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Affine Gelfand-Dickey brackets and holomorphic vector bundles

Pavel I. Etingof and Boris A. Khesin

Yale University
Department of Mathematics
New Haven, CT 06520 USA
e-mail: etingof @ math.yale.edu
khesin @ math.yale.edu

Abstract

We define the (second) Adler-Gelfand-Dickey Poisson structure on differential operators over an elliptic curve and classify symplectic leaves of this structure. This problem leads to the problem of classification of coadjoint orbits for double loop algebras, conjugacy classes in loop groups, and holomorphic vector bundles over the elliptic curve. We show that symplectic leaves have a finite but (unlike the traditional case of operators on the circle) arbitrarily large codimension, and compute it explicitly.

Introduction

In the seventies M.Adler[A], I.M.Gelfand and L.A.Dickey [GD] discovered a natural Poisson structure on the space of n-th order differential operators on the circle with highest coefficient 1 which is now called the (second) Gelfand-Dickey bracket. This bracket arises in the theory of nonlinear integrable equations under various names (nKdV-structure, classical W_n-algebra). B.L.Feigin proposed to consider and study symplectic leaves for the Gelfand-Dickey bracket – a problem motivated by the fact that for $n = 2$ these symplectic leaves are orbits of coadjoint representation of the Virasoro algebra. A classification of symplectic leaves for the Gelfand-Dickey bracket and a description of their adjacency were given in [OK]. It turned out that locally symplectic leaves are labeled by one of the following:

1) conjugacy classes in the group GL_n;

2) orbits of the coadjoint representation of the affine Lie algebra $\widehat{\mathfrak{gl}_n}$;

3) equivalence classes of flat vector bundles on the circle of rank n (these three things are in one-to-one correspondence).

Furthemore, the codimension of a symplectic leaf is equal to any of the following:

1) the dimension of the centralizer of the corresponding conjugacy class;

2) the codimension of the corresponding coadjoint orbit;

3) the dimension of the space of flat global sections of the bundle of endomorphisms of the corresponding flat vector bundle.

In Section 1 of this paper we define an "affine" analogue of the Gelfand-Dickey bracket. It is realized on the space of n-th order differential operators on an elliptic curve which are polynomials in $\overline{\partial}$ with smooth coefficients and highest coefficient 1. The main goal of the paper is to classify and study the symplectic leaves of the affine Gelfand-Dickey bracket.

In Section 2 we show that locally symplectic leaves of this bracket are labeled by

1) Conjugacy classes for the action of the loop group $LGL_n(\mathbb{C})$ on the semidirect product $\mathbb{C}^* \ltimes LGL_n(\mathbb{C})_0$ (where $LGL_n(\mathbb{C})_0$ denotes the connected component of the identity in the group $LGL_n(\mathbb{C})$);

2) orbits of the coadjoint representation of the "double"' affine Lie algebra – a central extension of the Lie algebra of $\mathfrak{gl}_n$-valued smooth functions on the elliptic curve [EF];

3) equivalence classes of holomorphic vector bundles of rank n and degree zero on the elliptic curve (as before, these three things are in one-to-one correspondence).

Since holomorphic vector bundles over an elliptic curve are completely classified [At], this result gives a good description of symplectic leaves.

In Section 3 we show that the codimension of a symplectic leaf is equal to

1) the dimension of the centralizer of the corresponding conjugacy class;

2) the codimesion of the corresponding coadjoint orbit;

3) the dimension of the space of holomorphic sections of the bundle of endomorphisms of the corresponding holomorphic vector bundle.

In particular, this implies that in the affine case the codimension of a symplectic leaf, though always finite, can be arbitrarily large, even for $n = 2$ (See Theorem 5B and Proposition 8B), unlike the finite dimensional case, in which it is bounded from above by $\dim GL_n = n^2$.

These results constitute a two dimensional (or affine) counterpart of the results of [OK] for Gelfand-Dickey brackets. Similarly to the non-affine case, they can be generalized to other classical Lie groups – SL_n, Sp_{2n}, SO_{2n+1} (see [OK]).

In Section 4 of the paper we discuss the question whether the map assigning an equivalence class of vector bundles to a symplectic leaf is surjective. This question is equivalent to the question whether any monodromy (=vector bundle) can be realized by an n-th order differential operator. For the usual Gelfand-Dickey bracket the answer to this question is positive (it follows, for example, from the results of M.Shapiro [S]). We prove that the answer is positive in the affine case as well.

In Section 5, we describe an explicit realization of all possible monodromies for $n = 2$, using Atiyah's classification of vector bundles over an elliptic curve.

In the Appendix we discuss the interesting problem of describing the global structure of the fibration of the space of differential operators by symplectic leaves.

It turns out that two distinct symplectic leaves may correspond to the same monodromy. In the finite-dimensional case, the problem of counting symplectic leaves with a given monodromy is defined geometrically by homotopy classification of quasiperiodic nonflattening curves on a real projective space [O,OK,KS]. The problem of counting symplectic leaves of the affine GL_2-Gelfand-Dickey bracket corresponding to the trivial rank 2 vector bundle reduces to the topological problem of classification of nowhere holomorphic maps from an elliptic curve to the complex projective line (i.e. maps f with nonvanishing $\overline{\partial} f$) up to homotopy. In the affine GL_n case we encounter the problem of homotopy classification of maps f from an elliptic curve to $\mathbb{C}P^{n-1}$ such that the vectors $\overline{\partial} f, ..., \overline{\partial}^{n-1} f$ are everywhere linearly independent. (These maps are the affine counterparts of nonflattening curves in $\mathbb{R}P^{n-1}$). At the moment a complete solution of this problem (even in the GL_2-case) is unknown to the authors.

Remarks. 1. The reason to consider affine Gelfand-Dickey brackets is a search for an appropriate two-dimensional counterpart of the theory of affine Lie algebras. One can show that the "affine" analogue of the Drinfeld-Sokolov reduction [DS] sends the linear Poisson bracket on the double loop algebra (cf.[EF]) into the quadratic Gelfand-Dickey bracket on the space of differential operators on the elliptic curve.

2. In the case $n = 2$, the problem of classification of symplectic leaves coincides with the problem of classification of orbits of the coadjoint representation of the complex Virasoro algebra defined in [EF] – the Lie algebra of pairs (f, a) where f is a smooth function on an elliptic curve M and a is a complex number, with the commutation law $[(f, a)(g, b)] = \left(f\overline{\partial} g - g\overline{\partial} f, \int_M f\overline{\partial}^3 g\right)$.

