimizers depending on the steepness of the initial function u.

1.B. What is helicity?

To describe the first obstruction to energy minimization in 3D we need the following notion.

Definition 1.1 (H.K. Moffatt 1969, [9]) *The* helicity *of the field ξ in a domain $M \subset \mathbb{R}^3$ is the number*

$$\mathcal{H}(\xi) = \int_M (\xi, \mathrm{curl}^{-1}\xi)\ d^3x,$$

where the vector field $\mathrm{curl}^{-1}\xi$ *is a divergence-free vector potential of the field ξ, i.e.,* $\nabla \times (\mathrm{curl}^{-1}\xi) = \xi$ *and* $\mathrm{div}(\mathrm{curl}^{-1}\xi) = 0$.

In the above example of a divergence-free field ξ confined to two narrow linked flux tubes the helicity can be found explicitly. Suppose that the tube axes are closed curves C_1 and C_2, the fluxes of the field in the tubes are Flux_1 and Flux_2, Fig.1. Assume also that there is no net twist within each tube or, more precisely, that the field trajectories foliate each of the tubes into pairwise unlinked circles. One can show that the helicity invariant of such a field is given by

$$\mathcal{H}(\xi) = 2\ lk(C_1, C_2) \cdot |\mathrm{Flux}_1| \cdot |\mathrm{Flux}_2|,$$

where $lk(C_1, C_2)$ is the (Gauss) *linking number* of C_1 and C_2, which explains the term "helicity" coined in [9]. Recall, that the number $lk(C_1, C_2)$ for two oriented closed curves is the signed number of the intersection points of one curve with an arbitrary oriented surface spanning the other curve.

Although helicity was defined above by using the Riemannian metric on M, it is actually a topological characteristic of a divergence-free vector field, depending only on the choice of a volume form on the manifold. Indeed, consider a simply connected manifold M (possibly with boundary) with a volume form μ, and let ξ be a divergence-free vector field on M (tangent to the boundary). The divergence-free condition means that the Lie derivative of μ along ξ vanishes: $L_\xi \mu = 0$, or, which is the same, the substitution $i_\xi \mu =: \omega_\xi$ of the field ξ into the 3-form μ is a closed 2-form: $d\omega_\xi = 0$. On a simply connected manifold M this means that ω_ξ is exact: $\omega_\xi = d\alpha$ for some 1-form α, called a potential. (If M is not simply connected, we have to require that the field ξ is null-homologous, i.e., that the 2-form ω_ξ is exact.)

Definition 1.2 (V.Arnold 1973, [2]) *The* helicity $\mathcal{H}(\xi)$ *of a null-homologous field* ξ *on a three-dimensional manifold* M *equipped with a volume element* μ *is the integral of the wedge product of the form* $\omega_\xi := i_\xi \mu$ *and its potential:*

$$\mathcal{H}(\xi) = \int_M d\alpha \wedge \alpha, \text{ where } d\alpha = \omega_\xi.$$

An immediate consequence of this purely topological (metric-free) definition is the following

Theorem 1.3 *The helicity* $\mathcal{H}(\xi)$ *is preserved under the action on* ξ *of a volume-preserving diffeomorphism of* M.

In this sense $\mathcal{H}(\xi)$ is a topological invariant: it was defined without coordinates or a choice of metric, and hence every volume-preserving diffeomorphism carries a field ξ into a field with the same helicity. Thus for magnetic fields frozen into (and hence, transported by) the medium, their helicity is preserved. Furthermore, the physical significance of helicity is due to the fact that it appears as a conservation law not only in magnetohydrodynamics, but also ideal fluid mechanics: Kelvin's law implies the invariance of helicity of the vorticity field for an ideal fluid motion (see Section 3.A below).

V. Arnold proposed the following ergodic interpretation of helicity in the general case of any divergence-free field (when the trajectories are not necessarily closed or confined to invariant tori) as the *asymptotic Hopf invariant*, i.e., the average linking number of the field's trajectories. Let ξ be a divergence-free field on M. We will associate to each pair of points in M a number that characterizes the "asymptotic linking" of the ξ-trajectories passing through these points. Given any two points x_1, x_2 in M and two large numbers T_1 and T_2, we consider "long segments" of the trajectories of ξ issuing from x_1 and x_2. For each of these two long trajectory segments, connect their endpoints by the shortest geodesics to obtain two closed curves, Γ_1 and Γ_2, see Fig.2. Assume that these curves do

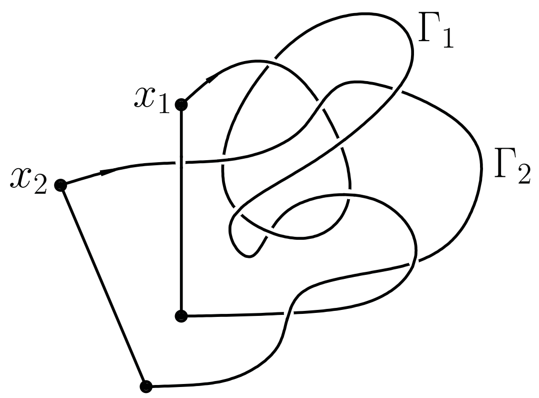

Figure 2: The long segments of the trajectories are closed by the "short paths."

not intersect (which is true for almost all pairs x_1, x_2 and for almost all T_1, T_2). Then the linking number $lk_\xi(x_1, x_2; T_1, T_2) := lk(\Gamma_1, \Gamma_2)$ of the curves Γ_1 and Γ_2 is well-defined.

Definition 1.4 *The* asymptotic linking number *of a pair of trajectories of the field ξ issuing from the points x_1 and x_2 is defined as the limit*

$$\lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{lk_\xi(x_1, x_2; T_1, T_2)}{T_1 \cdot T_2},$$

where T_1 and T_2 are to vary so that Γ_1 and Γ_2 do not intersect.

(T.Vogel (2003) showed that this limit exists as an element of the space $L^1(M \times M)$ of the Lebesgue-integrable functions and is independent of the system of geodesics, i.e., of the Riemannian metric, universally for any divergence-free field ξ.)

Theorem 1.5 (V.Arnold 1973, [2]) *For a divergence-free vector field ξ on a simply connected manifold M with a volume element μ, the average self-linking of trajectories of this field, i.e., the asymptotic linking number $\lambda_\xi(x_1, x_2)$ of trajectory pairs integrated over $M \times M$, is equal to the field's helicity:*

$$\int_M \int_M \lambda_\xi(x_1, x_2) \; \mu_1 \mu_2 = \mathcal{H}(\xi)$$

This elegant result promted numerous generalizations (see the survey in [4]).

1.C. Energy estimates.

It turns out that a nonzero helicity (or average self-linking of the trajectories) of a field ξ provides a lower bound for the energy.

Theorem 1.6 [2] *For a divergence-free vector field ξ*

$$E(\xi) \geq C \cdot |\mathcal{H}(\xi)|,$$

where C is a positive constant depending on the shape and size of the compact domain M.

The constant C can be taken as the reciprocal of the norm of the compact operator curl^{-1}, in the definition of helicity, on an appropriate space of vector fields. For instance, for any relaxation of the field confined to a pair of tori, the energy has a positive bound via helicity, once the linking number of tori is nonzero.

However, heuristically, there should be a lower bound for the energy for a field which has at least one linked pair of solid tori as in the example above, even if the total helicity vanishes. One of the best results in this direction is

Theorem 1.7 (M.Freedman and Z.X.He 1991) *Suppose a divergence-free vector field ξ in $\mathbb{R}^3$ has an invariant torus $\mathbf{K}$ forming an nontrivial knot K. Then*

$$E(\xi) \geq \left(\frac{16}{\pi \cdot \mathrm{Vol}(\mathbf{K})}\right)^{1/3} \cdot |\mathrm{Flux}\ \xi|^2 \cdot (2 \cdot \mathrm{genus}(K) - 1),$$

where Flux ξ *is the flux of ξ through a crossection of* $\mathbf{K}$*,* $\mathrm{Vol}(\mathbf{K})$ *is the volume of the solid torus, and* $\mathrm{genus}(K)$ *is the genus of the knot K.*

Recall, that for any knot its *genus* is the minimal number of handles of a spanning (oriented) surface for this knot. For an unknot the genus is 0, since one can take a disk as a spanning surface. For a nontrivial knot one has $\mathrm{genus}(K) \geq 1$ and, therefore, the above energy is bounded away from zero: $E(\xi) > 0$.

Note that this result has a wide range of applicability, since there is no restriction on the behavior of the field inside the invariant torus. In particular, it is sufficient for the field to have at least one closed knotted trajectory of elliptic type, i.e. a trajectory whose Poincaré map has two eigenvalues of modulus 1. Then the KAM theory implies that a generic elliptic orbit is confined to a set of nested invariant tori, and hence the energy of the corresponding field has a non-zero lower bound. The following question still remains one of the main challenges in this area:

Question 1.8 *Does the presence of* any *nontrivially knotted closed trajectory* (of any type: hyperbolic, non-generic, etc.) *or the presence of chaotic behavior of trajectories for a field provide a positive lower bound for the energy (even if the averaged linking of all trajectories totals zero) and therefore prevent a relaxation of the field to arbitrarily small energies?*

1.9 REMARK. The rotation field in the three-dimensional ball is an example of an opposite type: all its trajectories are *pairwise unlinked*. It was suggested by A.Sakharov and Ya.Zeldovich (in the 1970's) and proved by M.Freedman (1991), that this field can be transformed by a volume-preserving diffeomorphism to a field whose energy is less than any given ϵ. Physically this means that a neutron star with the rotation magnetic field can radiate all of its magnetic energy!

A somewhat opposite to the above *minimization* problem is the fast dynamo theory, which studies the *growth* of magnetic field in a given plasma flow. A bit more precisely, this theory regards the plasma velocity as given (stationary, periodic, etc.), neglecting the reciprocal (Lorentz) action of the magnetic field on the plasma velocity. It studies the rate of growth of the magnetic energy in time for sufficiently small magnetic diffusivity.

The non-zero diffusivity means that magnetic field lines are not precisely "frozen in," but rather "diffuse their topology," yet this problem exhibits a number of curious topological features, see [4].

1.D. Extremals and steady fluid flows.

One can explicitly describe the extremals in the above minimization problem. It turns out that these extremals appear in various parts of ideal fluid dynamics and magnetohydrodynamics.

Theorem 1.10 [2, 3] *The extremals of the above energy minimization problem are those divergence-free vector fields ξ on M which commute with their vorticities* curl ξ*. Moreover, these extremals are solutions of the stationary Euler equation in the domain M:*

$$(\xi \cdot \nabla)\xi = -\nabla p.$$

In 3D one can reformulate the above condition this way: the cross-product of the fields ξ and curl ξ is a potential vector field, i.e., $\xi \times \text{curl}\ \xi = -\nabla\alpha$. The extremal fields ξ have a very special topology [2, 3]. For instance, for a closed manifold M, every non-critical set of the function α is a torus, while the commuting fields ξ and curl ξ are tangent to these tori and define the $\mathbb{R}^2$ action on them. This way a steady 3D flow looks like a completely integrable Hamiltonian system with two degrees of freedom. In the case of M with boundary, the non-critical levels of α must be either tori or annuli, while the flow lines of ξ on the annuli are all closed.