3. The key tool in the study of Gelfand-Dickey brackets is the notion of monodromy of a differential operator. For the case of the circle, monodromy is a conjugacy class in the group GL_n. For the case of an elliptic curve, monodromy is a conjugacy class in the affine GL_n (more precisely, a conjugacy class of the action of $LGL_n(\mathbb{C})$ in the one-dimensional extension $\mathbb{C}^* \ltimes LGL_n(\mathbb{C})_0$ of the loop group of GL_n). This justifies the name "affine Gelfand-Dickey bracket".

4. One can define versal deformations of symplectic leaves following [LP],[OK]. They are equivalent to the deformations of the corresponding monodromies. This implies that adjacency of symplectic leaves is the same as that of orbits, conjugacy classes, and vector bundles.[1]

5. It would be interesting to find the counterpart of the affine GD bracket for surfaces of higher genus. A good definition of this object should lead to symplectic

[1] A symplectic leaf, coadjoint orbit, conjugacy class, vector bundle $\mathcal{O}_1$ is called adjacent to $\mathcal{O}_2$ if the closure of $\mathcal{O}_2$ contains $\mathcal{O}_1$; for symplectic leaves, orbits, and conjugacy classes, the closure is in the C^∞-sense, and for vector bundles it is in the sense of Zariski topology on the moduli space of bundles

leaves of finite codimension, like in the case of an elliptic curve. These symplectic leaves should be labeled by coadjoint orbits of the central extension of the Lie algebra of matrix-valued functions on the surface described in [EF], or by equivalence classes of holomorphic vector bundles over the surface (it is shown in [EF] that these two things are in one-to-one correspondence).

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1. Gelfand-Dickey brackets.

We start by recalling the definition of the Gelfand-Dickey structures (see [A,GD,DS]).

Let M be a compact smooth orientable closed manifold, $k=\mathbb{R}$ or $\mathbb{C}$, $C^\infty(M,k)$ be the algebra of smooth k-valued functions on M, ω be a volume form on M. Let D be a differential operator on $C^\infty(M,k)$ such that $\int_M (Df)\omega = 0$ and $D(fg) = (Df)g + f(Dg)$ for any $f,g\in C^\infty(M,k)$.

Define the vector space $\tilde{\mathcal{L}}$ as follows:

$$\tilde{\mathcal{L}} = \{P = \sum_{m=0}^{n-1} u_{m+1}D^m | u_m \in C^\infty(M,k)\}. \tag{1.1}$$

To realize the dual space to $\tilde{\mathcal{L}}$, we need to introduce pseudodifferential symbols. They are formal expressions of the form $\sum_{m=m_0}^{\infty} a_m D^{-m}$, $m_0\in\mathbb{Z}$, $a_m\in C^\infty(M,k)$. It is known that such symbols form an associative algebra: two symbols A,B can be multiplied with the help of the rules $D\circ f = f\circ D + Df$, $D^{-1}\circ f = f\circ D^{-1} - f'\circ D^{-2} + f''\circ D^{-3} - ...$, for any $f\in C^\infty(M,k)$.

We realize (the regular part of) the dual space to $\tilde{\mathcal{L}}$ as follows:

$$\mathcal{A} = \{A = \sum_{m=1}^{n} a_m D^{-m} | a_m \in C^\infty(M,k)\}, \tag{1.2}$$

and the pairing $\tilde{\mathcal{L}}\otimes\mathcal{A}\to k$ is given by the formula

$$< P, A >= \int_M \mathrm{Res}(PA)\omega, \tag{1.3}$$

where $\mathrm{Res}(X)$ is the coefficient to D^{-1} in a pseudodifferential operator X. It is clear that any regular linear functional on $\tilde{\mathcal{L}}$ has this form.

Note that $\mathrm{Res}(PA-AP) = Df$, where f is some function on M, which implies that $\int_M \mathrm{Res}(PA)\omega = \int_M \mathrm{Res}(AP)\omega$.

Let $\mathcal{L}$ be the affine space of all operators of the form $L = D^n + P$, $P \in \tilde{\mathcal{L}}$. Clearly, the tangent space to $\mathcal{L}$ at any point is naturally identified with $\tilde{\mathcal{L}}$.

Following Adler, Gelfand and Dickey, let us assign a vector field V_A on $\mathcal{L}$ to every regular linear functional A on $\tilde{\mathcal{L}}$. Its value at a point $L \in \mathcal{L}$ is:

$$V_A(L) = L(AL)_+ - (LA)_+ L, \tag{1.4}$$

where X_+ denotes the differential part of X.

Let $\mathcal{C}$ denote the algebra of smooth functions on $\mathcal{L}$ for $k = \mathbb{R}$, and the algebra of holomorphic functions on $\mathcal{L}$ for $k = \mathbb{C}$. Then assignment (1.4) allows one to define a Poisson bracket on $\mathcal{C}$:

$$\{f, g\}(L) = < dg\mid_L, V_{df\mid_L}(L) > . \tag{1.5}$$

Let us call this bracket *the Gelfand-Dickey (GD) bracket.* It equips $\mathcal{L}$ with a structure of a Poisson manifold.

Let us now define symplectic leaves of the GD bracket and their codimensions (cf. [K2],[We]).

Let $L \in \mathcal{L}$. A vector $v \in T_L\mathcal{L} = \tilde{\mathcal{L}}$ is called a Hamiltonian vector if there exists $A \in \mathcal{A}$ such that $v = V_A(L)$.

Define the symplectic leaf $\mathcal{O}_L$ to be the set of all points $L' \in \mathcal{L}$ such that there exists a smooth curve $\gamma : [0, 1] \to \mathcal{L}$ such that $\gamma(0) = L$, $\gamma(1) = L'$, and $\frac{d\gamma}{dt}$ is a Hamiltonian vector for any t. It is clear that two symplectic leaves are either disjoint or identical. Therefore, the space $\mathcal{L}$ becomes a disjoint union of symplectic leaves.

The tangent space $T_L\mathcal{O}_L \subset \tilde{\mathcal{L}}$ to the symplectic leaf $\mathcal{O}_L$ at L is obviously the space of all Hamiltonian vectors at L. Define the codimension of $\mathcal{O}_L$ to be the codimension of this tangent space in $\tilde{\mathcal{L}}$. This definition makes sense because the codimension is the same at all points of $\mathcal{O}_L$.

We will be concerned with the following two special cases of GD brackets.

Main definition.

Case 1. *$M = S^1$, $k = \mathbb{R}$ or $\mathbb{C}$, $D = \frac{d}{dx}$, $\omega = dx$. The GD bracket corresponding to this situation is called the $GL_n(k)$-GD bracket* [GD].

Case 2. *M is a nondegenerate elliptic curve over $\mathbb{C}$: $M = \mathbb{C}/\Gamma$, where Γ is a lattice generated by 1 and τ, where $Im\ \tau > 0$, $k = \mathbb{C}$, $D = \overline{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, where $z = x+iy$ is the standard complex coordinate on $\mathbb{C}$, $\omega = \frac{i}{2} dz \wedge d\bar{z}$. The space $\mathcal{L}$ consists of differential operators $\overline{\partial}^n + \sum_{j=0}^{n-1} u_{j+1}(z, \bar{z})\overline{\partial}^j$, where $u_i \in C^\infty(\mathbb{C}/\Gamma, \mathbb{C})$. We call the GD bracket corresponding to this case the affine GL_n-GD bracket.*

Symplectic leaves of the GL_n-GD bracket are described in [OK]. In this paper, a similar description is given for symplectic leaves of the affine GL_n-GD bracket.

To emphasize the parallel between the non-affine and affine theories, we give an exposition of both of them, marking definitions and statements from the non-affine theory by the letter A and from the affine theory by the letter B.

2. Local classification of symplectic leaves

Definition 1AB. *Let $\mathbf{f} = (f_1, ..., f_n)$ be a smooth k^n-valued function on some covering of M ($k = \mathbb{R}$ or $\mathbb{C}$). The matrix-valued function $W(\mathbf{f}) = (w_{ij})$, $w_{ij} = D^{i-1} f_j$ is called the Wronski matrix of $\mathbf{f}$.*

We start by recalling a standard statement from the theory of ordinary differential equations.

Proposition 1A. *Let L be a differential operator of the form $L = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} u_{j+1} \frac{d^j}{dx^j}$, $u_j \in C^\infty(S^1, k)$. Then:*

(i) there exists a set of n solutions $\mathbf{f} = (f_1, ..., f_n)$ of the equation $L\phi = 0$ belonging to $C^\infty(\mathbb{R}, k)$ whose Wronski matrix is everywhere nondegenerate (here $\mathbb{R}$ is regarded as a cover of S^1);

(ii) if $\tilde{\mathbf{f}} = (\tilde{f}_1, ..., \tilde{f}_n)$ is another set of solutions satisfying (i) then there exists a unique matrix $R \in GL_n(k)$ such that $\tilde{\mathbf{f}} = \mathbf{f} R$;

(iii) if $\mathbf{f} = (f_1, ..., f_n)$ is any set of smooth k-valued functions on the real line such that its Wronski matrix is everywhere nondegenerate, and if $\mathbf{f}(x+1) = \mathbf{f}(x) R$ for some $R \in GL_n(k)$, then there exists a unique differential operator $L = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} u_{j+1} \frac{d^j}{dx^j}$ with periodic coefficients such that $L f_i = 0$ for all i.

Let $\Sigma = \mathbb{C}/\mathbb{Z}$ be a cylinder. It has a natural structure of an abelian group, is equivalent to $\mathbb{C}^*$ as a complex manifold, and naturally covers the elliptic curve $M = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. From now on we do not make a distinction between Σ and $\mathbb{C}^*$.

Before we formulate the affine analogue of Proposition 1A, we need to define loop groups. We will need three versions of a loop group for $GL_n(\mathbb{C})$:

Notation. *$LGL_n(\mathbb{C})$ is the group of holomorphic $GL_n(\mathbb{C})$-valued functions on Σ. $LGL_n(\mathbb{C})_0$ is the connected component of identity in $LGL_n(\mathbb{C})$. $\overline{GL_n}(\mathbb{C})$ is the semidirect product $\Sigma \ltimes LGL_n(\mathbb{C})_0$, where Σ acts on $LGL_n(\mathbb{C})_0$ by $(z \circ g)(w) = g(w + z)$.*

The group $\overline{GL_n}(\mathbb{C})$ should be regarded as the group of pairs $(g(\cdot), \tau)$, $g \in LGL_n(\mathbb{C})_0$, $\tau \in \Sigma$, with the multiplication law $(g(z), \tau)(h(z), \theta) = (g(z)h(z+\tau), \tau + \theta)$. It is clear that $LGL_n(\mathbb{C})_0$ is embedded into $\overline{GL_n}(\mathbb{C})$ by the map $g(\cdot) \to (g(\cdot), 0)$.

Consider the action of $LGL_n(\mathbb{C})$ on $\overline{GL_n}(\mathbb{C})$ by conjugacy. We will call the orbits of this action *restricted conjugacy classes.*

Proposition 1B. *Let L be a differential operator of the form $L = \overline{\partial}^n + \sum_{j=0}^{n-1} u_{j+1} \overline{\partial}^j$, $u_j \in C^\infty(M, \mathbb{C})$, where M is an elliptic curve. Then:*

(i) there exists a set of n solutions $\mathbf{f} = (f_1, ..., f_n)$ of the equation $L\phi = 0$ belonging to $C^\infty(\Sigma, \mathbb{C})$ whose Wronski matrix is everywhere nondegenerate (here Σ is regarded as a cover of M);

(ii) if $\tilde{\mathbf{f}} = (\tilde{f}_1, ..., \tilde{f}_n)$ is another set of solutions satisfying (i) then there exists a unique matrix $R(z) \in LGL_n(\mathbb{C})$ such that $\tilde{\mathbf{f}} = \mathbf{f}R$.

(iii) if $\mathbf{f} = (f_1, ..., f_n)$ is any set of smooth complex-valued functions on Σ such that its Wronski matrix is everywhere nondegenerate, and if $\mathbf{f}(z+\tau) = \mathbf{f}(z)R(z)$ for some $R(z) \in LGL_n(\mathbb{C})$, then there exists a unique differential operator $L = \overline{\partial}^n + \sum_{j=0}^{n-1} u_{j+1}\overline{\partial}^j$ such that $Lf_i = 0$ for all i.