Of special interest is the case, where an ξ is eigen field for the curl operator: curl $\xi = \lambda\xi$. (This corresponds to a constant function α, or to collinear fields ξ and curl ξ.) For instance, the so called ABC flows on a 3D torus are eigen fields for the curl operator. They exhibit chaotic behavior and draw special attention in fast dynamo constructions.

In the 2D fluid, the extremal fields, or stationary solutions of the Euler equation, obey the following condition: The gradients of the functions u and Δu are collinear at every point of the Riemannian manifold M. In other words, the extremal functions u have the "same" level curves as their Laplacians: Locally there is a function $F : \mathbb{R} \to \mathbb{R}$ such that $\Delta u = F(u)$. This can be thought of as a two-dimensional reformulation of the collinearity of the field and its vorticity.

2 Euler equations and geodesics.

2.A. Example: fluid motion.

Imagine an incompressible fluid occupying a domain M in $\mathbb{R}^n$. The fluid motion is described by a velocity field $v(t, x)$ and a pressure field $p(t, x)$ which satisfy the classical Euler equation:

$$\partial_t v + (v \cdot \nabla)v = -\nabla p, \tag{1}$$

where div $v = 0$ and the field v is tangent to the boundary of M. The function p is defined uniquely modulo an additive constant by the condition that v has zero divergence. (Note that stationary Euler flows are defined by the equation $(v \cdot \nabla)v = -\nabla p$, discussed in the preceding section.)

The flow $(t, x) \to g(t, x)$ describing the motion of fluid particles is defined by its velocity field $v(t, x)$:

$$\partial_t g(t, x) = v(t, g(t, x)),\ \ g(0, x) = x.$$

The chain rule immediately gives $\partial_t^2 g(t, x) = (\partial_t v + (v \cdot \nabla)v)(t, g(t, x))$, and hence the Euler equation is equivalent to

$$\partial_t^2 g(t, x) = -(\nabla p)(t, g(t, x)),$$

while the incompressibility condition is $\det(\partial_x g(t, x)) = 1$. The latter form of the Euler equation (for a smooth flow $g(t, x)$) says that the acceleration of the flow is given by a gradient and hence it is L^2-orthogonal to the set of volume-preserving diffeomorphisms (or, rather, to its tangent space of divergence-free fields). In other words, the fluid motion $g(t, x)$ is a geodesic line on the set of such diffeomorphisms of the domain M with respect to the induced L^2-metric. The same equation describes the motion of an ideal incompressible fluid filling an arbitrary Riemannian manifold M equipped with a volume form μ, [1, 6]. In the latter case v is a divergence-free vector field on M, while $(v \cdot \nabla)v$ stands for the Riemannian covariant derivative of v in the direction of itself.

2.1 REMARK. Note that the dynamics of an ideal fluid is, in a sense, dual to the Monge–Kantorovich mass transport problem, which asks for the most economical way to move, say, a pile of sand to a prescribed location. Mass (or density) is transported most effectively by gradient vector fields. The latter are L^2-orthogonal to divergence-free ones, which, in turn, preserve volume (or mass). The corresponding transportation (or, Wasserstein) metric on the space of densities and the L^2-metric on volume-preserving diffeomorphisms can be viewed as a natural extensions of each other (F.Otto 2001, [5]).

2.B. Geodesics on Lie groups and equations of mathematical physics.

V. Arnold (1966) proposed the following general framework for the Euler equation on an arbitrary group, which describes the geodesic flow with respect to a suitable one-sided invariant Riemannian metric on this group.

Consider a (possibly infinite-dimensional) Lie group G, which can be thought of as the configuration space of some physical system. (Examples from [1]: $SO(3)$ for a rigid body or the group $\mathrm{SDiff}(M)$ of volume-preserving diffeomorphisms for an ideal fluid filling a domain M.) The tangent space at the identity of the Lie group G is the corresponding Lie algebra $\mathfrak{g}$. Fix some (positive definite) quadratic form, the *energy*, on $\mathfrak{g}$ and extend it through right translations to the tangent space at each point of the group (the "translational symmetry" of the energy). This way the energy defines a right-invariant Riemannian metric on the group G. The geodesic flow on G with respect to this metric represents the extremals of the least action principle, i.e., the actual motions of the physical system. (For a rigid body one has to consider left translations, but in our exposition we stick to the right-invariant case in view of its applications to the groups of diffeomorphisms.)

Given a geodesic on the *Lie group* with an initial velocity $v(0)$, we can right-translate its velocity vector at any moment t to the identity of the group. This way we obtain the evolution law for $v(t)$ given by a (non-linear) dynamical system $dv/dt = F(v)$ on the *Lie algebra* $\mathfrak{g}$.

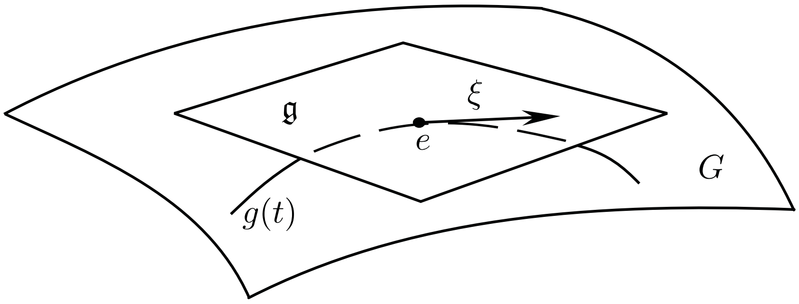

Figure 3: The vector ξ in the Lie algebra $\mathfrak{g}$ is the velocity at the identity e of a geodesic $g(t)$ on the Lie group G.