Proof. First of all, statements (i) and (ii) are true in a small enough neighborhood U_p of every point $p \in \Sigma$ [AB]. Let $\mathbf{g}^p = (g_1^p, ..., g_n^p)$ be the corresponding sets of solutions. By the local version of statement (ii), whenever U_p and U_q intersect, $g_j^p = \sum_{i=1}^n g_i^q Q_{ij}^{pq}$, where $Q^{pq}(z)$ are holomorphic $GL_n(\mathbb{C})$-valued functions on $U_p \cap U_q$. These functions satisfy the conditions: $Q^{pq}Q^{qp} = 1$, $Q^{pq}Q^{qr}Q^{rp} = 1$, which imply that they are clutching transformations of some holomorphic vector bundle E_L of rank n on Σ.

Since Σ is equivalent to $\mathbb{C}^*$ as a complex manifold, any holomorphic vector bundle over Σ has to be trivial. This, of course, applies to E_L, which implies that E_L has n global holomorphic sections $s_1, ..., s_n$ which are everywhere linearly independent. That is to say, for every $p \in \Sigma$ there exists a holomorphic function $S^p(z)$ on U_p with values in $GL_n(\mathbb{C})$ such that $S^p = Q^{pq}S^q$ on $U_p \cap U_q$ for any $p, q \in \Sigma$ (s_i are the columns of S). Therefore, the functions $f_j^p = \sum_i g_i^p S_{ij}^p$ satisfy the condition $f_j^p = f_j^q$ on $U_p \cap U_q$. This means, we have a globally defined vector-function $\mathbf{f} = (f_1, ..., f_n)$, such that $f_j \mid_{U_p} = f_j^p$. Since the functions $S_{ij}^p(z)$ are holomorphic, the functions f_j satisfy the equation $Lf_j = 0$. Also, $W(\mathbf{f}) = W(\mathbf{g}^p)S^p$ in every U_p, which implies $W(\mathbf{f})$ is everywhere nondegenerate. This settles (i).

Now let ϕ be any smooth complex function on Σ. Consider the column vector $\Phi = (\phi, \overline{\partial}\phi, ..., \overline{\partial}^{n-1}\phi)^t$. It is obvious that ϕ is a solution of $L\phi = 0$ if and only if Φ satisfies the first order $n \times n$-matrix equation $\overline{\partial}\Phi = A_L\Phi$, where A_L is the Frobenius matrix corresponding to L:

$$A_L = \begin{pmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 \\ -u_1 & -u_2 & \dots & \dots & -u_n \end{pmatrix}, \text{ i.e. } (A_L)_{ij} = \begin{cases} 1 & j-i=1 \\ -u_j & i=n \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

This implies that if $\mathbf{f} = (f_1, ..., f_n)$ is a set of solutions to $L\phi = 0$ then the Wronski matrix $W(\mathbf{f})$ satisfies the equation

$$\overline{\partial}W = A_L W. \tag{2.2}$$

To prove (ii), define the matrix function R on Σ by $W(\tilde{\mathbf{f}}) = W(\mathbf{f})R$. This matrix is obviously always in $GL_n(\mathbb{C})$, and it is holomorphic on Σ because both $W(\tilde{\mathbf{f}})$ and $W(\mathbf{f})$ satisfy the equation $\overline{\partial} W = A_L W$. Thus, $R \in LGL_n(\mathbb{C})$.

To establish (iii), for any $\mathbf{f}$ satisfying the conditions of (iii) define the vector-function $\mathbf{u} = (u_1, ..., u_n)$ on Σ by the formula

$$\mathbf{u} = (\overline{\partial}^n \mathbf{f}) W(\mathbf{f})^{-1}. \tag{2.3}$$

This vector function exists and is unique because of the nondegeneracy of W. Moreover, it is τ-periodic since both $\overline{\partial}^n \mathbf{f}$ and $W(\mathbf{f})$ multiply by R from the right when z is replaced by $z + \tau$. Now set $L = \overline{\partial}^n + \sum_{j=0}^{n-1} u_{j+1} \overline{\partial}^j$. It is obvious that (2.3) is equivalent to the condition that $L f_i = 0$ for all i, which implies (iii). □

Propositions 1A and 1B have a simple geometric reformulation:

Proposition 1AB. *For every vector-function $\mathbf{f}$ with a nondegenerate Wronski matrix there exists a unique differential operator $L_{\mathbf{f}} \in \mathcal{L}$ such that $L_{\mathbf{f}} f_i = 0$, and the assignment $\mathbf{f} \to L_{\mathbf{f}}$ is a principal fibration over $\mathcal{L}$ whose fiber is $GL_n(k)$ in Case 1 and $LGL_n(\mathbb{C})$ in Case 2.*

Corollary 2AB. *Let $L(t)$ be any smooth curve in $\mathcal{L}$. Then there exists a smooth family of vector-functions $\mathbf{f}^t$ with a nondegenerate Wronski matrix such that $L(t) f_i^t = 0$ for all i and for all t.*

Proof. This is just the statement that any path on the base of a fiber bundle can be covered by a path on the total space. □

Let us now define the notion of monodromy of a differential operator.

Definition 2A. *Let L be a differential operator of the form $L = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} u_{j+1} \frac{d^j}{dx^j}$, $u_j \in C^\infty(\mathbb{R}/\mathbb{Z}, k)$. Let $\mathbf{f} = (f_1, ..., f_n)$ be a set of solutions of the equation $L\phi = 0$ belonging to $C^\infty(\mathbb{R}, k)$ whose Wronski matrix is everywhere nondegenerate. Let $R \in GL_n(k)$ be the matrix such that $\mathbf{f}(x+1) = \mathbf{f}(x) R$ (it exists because of Prop. 1A (ii)). Then the conjugacy class of R in $GL_n(k)$ is called the monodromy of L.*

Note that the matrix R itself (unlike the conjugacy class of R, cf. Proposition 1A (ii)) is not well defined since it relies on the choice of the set of solutions $\mathbf{f}$.

Definition 2B. *Let L be a differential operator of the form $L = \overline{\partial}^n + \sum_{j=0}^{n-1} u_{j+1} \overline{\partial}^j$, $u_j \in C^\infty(M, \mathbb{C})$ (M is an elliptic curve). Let $\mathbf{f} = (f_1, ..., f_n)$ be a set of solutions of the equation $L\phi = 0$ belonging to $C^\infty(\Sigma, \mathbb{C})$ whose Wronski matrix is everywhere nondegenerate. Let $R \in LGL_n(\mathbb{C})$ be the matrix such that $\mathbf{f}(z+\tau) = \mathbf{f}(z) R(z)$ (it exists because of Prop. 1B (ii)). Then the restricted conjugacy class of the element (R, τ) in $\overline{GL_n}(\mathbb{C})$ is called the monodromy of L.*

Remarks. 1. The reason for Definition 2B is the following: if $\mathbf{g}(z) = \mathbf{f}(z) Q(z)$ is another set of solutions (i.e. $Q(z) \in LGL_n(\mathbb{C})$), then $\mathbf{g}(z+\tau) = \mathbf{g}(z) \tilde{R}(z)$, where

$\tilde{R}(z) = Q^{-1}(z)R(z)Q(z+\tau)$, which corresponds to conjugation of the element $(R,\tau) \in \overline{GL_n}(\mathbb{C})$ by $(Q^{-1},0)$. Since any set of solutions has the form $\mathbf{f}(z)Q(z)$, where Q is a holomorphic matrix (Proposition 1B, part (ii)), monodromy is well defined, i.e. does not depend on the choice of $\mathbf{f}$.

2. Note that for differential equations on the line there is a canonical choice of a set of solutions $\mathbf{f}$ – the set whose Wronski matrix is the identity matrix at a fixed point x_0 of the line (the fundamental system of solutions). The notion of a fundamental system of solutions does not have a natural analogue in two dimensions.

3. Observe that in Case 2 the monodromy matrix $R(z)$ is always in $LGL_n(\mathbb{C})_0$. Indeed, $\det R(z) = \frac{\det W(\mathbf{f})(z+\tau)}{\det W(\mathbf{f})(z)}$, which means that the map $z \to \det R(z)$ is homotopic to the identity: the homotopy is $\phi_s(z) = \frac{\det W(\mathbf{f})(z+s\tau)}{\det W(\mathbf{f})(z)}$, $s \in [0,1]$. For a similar reason, in Case 1 if $k=\mathbb{R}$ then the determinant of R is always positive.

Now we are ready to formulate the main theorem about the local structure of the fibration of $\mathcal{L}$ into symplectic leaves.

Theorem 3AB. *Let $L(t), a<t<b$ be a smooth curve in $\mathcal{L}$. Then $L(t)$ lies inside a single symplectic leaf if and only if the monodromy of $L(t)$ is the same for all t.*

The proof of this theorem for Case 1 was given in [OK]. Before proving Case 2, let us give a reformulation of the isomonodromic condition in terms of vector bundles and in terms of coadjoint orbits of double loop algebras.

Define the rank n vector bundle $\mathcal{E}_L$ on M corresponding to a differential operator $L \in \mathcal{L}$. It will be a flat k-bundle in Case 1 and a holomorphic bundle in Case 2.

For every $p \in M$ let U_p be the neighborhood of p such that there exists a set $\mathbf{f} = (f_1^p, ..., f_n^p)$ of n solutions of the equation $L\phi = 0$ defined in U_p whose Wronski matrix is nondegenerate in U_p. Let the matrices Q^{pq} (belonging to $GL_n(k)$ in Case 1 and $LGL_n(\mathbb{C})$ in Case 2) be defined by the condition $\mathbf{f}^q = \mathbf{f}^p Q^{pq}$. Then Q^{pq} satisfy the conditions $Q^{pq}Q^{qp}=1$, $Q^{pq}Q^{qr}Q^{rp}=1$.

Definition 3AB. *The vector bundle $\mathcal{E}_L$ is the bundle on M defined by the set of transition functions Q^{pq}.*

There is another, more explicit construction of the vector bundle $\mathcal{E}_L$. Let R be a monodromy matrix for L. Let $\hat{M}$ be the interval $[0,1]$ in Case 1 and the annulus $\{x+\tau y \in \Sigma | 0 \le y \le 1\}$ in Case 2. Define the vector bundle $\mathcal{E}_L$ on M as follows. Take a trivial rank n bundle over $\hat{M}$ and glue the two boundaries of $\hat{M}$ together: $0 \sim 1$ in Case 1, $x \sim x+\tau$ in Case 2 (this will transform $\hat{M}$ into M), identifying the fibers over corresponding points by means of the monodromy matrix R. It is easy to check that the obtained flat (holomorphic) vector bundle over M is isomorphic to $\mathcal{E}_L$.

Thus, global smooth sections of $\mathcal{E}_L$ can be realized as quasiperiodic vector-functions on $\mathbb{R}$ (respectively on Σ), i.e. as such functions $\mathbf{f}$ that $\mathbf{f}(x+1) = \mathbf{f}(x)R$ (respectively $\mathbf{f}(z+\tau) = \mathbf{f}(z)R(z)$).

Let us now define affine and double affine Lie algebras. Let $\mathfrak{g}(M) = C^\infty(M, \mathfrak{gl}_n(k)) \oplus \mathbb{C}$ be the one dimensional central extension of $C^\infty(M, \mathfrak{gl}_n(k))$ by means of the cocycle $\Omega(f,g) = \int_M \operatorname{tr}(fDg)\omega$. In the one-dimensional case it is the usual affine Lie algebra. In the two-dimensional case it is the double affine algebra considered in [EF].

It is known that the Lie algebra $\mathfrak{g}(M)$ integrates to a Lie group $G(M)$. ([PS] for Case 1, [EF] for Case 2). The coadjoint representation of this group can be realized as the space of differential operators $\lambda D + f$ ($\lambda \in k$), where f is a smooth function on M with values in $\mathfrak{gl}_n(k)$, in which the action of the group $G(M)$ reduces to the action of $C^\infty(M, GL_n(k))$ by conjugation (the so called gauge action): $g \circ (\lambda D + f) = \lambda D + Dg \cdot g^{-1} + gfg^{-1}$. The coadjoint orbit containing the operator $\Delta = \lambda D + f$ will be denoted by $\mathcal{O}_\Delta$.

The notion of monodromy for operators of the form $\lambda D + f$ ($\lambda \neq 0$), where f is matrix-valued, is analogous to that for higher order scalar operators. For $D = d/dx$ this notion is standard; for $D = \overline{\partial}$, monodromy is the restricted conjugacy class in $\overline{GL_n}(\mathbb{C})$ of an element $(g(z), \tau)$ such that there exists a nondegenerate matrix solution $B(z)$ of the equation $\lambda\overline{\partial}B + fB = 0$ defined on the cylinder Σ and such that $B(z+\tau) = B(z)g(z)$ [EF].