Definition 2.2 *The system on the Lie algebra $\mathfrak{g}$, describing the evolution of the velocity vector along a geodesic in a right-invariant metric on the Lie group G, is called the* Euler equation *corresponding to this metric on G.*

Many conservative dynamical systems in mathematical physics describe geodesic flows on appropriate Lie groups. Here we list several examples of such systems to emphasize the range of applications of this approach. The choice of a group G (column 1) and an energy metric E (column 2) defines the corresponding Euler equation (column 3)

Group	*Metric*	*Equation*
$SO(3)$	$< \omega, I\omega >$	Euler top
$SO(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equations for a body in a fluid
$SO(n)$	Manakov$'$s metrics	$n-$dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa – Holm equation
Virasoro	$\dot{H}^1$	Hunter – Saxton (or Dym) equation
$\mathrm{SDiff}(M)$	L^2	Euler ideal fluid
$\mathrm{SDiff}(M)$	H^1	Averaged Euler flow
$\mathrm{SDiff}(M) \ltimes \mathrm{SVect}(M)$	$L^2 + L^2$	Magnetohydrodynamics
$\mathrm{Maps}(S^1, SO(3))$	H^{-1}	Heisenberg magnetic chain

This list is by no means complete. There are many other interesting conservative systems, e.g., the super-KdV equations or equations of gas dynamics. We refer to [4] for more details.

2.3 Remark. It is curious to note that the similarity pointed out by V. Arnold between the Euler top on the group $SO(3)$ and Euler ideal fluid equations on $\mathrm{SDiff}(M)$, has a "magnetic analog": a similarity between the Kirchhoff and magneto-hydrodynamics equations, which are related to the *semi-direct product* groups. The Kirchhoff equations for a rigid body dynamics in a fluid is associated with the group $E(3) = SO(3) \ltimes \mathbb{R}^3$ of Euclidean motions of $\mathbb{R}^3$. The latter are described by pairs (a, b) consisting of a rotation $a \in SO(3)$ and a translation by a vector $b \in \mathbb{R}^3$. Similarly, magneto-hydrodynamics is governed by the group $S\mathrm{Diff}(M) \ltimes S\mathrm{Vect}(M)$, where elements (g, B) consist of a fluid configuration g and a magnetic field B (S.Vishik & F.Dolzhanskii 1978, [7]).

2.4 Remark. The differential-geometric description of the Euler equation as a geodesic flow on a Lie group has a Hamiltonian reformulation. Namely, identify the Lie algebra $\mathfrak{g}$ and its dual with the help of the energy quadratic form $E(v) = \frac{1}{2}\langle v, Iv\rangle$. This identification $I : \mathfrak{g} \to \mathfrak{g}^*$ (called the *inertia operator*) allows one to rewrite the Euler equation on the dual space $\mathfrak{g}^*$. It turns out that the Euler equation on $\mathfrak{g}^*$ is Hamiltonian with respect to the natural Lie–Poisson structure on the dual space. This means, in particular, that the trajectories of this dynamical system on the dual space are always tangent to the orbits of coadjoint action of the group, while invariants of the group action (called Casimir functions) provide a source of first integrals for the Euler equation.

3 Applications of the geometric approach.

3.A. Conservation laws in ideal hydrodynamics.

As the first application of the group-geodesic point of view, developed in [1], consider the construction of first integrals for fluid motion on manifolds of various dimensions. The Euler equation for an ideal fluid (1) filling a three-dimensional simply connected manifold has the helicity (or Hopf) invariant, discussed in Section 1. This invariant describes the mutual linking of the trajectories of the vorticity field curl v, and in the Euclidean space $\mathbb{R}^3$ it has the form

$$J(v) := \mathcal{H}(\mathrm{curl}\ v) = \int_{\mathbb{R}^3} (\mathrm{curl}\ v,\ v)\ d^3x\,.$$

Besides the energy integral, the helicity is essentially the only differential invariant for 3D flows (D. Serre 1979).[2]

On the other hand, for an ideal 2D fluid one has an infinite number of conserved quantities. For example, for the standard metric in $\mathbb{R}^2$ there are the *enstrophy invariants*

$$J_k(v) := \int_{\mathbb{R}^2} (\mathrm{curl}\ v)^k\ d^2x = \int_{\mathbb{R}^2} (\Delta\psi)^k\ d^2x \quad \text{for } k = 1, 2, \ldots,$$

where curl $v := \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}$ is the vorticity function, the Laplacian of the stream function ψ of the flow.

It turns out that helicity-type integrals do exist for all odd-dimensional ideal fluid flows, as do enstrophy-type integrals for all even-dimensional flows. (In a sense, the situation here is similar to the dichotomy of contact and symplectic geometry in odd- and even-dimensional spaces.) To describe the first integrals, consider the motion of an ideal fluid in a Riemannian manifold M equipped with a volume form μ. Define the 1-form u on M by lifting the indexes of the velocity field v using the Riemannian metric: $u(\xi) = (v, \xi)$ for all $\xi \in T_xM$.

[2] Although one can extract more subtle ergodic invariants from the asymptotic linking of trajectories of curl v.

Theorem 3.1 (D.Serre and L.Tartar (1984) for $\mathbb{R}^n$; V.Ovsienko, B.Khesin, & Yu.Chekanov (1989) for any M) *The Euler equation of an ideal incompressible fluid on an n-dimensional Riemannian manifold M (possibly with boundary) with a measure form μ has*

(i) the first integral

$$J(v) = \int_M u \wedge (du)^m$$

in the case of an arbitrary odd-dimensional manifold M $(n = 2m + 1)$; and

(ii) an infinite number of functionally independent first integrals

$$J_k(v) = \int_M \left(\frac{(du)^m}{\mu}\right)^k \mu \quad \text{for} \quad k = 1, 2, \ldots$$

in the case of an arbitrary even-dimensional manifold M $(n = 2m)$, where the 1-form u and the vector field v are related by means of the metric on M.