Consider now the affine linear map $\Delta : \mathcal{L} \to \mathfrak{g}(M)^*$ given by the formula $L \to D - A_L$, where A_L is defined by (2.1) (for both Case 1 and Case 2). This map takes values in the hyperplane $\lambda = 1$.

Proposition 4AB. *The following three conditions on two differential operators $L_1, L_2 \in \mathcal{L}$ are equivalent:*

(i) L_1 and L_2 have the same monodromy;

(ii) The flat (respectively holomorphic) vector bundles $\mathcal{E}_{L_1}$ and $\mathcal{E}_{L_2}$ are isomorphic.

(iii) The points $\Delta(L_1)$ and $\Delta(L_2)$ are in the same orbit of coadjoint representation of $G(M)$.

Proof. It is clear that the monodromy of the operator L is the same as the monodromy of $\Delta(L)$.

Case 1. The equivalence of (i) and (ii) is obvious; the equivalence of (ii) and (iii) was observed in [F], [RS], [Se].

Case 2. The equivalence of (i) and (ii) is an observation of E.Loojienga (cf. [EF]) (he observed that conjugacy classes in the extended loop group correspond to holomorphic bundles over an elliptic curve). The equivalence of (ii) and (iii) follows from [EF]. □

Remark. In Case 2 the vector bundle $\mathcal{E}_L$ is always of degree zero since it is obtained from the trivial bundle on the annulus by gluing with the help of a transition matrix $R(z) \in LGL_n(\mathbb{C})_0$ which is homotopic to the identity.

Proof of Theorem 3AB for Case 2. The proof given below follows the method of [OK].

Let $L(t)$ be a smooth curve on $\mathcal{L}$. Pick a smooth family of vector-functions $\mathbf{f}^t$ with a nondegenerate Wronski matrix such that $L(t)f_i^t = 0$ for all t, i. This is possible because of Corollary 2AB. Let $R^t(z) \in LGL_n(\mathbb{C})_0$ be the monodromy matrix of this set of solutions: it is defined by the formula $\mathbf{f}^t(z+\tau) = \mathbf{f}^t(z)R^t(z)$.

If. We must show that $L'(t)$ is a Hamiltonian vector for any t.

We know that all elements $(R^t(z), \tau)$ are in the same restricted conjugacy class in $\overline{GL_n}(\mathbb{C})$, i.e. are conjugate to the same element $(R(z), \tau)$. Therefore, $(R^t(z), \tau)$ is a smooth curve on the restricted conjugacy class of $(R(z), \tau)$. Since the group $LGL_n(\mathbb{C})$ is the total space of a principal fibration over this restricted conjugacy class whose fiber is the centralizer of $(R(z), \tau)$ in $LGL_n(\mathbb{C})$ (this is a finite-dimensional complex Lie group), the curve $(R^t(z), \tau)$ can be lifted to a smooth curve $C^t(z)$ on $LGL_n(\mathbb{C})$. In other words, there exists a function $C^t(z)$ taking values in $LGL_n(\mathbb{C})$ which is smooth in t and satisfies the relation

$$R^t(z) = C^t(z)R(z)(C^t)^{-1}(z+\tau). \tag{2.4}$$

Define a new vector function $\mathbf{g}^t = \mathbf{f}^t C^t$. Obviously, its components are still solutions of $L(t)\phi = 0$, and its Wronski matrix is nondegenerate. But now we have an additional property – the monodromy matrix of $\mathbf{g}^t$ does not depend on t: $\mathbf{g}^t(z+\tau) = \mathbf{g}^t(z)R(z)$.

Let $t_0 \in (a, b)$. Let $\mathbf{g}^t = \mathbf{g} + (t-t_0)\mathbf{g}' + \mathcal{O}((t-t_0)^2)$ as $t \to t_0$. Also let $L(t) = L + (t-t_0)L' + \mathcal{O}((t-t_0)^2)$ as $t \to t_0$. Let us differentiate the relation $L(t)\mathbf{g}^t = 0$ by t at $t = t_0$. We get

$$L\mathbf{g}' + L'\mathbf{g} = 0. \tag{2.5}$$

In order to show that L' is a Hamiltonian vector, we must find a pseudodifferential symbol A such that $L' = V_A(L) = L(AL)_+ - (LA)_+L$. This is the same as to find an A such that

$$L\mathbf{g}' + (L(AL)_+ - (LA)_+L)\mathbf{g} = 0. \tag{2.6}$$

Indeed, the equation $L\mathbf{g}' + F\mathbf{g} = 0$ with respect to an $(n-1)$-th order differential operator F has a unique solution: $F = \sum_{j=1}^n c_j \overline{\partial}^{j-1}$, where $\mathbf{c} = (c_1, ..., c_n)$ is equal to $-(L\mathbf{g}')W(\mathbf{g})^{-1}$ (note that to apply a differential operator of order $n-1$ to a set of n functions $\mathbf{h}$ is the same as to multiply the row vector of coefficients of this operator by the Wronski matrix $W(\mathbf{h})$).

Since $L\mathbf{g} = 0$, equation (2.6) is equivalent to

$$L(\mathbf{g}' + (AL)_+\mathbf{g}) = 0. \tag{2.7}$$

This means that it is enough to find an A such that

$$\mathbf{g}' + (AL)_+\mathbf{g} = 0. \tag{2.8}$$

That is, to find an A such that

$$(AL)_+ = \sum_{j=1}^{n} b_j \overline{\partial}^{j-1}, \tag{2.9}$$

where $\mathbf{b} = (b_1, ..., b_n)$ is defined as follows:

$$\mathbf{b} = -\mathbf{g}' W(\mathbf{g})^{-1}. \tag{2.10}$$

Since $\mathbf{g}$ and $\mathbf{g}'$ have the same monodromy matrix, it follows from (2.10) that $\mathbf{b}$ is doubly periodic: $b_i \in C^\infty(M, \mathbb{C})$.

In order to prove the existence of A satisfying (2.9), it suffices to show that the linear map $\chi : \mathcal{A} \to \tilde{\mathcal{L}}$ given by $\chi(A) = (AL)_+$ is an isomorphism. But this is obvious: the coefficients of the operator $(AL)_+$, have the triangular form $a_i + P_i$, where P_i is a differential polynomial in $a_1, ..., a_{i-1}$, and hence the coefficients a_i of the solution of the equation $(AL)_+ = \Lambda$, $\Lambda \in \tilde{\mathcal{L}}$, can be uniquely determined recursively starting from a_1.

Only if. Differentiating the equation $L(t)\mathbf{f}^t = 0$, we get

$$L\mathbf{f}' + L'\mathbf{f} = 0. \tag{2.11}$$

(we use the shortened notation $\mathbf{f}$ for $\mathbf{f}^t$). We know that $L' = V_A(L)$ for some A. This implies:

$$L(\mathbf{f}' + (AL)_+\mathbf{f}) = 0. \tag{2.12}$$

This means that

$$\mathbf{f}' + (AL)_+\mathbf{f} = \mathbf{h}, \tag{2.13}$$

where $\mathbf{h}$ satisfies the equation $L\mathbf{h} = 0$.

Let us show that we could have chosen $\mathbf{f}^t$ in such a way that $\mathbf{h} = 0$. Indeed, let $\mathbf{g}^t$ be another set of solutions of $L\phi = 0$ given by

$$\mathbf{g}^t = \mathbf{f}^t (C^t)^{-1}, \tag{2.14}$$

where $C^t \in LGL_n(\mathbb{C})$. Substituting (2.14) in (2.13), we get

$$\mathbf{g}'C + \mathbf{g}C' + (AL)_+ \mathbf{g}C = \mathbf{h} \tag{2.15}$$

(here we used the shortened notation $\mathbf{g}$ for $\mathbf{g}^t$, and C for C^t). We want to have the relation $\mathbf{g}' + (AL)_+\mathbf{g} = 0$. This is equivalent to the relation $\mathbf{g}C' = \mathbf{h}$, or, in terms of $\mathbf{f}$, $\mathbf{f}C^{-1}C' = \mathbf{h}$. This happens if and only if $C^{-1}C' = W(\mathbf{f})^{-1}W(\mathbf{h})$, or $C' = CW(\mathbf{f})^{-1}W(\mathbf{h})$. This is a first order differential equation on $LGL_n(\mathbb{C})$ (since $W(\mathbf{f})^{-1}W(\mathbf{h})$ is a holomorphic matrix-valued function), and it has a unique solution with the initial condition $C(t_0) = \mathrm{Id}$.

Therefore, we may assume that $\mathbf{h}$ in (2.13) is equal to 0.

We have

$$\mathbf{f}'(z) = -(AL)_+ \mathbf{f}(z). \tag{2.16}$$

Changing z to $z+\tau$ and using the monodromy relation $\mathbf{f}(z+\tau) = \mathbf{f}(z)R(z)$ ($R = R^t$), we get

$$\mathbf{f}'(z)R(z) + \mathbf{f}(z)\frac{\partial R}{\partial t}(z) = -(AL)_+\mathbf{f}(z)R(z), \tag{2.17}$$

which, together with (2.16), implies $\mathbf{f}(z)\frac{\partial R}{\partial t}(z) = 0$. Therefore, $W(\mathbf{f})\frac{\partial R}{\partial t} = 0$, which means $\frac{\partial R}{\partial t} = 0$, or $R^t(z)$ is independent of t. Thus, the monodromy of $L(t)$ is independent of t Q.E.D. □

3. Codimension of symplectic leaves.

Theorem 5AB. *Let $L \in \mathcal{L}$ be a differential operator. Then the following four numbers coincide:*

(i) the codimension of the symplectic leaf $\mathcal{O}_L$;

(ii) the dimension of the centralizer of the monodromy matrix of L;

(iii) the codimension of the orbit $\mathcal{O}_{\Delta(L)}$ in the hyperplane $\lambda = 1$ in the coadjoint representation of the group $G(M)$ (see Section 2);

(iv) the dimension of the space of global sections of the vector bundle $End(\mathcal{E}_L) = \mathcal{E}_L \otimes \mathcal{E}_L^*$ (flat sections for Case 1, holomorphic sections for Case 2).

Remarks. 1. By the codimension of an orbit of the coadjoint representation we mean the codimension (in the hyperplane $\lambda = 1$) of the tangent space to the orbit at any point.

2. We call the dimension of the centralizer of a (restricted) conjugacy class the *codimension* of this conjugacy class.

3. For Case 1, it is easy to show that the number (i)-(iv) is finite. In Case 2, it follows from algebraic geometry that (iv) is finite, and Theorem 5AB implies that so are (i),(ii),(iii).

4. We have seen that symplectic leaves of the classical (respectively, affine) GD bracket are labeled by conjugacy classes in $GL_n(k)$ (respectively, $\overline{GL_n}(\mathbb{C})$). It turns out, however, that in the affine case conjugacy classes close enough to the "identity" (Id, τ) in $\overline{GL_n}(\mathbb{C})$ can be labeled by congugacy classes of the finite-dimensional group $GL_n(\mathbb{C})$. Indeed, near the "identity" the group $\overline{GL_n}(\mathbb{C})$ is identified with a region in its Lie algebra by the exponential map. The Lie algebra of $\overline{GL_n}(\mathbb{C})$ can be thought of as the coadjoint representation of the affine Lie algebra $\widehat{\mathfrak{gl}_n}$ (i.e. the space of differential operators $\lambda\frac{d}{dz} - A(z)$). Therefore, the conjugacy classes become coadjoint orbits for the affine Lie algebra $\widehat{\mathfrak{gl}_n}$, and those are enumerated by λ and the monodromy of the corresponding operators $\lambda\frac{d}{dz} - A(z)$ (see [F],[RS]).

Proof of Theorem 5AB.

(i)=(ii). Let $L \in \mathcal{L}$.

Let $\mathbf{f}$ be a set of solutions of $L\phi = 0$ with a nondegenerate Wronski matrix. Let R be the monodromy matrix of $\mathbf{f}$: $\mathbf{f}(x+1) = \mathbf{f}(x)R$, $R \in GL_n(k)$ (Case 1), $\mathbf{f}(z+\tau) = \mathbf{f}(z)R(z)$, $R \in LGL_n(\mathbb{C})_0$ (Case 2).

We will describe the tangent space $T_L\mathcal{O}_L$ as the image of a certain operator.

Consider the linear operator $\hat{L}(\mathbf{g}) = (L\mathbf{g})W(\mathbf{f})^{-1}$ sending the space of vector-functions $\mathbf{g} = (g_1, ..., g_n)$ such that

$$\mathbf{g}(z+\tau) = \mathbf{g}(z)R(z). \tag{3.1}$$

to the space of doubly periodic vector-functions.