One can see that for domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ the integrals above become the helicity and enstrophy invariants. Furthermore, the geometric viewpoint implies that in the odd-dimensional case $n = 2m + 1$ the vorticity field ξ defined by $i_\xi \mu = (du)^m$ is "frozen into the fluid," i.e., transported by the flow. In the even-dimensional case $n = 2m$ the function $(du)^m/\mu$ is transported pointwise.

3.2 Remark. The above first integrals arise naturally in the Hamiltonian framework of the Euler equation for incompressible flows. Namely, for an ideal fluid the Lie algebra $\mathfrak{g} = \mathrm{SVect}(M)$ consists of divergence-free vector field in M. The 1-forms u (defined modulo function differentials) can be thought of as elements of the corresponding dual space $\mathfrak{g}^*$, while the lifting of indexes is the inertia operator $I : \mathfrak{g} \to \mathfrak{g}^*$. The invariance of the integrals in the theorem above essentially follows from their coordinate-free definition on this dual space. The Euler equation on $\mathfrak{g}^*$ can be rewritten as an equation on 1-forms:

$$\partial_t u + L_v u = -dp,$$

where one can recognize all the terms of the Euler equation (1) for an ideal fluid.

3.B. Stability of fluid motion.

The following stability experiment was apparently tried by everyone: watch the rotation of a tennis racket (or a book) thrown into the air. One immediately observes that the racket rotates stably about the axis through the handle, as well as the axis orthogonal to the racket surface. However, a tennis racket thrown up into the air rotating about the third axis (parallel to the surface, but orthogonal to the handle) makes unpredictable wild moves.

To describe the free motion of a rigid body look at its inertia ellipsoid. In general, it is not an ellipsoid of revolution, and it "approximates" the shape of the body. The stable stationary rotations about the two axes correspond to the longest and shortest axes of the inertia ellipsoid, while the rotation about the middle axis is always unstable. It turns out that our geodesic point of view is helpful in detecting stability of the corresponding stationary solutions, and, in the particular case of fluid motions, it yields sufficient conditions for stability in 2D ideal hydrodynamics (V.Arnold 1969, [3]).

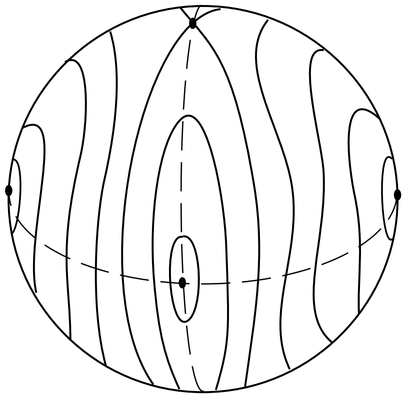

Figure 4: Energy levels on a coadjoint orbit of the Lie algebra $so(3)$ of a rigid body.

Suppose a (finite-dimensional) dynamical system has both an invariant foliation and a first integral E. Consider a point x_0 which is critical for the restriction of E to one of the leaves and suppose that the foliation is regular at that point. One can show that x_0 is a (Lyapunov) stable stationary point for the dynamical system, provided that the second differential of E restricted to the leaf of x_0 is positively or negatively defined. (Note that the converse is not true: a sign-indefinite second variation does not, in general, imply instability, as an example of a Hamiltonian system with $E = \omega_1(p_1^2 + q_1^2) - \omega_2(p_2^2 + q_2^2)$ shows.)

A similar consideration for any Lie algebra suggests the following sufficient condition for stability. As we discussed above, the Euler equation on a dual Lie algebra is always Hamiltonian and the corresponding dynamical system keeps the coadjoint orbits invariant. These orbits will play the role of the foliation, while the Hamiltonian function (the energy) is the first integral E. In the case of the rigid body the coadjoint orbits of the algebra $\mathfrak{g} = so(3)$ are spheres centered at the origin, while the energy levels form a family of ellipsoids. The energy restricted to each orbit has 6 critical points (being points of tangency of the sphere with the ellipsoids): 2 maxima, 2 minima, and 2 saddles (Fig.4). The maxima and minima correspond to stable rotations of the rigid body about the shortest and the longest axes of the inertia ellipsoid. The saddles correspond to unstable rotations about its middle axis.

This stability consideration can be developed in the infinite-dimensional situation of fluids, where one can justify the final conclusion about stability of flows without having to justify all of the intermediate constructions. The analogy between the equations of a rigid body and of an incompressible fluid enables one to study stability of steady flows by considering critical points of the energy function on the sets of isovorticed vector fields, which form the coadjoint orbits of the diffeomorphism group.

First, recall that in the 2D case stationary flows have the property that locally the stream function ψ is a function of vorticity, i.e. of the Laplacian of the stream function $\Delta\psi$. In other words, the gradient vectors of the stream function and of its Laplacian are collinear and, in particular, the ratio $\nabla\psi/\nabla\Delta\psi$ makes sense.

Theorem 3.3 [3] *Suppose that the stream function of a stationary flow, $\psi = \psi(x, y)$, in a region M is a function of the vorticity function $\Delta\psi$ not only locally, but globally. Then the*

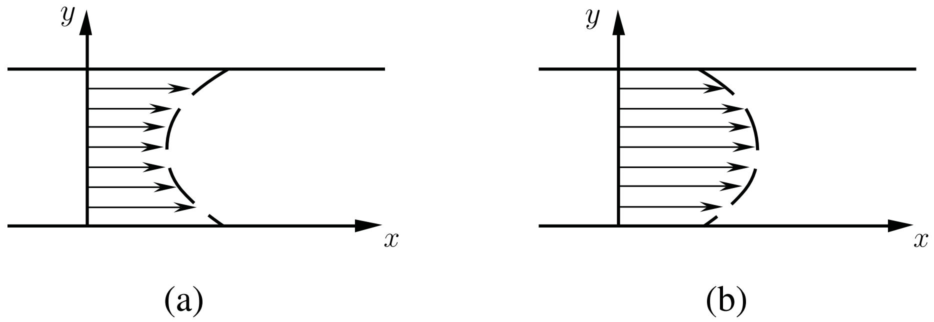

Figure 5: Lyapunov stable fluid flows in a strip. Profiles with the ratio (a) $v/v_{yy} > 0$ and (b) $v/v_{yy} < 0$.

stationary flow is stable provided that its stream function satisfies the following inequality:

$$0 < c \leq \frac{\nabla\psi}{\nabla\Delta\psi} \leq C < \infty$$

for some constants c and C. Moreover, there is an explicit estimate of the (time-dependent) deviation from the stationary flow in terms of the perturbation of the initial condition.

The above condition implies that the second variation $\delta^2 E$ of energy restricted to isovorticed fields is positive definite. A similar statement exists also for the negative-definite second variation, although to ensure the latter one has to impose not only some condition on the ratio $\nabla\psi/\nabla\Delta\psi$, but also on the geometry of the domain, see [3]. The underlying heuristic idea of the proof is that the first integral, which has a nondegenerate minimum or maximum at the stationary point ψ, after a normalization can be regarded as a "norm" which allows one to control the flow trajectories on the set of isovorticed fields. Note that invariants of such fields (i.e., Casimir functions of the group of area-preserving diffeomorphisms) play the role of Lagrange multipliers in the above study of the conditional extremum. We refer to the surveys [4, 7] and references therein for further applications and a study of stability by combining the energy function with Casimir functions for a number of physically interesting infinite-dimensional dynamical systems.

3.4 Example. [1, 3] Consider a steady planar shear flow in a horizontal strip in the (x, y)-plane with a velocity field $(v(y), 0)$, Fig.5. The form $\delta^2 E$ is positively or negatively defined if the velocity profile $v(y)$ has no zeroes and no points of inflection (i.e., $v \neq 0$ and $v_{yy} \neq 0$). The conclusion, that the planar parallel flows are stable, provided that there are no inflection points in the velocity profile, is a nonlinear analogue of the so-called Rayleigh theorem. Profiles with the ratio $v/v_{yy} > 0$ and $v/v_{yy} < 0$ are sketched in Figs.5a and 5b, respectively.

It turns out that the stability test for steady flows based on the second variation $\delta^2 E$ is inconclusive in dimensions greater than two: The second variation of the kinetic energy is never sign-definite in that case (see the discussion and references in [4]).

3.5 Remark. One should emphasize that the question under discussion is not stability "in a linear approximation," but the actual Lyapunov stability (i.e., with respect to finite perturbations in the nonlinear problem). The difference between these two forms of stability is substantial in this case, since the Euler equation is Hamiltonian. For Hamiltonian systems asymptotic stability is impossible, so stability in a linear approximation is always neutral and inconclusive about the stability of an equilibrium position of the nonlinear problem.

3.C. Bihamiltonian and Euler properties of the KdV, CH, and HS equations.

As we discussed above, the Eulerian nature of an equation implies that it is necessarily *Hamiltonian*, although, of course, not necessarily integrable (e.g., the equations of ideal fluids or magnetohydrodynamics). However, on certain lucky occasions, the Euler equations for some metrics and groups turn out to be *bihamiltonian* (and so completely integrable), while the geodesic description provides an insight into the corresponding structures.

This is the case, for example, with the family of equations

$$\alpha(u_t + 3uu_x) - \beta(u_{txx} + 2u_x u_{xx} + uu_{xxx}) - cu_{xxx} = 0 \tag{2}$$

on a function $u = u(t,x)$, $x \in S^1$, which for different values of parameters α, β and c combines several extensively studied nonlinear equations of mathematical physics, related to various hydrodynamical approximations. For non-zero c these are the Korteweg-de Vries equation ($\alpha = 1, \beta = 0$), the shallow water Camassa-Holm equation ($\alpha = \beta = 1$), and the Hunter–Saxton equation ($\alpha = 0, \beta = 1$), see the Table above. (Note that as a very degenerate case $c = \beta = 0$ this family also includes the Hopf, or inviscid Burgers, equation.) All these equations are known to possess infinitely many conserved quantities, as well as remarkable soliton or soliton–like solutions. It turns out that they all have a common symmetry group, the *Virasoro group*.

Definition 3.6 *The* Virasoro algebra *is a one-dimensional extension of the Lie algebra of vector fields on the circle, where the elements are the pairs* (a vector field $v(x)\partial_x$, a real number a) *and the commutator between such pairs is given by*

$$[(v\partial_x, a), (w\partial_x, b)] = \left(\ (-vw_x + v_x w)\partial_x, \int_{S^1} vw_{xxx}\, dx \ \right).$$

Note that the commutator does not depend on a and b, which means that the Virasoro algebra is a *central* extension of vector fields. The Virasoro group Vir is the corresponding extension of the diffeomorphism group of the circle. Given any α and β, equip this group with the right-invariant metric, generated by the following quadratic form, "$H^1_{\alpha,\beta}$-energy," on the Virasoro algebra:

$$\langle (v\partial_x, a), (w\partial_x, b)\rangle_{H^1_{\alpha,\beta}} = \int_{S^1} (\alpha\, vw + \beta\, v_x w_x)\, dx + ab\,.$$

For different values of α and β this family includes the L^2, H^1 and homogeneous $\dot{H}^1$-metrics. It turns out that the above equations can be regarded as equations of the geodesic flow related to different right-invariant metrics on the Virasoro group.

Theorem 3.7 (B.Khesin & G.Misiołek 2003)*For any α and β, the equation (2) is the Euler equation of the geodesic flow on the Virasoro group for the right-invariant $H^1_{\alpha,\beta}$-energy.*

This equation is bihamiltonian, possessing two Poisson structures: the linear Lie–Poisson structure (universal for all Euler equations) and a constant Poisson structure, depending on α and β. Moreover, the KdV, CH, and HS equations exactly correspond to (the choice of this constant structure at) three generic types of the Virasoro coadjoint orbits.

In particular, the KdV equation corresponds to the L^2-metric ($\alpha = 1$, $\beta = 0$, V.Ovsienko & B.Khesin 1987), while the Camassa–Holm equation corresponds to H^1 ($\alpha = \beta = 1$, G.Misiołek 1998). The Hunter–Saxton equation is related to the $\dot{H}^1$-norm ($\alpha = 0$, $\beta = 1$) defining a nondegenerate metric on the homogeneous space $Vir/\mathrm{Rot}(S^1)$.

The main feature of bihamiltonian systems is that they admit an infinite sequence of conserved quantities (obtained by the expansion of Casimir functions in the parameter interpolating between the Poisson structures) together with the whole hierarchy of commuting flows associated to them. The same family of equations also appears as a continuous limit of generic *discrete* Euler equations on the Virasoro group (A.Veselov and A.Penskoi 2003).

4 Geometry of the diffeomorphism groups

In the preceding two sections we were mostly concerned with similarities between the finite and infinite-dimensional groups and Hamiltonian systems and their hydrodynamical implications. However, the dynamics of an ideal fluid has many distinct and very peculiar properties (such as the existence of weak solutions not preserving the energy), while the corresponding configuration space, the group of volume-preserving diffeomorphisms, exhibits a very subtle differential geometry, that partially explains why the analysis of hydrodynamics equations is so difficult. In this section we survey several related results.

4.A. The diffeomorphism group as a metric space.

Consider a volume-preserving diffeomorphism of a bounded domain and think of it as a final fluid configuration for a fluid flow starting at the identity diffeomorphism. In order to reach the position prescribed by this diffeomorphism, every fluid particle has to move along some path in the domain. The distance of this diffeomorphism from the identity in the diffeomorphism group is the averaged characteristic of the path lengths of the particles.

It turns out that the geometry of diffeomorphism groups of two-dimensional manifolds differs drastically from that of higher-dimensional ones. This difference is due to the fact that in three (and more) dimensions there is enough space for particles to move to their final positions without hitting each other. On the other hand, the motion of the particles in the plane might necessitate their rotations about one another. The latter phenomenon of "braiding" makes the system of paths of particles in 2D necessarily long, in spite of the boundedness of the domain. The distinction between different dimensions can be formulated in terms of properties of $\mathrm{SDiff}(M)$ as a *metric space.*

Recall that on a Riemannian manifold M^n the group $\mathrm{SDiff}(M^n)$ of volume-preserving diffeomorphisms is equipped with the right-invariant L^2-metric, which is defined at the identity by the energy of vector fields. In other words, to any path $g(t,.)$, $\quad 0 \le t \le 1$, on $\mathrm{SDiff}(M)$ we associate its *length:*

$$\ell\{g(t,.)\} = \int_0^1 \left(\int_{M^n} |\partial_t g(t,x)|^2 d^n x \right)^{1/2} dt.$$

Then the *distance* between two fluid configurations $f, h \in \mathrm{SDiff}(M)$ is the infimum of the lengths of all paths in $\mathrm{SDiff}(M)$ connecting them: $\mathrm{dist}_{\mathrm{SDiff}}(f,h) = \inf \ell\{g(t,.)\}$. It is natural to call the *diameter* of the group $\mathrm{SDiff}(M)$ the supremum of distances between any two of its elements:

$$\mathrm{diam}\ (\ \mathrm{SDiff}(M)\ ) = \sup_{f,h \in \mathrm{SDiff}(M)} \mathrm{dist}_{\mathrm{SDiff}}(f,h).$$

Theorem 4.1 *i*) (A.Shnirelman 1985, 1994, [11]) *For a unit n-dimensional cube M^n where $n \ge 3$, the diameter of the group of smooth volume-preserving diffeomorphisms* $\mathrm{SDiff}(M)$ *is finite in the right-invariant metric* $\mathrm{dist}_{\mathrm{SDiff}}$*:*

$$\mathrm{diam}\ (\ \mathrm{SDiff}(M^n)\ ) \le 2\sqrt{\frac{n}{3}}.$$

ii) (Ya.Eliashberg & T.Ratiu 1991) *For an arbitrary manifold M of dimension $n = 2$, the diameter of the group* $\mathrm{SDiff}(M)$ *is infinite.*

Finiteness of the diameter holds for an arbitrary simply connected manifold M of dimension three or higher. However, the diameter can become infinite if the fundamental group of M is non-trivial (Ya.Eliashberg & T.Ratiu 1991). The two-dimensional case is completely different: the infiniteness of the diameter is of "local" nature. The main difference between the geometries of the groups of diffeomorphisms in two and three dimensions is based on the observation that for a long path on $\mathrm{SDiff}(M^3)$, which twists the particles in space, there always exists a "shortcut" untwisting them by making use of the third coordinate. (Compare this with the corresponding linear problems: $\pi_1(SL(2)) = \mathbb{Z}$, while $\pi_1(SL(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \ge 3$.)

4.2 REMARK. More precisely, for an n-dimensional cube ($n \ge 3$) the distance between two volume-preserving diffeomorphisms $f, h \in \mathrm{SDiff}(M)$ is bounded above by some power of the L^2-norm of the "difference" between them:

$$\mathrm{dist}_{\mathrm{SDiff}}(f,h) \le C \cdot ||f - h||^{\alpha}_{L^2(M)},$$

where the exponent α in this inequality is at least $2/(n+4)$, and, presumably, this estimate is sharp (A.Shnirelman 1994, [11]). This property means that the embedding of the group $\mathrm{SDiff}(M^n)$ into the vector space $L^2(M, \mathbb{R}^n)$ for $n \ge 3$ is "Hölder-regular" and, apparently, far from being smooth. Certainly, this Hölder property implies the finiteness of the diameter of the diffeomorphism group. A similar estimate exists for a simply connected higher-dimensional M.

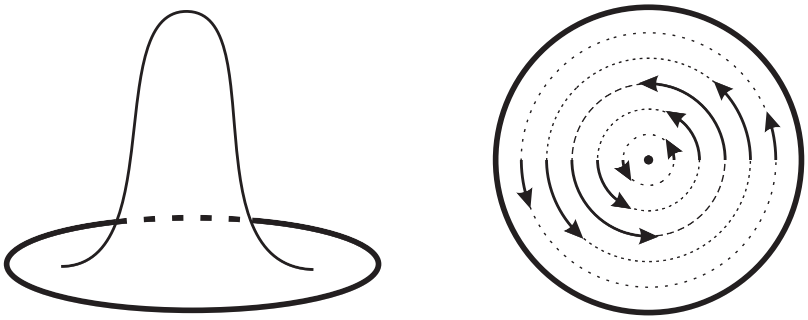

Figure 6: Profile of the Hamiltonian function (left) whose flow (right) for sufficiently long time provides "a long path" on the area-preserving diffeomorphism group in 2D.

However, no such estimate holds for $n = 2$: one can find a pair of volume-preserving diffeomorphisms arbitrarily far from each other on the group $\mathrm{SDiff}(M^2)$, but close in the L^2-metric on the square or a disk. For instance, an explicit example of a long path on this group is given by the following flow for sufficiently long time t: in polar coordinates it is defined by

$$(r, \phi) \mapsto (r, \phi + t \cdot v(r)),$$

where the angular velocity $v(r)$ is *non-constant*, see Fig.6. One can show that the distance of this diffeomorphism from the identity in the group grows linearly in time. As a matter of fact, the lengths of paths on the area-preserving diffeomorphism group in 2D is bounded below by the Calabi invariant in symplectic geometry (Ya.Eliashberg & T.Ratiu 1991, [10]).

4.B. Shortest paths and geodesics.

The above properties imply the following curious property of non-existence of the shortest path in the diffeomorphism groups:

Theorem 4.3 (Shnirelman 1985, [11]) *For a unit cube M^n of dimension $n \geq 3$, there exist a pair of volume-preserving diffeomorphisms that cannot be connected within the group $\mathrm{SDiff}(M)$ by a shortest path, i.e., for every path connecting the diffeomorphisms there always exists a shorter path.*

While the long-time existence and uniqueness for the Cauchy problem of the 3D Euler hydrodynamics equation is still an open problem (see the survey P.Constantin 1995), the above theorem proves the non-existence for the corresponding two-point boundary problem. Thus, the attractive variational approach to constructing solutions of the Euler equations is not directly available in the hydrodynamical situation. Y.Brenier (1989) found a natural class of "generalized incompressible flows" for which the variational problem is always solvable (a shortest path always exists) and developed their theory. Generalized flows are a far-reaching generalization of the classical flows, where fluid particles are not only allowed to move independently from each other, but also their trajectories may meet each other: the particles may split and collide. In a sense, the particles are replaced by "clouds of particles" with the only restrictions that the density of particles remains constant all the time and that the mean kinetic energy is finite [5].

On the other hand, in 2D the corresponding shortest path problem always has a solution in terms of *continual braids*, yet another "intrusion" of topology to fluid dynamics (A.Shnirelman 2001). These shortest braids have a well-defined L^2-velocity, which gives a weak solution of the 2D Euler equation. (One can compare this with the long-time existence result in the 2D ideal hydrodynamics (V.Yudovich 1963).) Furthermore, shortest braids provide minimizers of magnetic energy in a cylinder or in a narrow 3D ring, i.e., give partial answers in the energy relaxation problem discussed in the beginning of the paper!

4.4 Remark. The Riemannian geometry of the group SDiff(M) not only defines the geodesics, solutions to the Euler hydrodynamics equation, but also sheds the light on their properties. D.Ebin & J.Marsden (1970) established the smoothness of the geodesic spray on this group, which yielded local existence and uniqueness results in Sobolev spaces. They also showed that in any dimension any two sufficiently close diffeomorphisms can always be connected by a shortest path. The existence of conjugate points along the geodesics, where they cease to be length minimizing, was addressed by G.Misiołek (1996).

The study of sectional curvatures for the right-invariant L^2-metric showed that the diffeomorphism group looks rather like a negatively curved manifold and allowed one to give explicit estimates on the divergence of geodesics on the group (V.Arnold 1966, A.Lukatsky 1979, S.Preston 2002). Regarding the Earth atmosphere, with the equator length of 40,000km and the characteristic depth of 14km, as an ideal 2D fluid on a sphere, one obtains, in particular, the low predictability of motion of atmospheric flows for two weeks, as discussed in the introduction. Curiously, at a recent lecture a former head of the UK Meteorological Office said that he would not trust any weather forecast beyond 3 days!

Acknowledgments

The author is indebted to A. Shnirelman and G. Misiołek for numerous suggestions on the manuscript and to D. Novikov for drawing the figures. The present work was partially sponsored by an NSERC research grant. The image of the Earth atmosphere was taken by the NASA satelite GOES-7.

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