Lemma. The tangent space $T_L\mathcal{O}_L$ is the set of all differential operators of the form $\sum_{i=0}^{n-1} p_{i+1}D^i$, such that the vector $\mathbf{p} = (p_1, ..., p_n)$ belongs to the image of $\hat{L}$.

Proof of the Lemma Applying equation (1.4) to $\mathbf{f}$, we get

$$V_A(L)\mathbf{f} = L(AL)_+\mathbf{f}. \tag{3.2}$$

Let $V_A(L) = \sum_{i=0}^{n-1} p_{i+1}D^i$, and let $\mathbf{p} = (p_1, ..., p_n)$. Then (3.2) can be rewritten in the form

$$\mathbf{p}W(\mathbf{f}) = L(AL)_+\mathbf{f}. \tag{3.3}$$

We know that $(AL)_+$ can be any differential operator of the form $\sum_{i=0}^{n-1} b_{i+1}D^i$, $b_i \in C^\infty(M, k)$. Therefore, the set of possible values of the expression $(AL)_+\mathbf{f}$ is the set of all vector-functions $\mathbf{g}$ on the cylinder satisfying (3.1). Indeed, (3.1) clearly must be satisfied, and whenever $\mathbf{g}$ does satisfy (3.1), one can set $\mathbf{b} = \mathbf{g}W(\mathbf{f})^{-1}$ and get a doubly periodic vector-function.

This consideration implies that the set of possible values of $\mathbf{p}$ is the image of the operator $\hat{L}$, Q.E.D. □

The Lemma shows that the set of possible values of $\mathbf{p}W(\mathbf{f})$ is the image of the operator L regarded as an operator on the space of vector-functions $\mathbf{g}$ satisfying (3.1), i.e. on the space of smooth sections of the vector bundle $\mathcal{E}_L$. The codimension of $T_L\mathcal{O}_L$ is therefore equal to the codimension of this image, since $W(\mathbf{f})$ is just an automorphism of $\tilde{\mathcal{L}}$.

The operator $L : \Gamma(\mathcal{E}_L) \to \Gamma(\mathcal{E}_L)$ is an elliptic operator on the circle (torus), so its index is equal to zero. Therefore, the dimension of its kernel is equal to the codimension of its image. Thus, it remains to compute the dimension of the kernel of L.

An element that undoubtedly belongs to Ker$L$ is $\mathbf{f}$. Furthermore, any other element $\mathbf{g}$ of this kernel, according to Proposition 1AB, can be represented in the form $\mathbf{g} = \mathbf{f}C$, where C is an $n \times n$-matrix in Case 1 and a holomorphic $n \times n$-matrix valued function on Σ in Case 2. The matrix C has to satisfy the relation

$$C = R^{-1}CR \text{ (Case 1)}$$
$$C(z+\tau) = R^{-1}(z)C(z)R(z) \text{ (Case 2)}, \tag{3.4}$$

which is equivalent to C being in the Lie algebra of the centralizer of the monodromy of L. This shows that Ker$L$ is isomorphic to the Lie algebra of the centralizer, i.e. their dimensions are the same.

(ii)=(iv) The solutions of (3.4) are exactly the flat (respectively holomorphic) sections of the vector bundle $\mathrm{End}(\mathcal{E}_L) = \mathcal{E}_L \otimes \mathcal{E}_L^*$, and vice versa.

(iii)=(iv) Let $\Delta = D - A \in \mathfrak{g}(M)^*$. Then the tangent space to the coadjoint orbit at Δ, $T_\Delta\mathcal{O}_\Delta$, consists of vectors of the form $DX - [A, X]$, where X is an arbitrary matrix-valued function on M. Therefore, the codimension of the orbit is equal to the codimension of the image of the operator $D - \mathrm{ad}A$ in $C^\infty(M, \mathfrak{gl}_n(k))$. Since this operator is elliptic, its index is zero, so the codimension of its image equals the dimension of its kernel. But the kernel of this operator consists of flat (respectively holomorphic) sections of the bundle $\mathcal{E}_L \otimes \mathcal{E}_L^*$ and only of them. Therefore, the dimensions of the kernel and the space of sections coincide. □

Proposition 6AB. *The codimension of every symplectic leaf (coadjoint orbit, conjugacy class) is congruent to n modulo 2.*

Motivation. Thanks to Theorem 5AB, it is enough to consider coadjoint orbits. Coadjoint orbits have a natural symplectic (or holomorphic symplectic) structure – the Kirillov-Kostant structure. Therefore, they must all be "even-dimensional", i.e. their codimensions must have the same parity. Also, the orbit corresponding to $\Delta = \overline{\partial}$ has codimension n^2, which is congruent to n modulo 2. Therefore, all codimensions must be congruent to n modulo 2. □

It is not obvious how to make this argument into a rigorous proof, so we give a different (algebraic) proof.

Proof. Case 1. Because of Theorem 5AB, it is enough to show that codimensions of all conjugacy classes in $GL_n(k)$ have the same parity. This follows from the fact that all conjugacy classes in $GL_n(k)$ are even-dimensional – a standard fact from linear algebra.

Case 2. Because of Theorem 5AB, Proposition 6AB is equivalent to the assertion that for any rank n holomorphic vector bundle E of degree zero over an elliptic curve M the dimension of the space $H^0(M, E \otimes E^*)$ of global holomorphic sections of the bundle $E \otimes E^*$ is congruent to n modulo 2. This assertion is a corollary of the following Lemma.

Lemma. Let E be a holomorphic vector bundle over an elliptic curve M of rank r and degree d. Then $\dim H^0(M, E \otimes E^*) \equiv rd + r + d \bmod 2$.

Proof of the Lemma. Let V be a holomorphic vector bundle over M of degree d. Then by the Riemann-Roch theorem $\dim H^0(M, V) - \dim H^1(M, V) = d$. Also, Serre's duality tells us that $H^0(M, V^*) = H^1(M, V)^*$. Combining these two facts, we get:

$$\dim H^0(M, V \oplus V^*) \equiv d \bmod 2. \tag{3.5}$$

The proof of the Lemma is by induction. For line bundles the statement is obvious. We assume that we know the Lemma is true for bundles of rank $l < m$. Let E be a bundle of rank m. We consider two possibilities.

1) E is indecomposable. Then a theorem of Atiyah's [At] tells us that $\dim H^0(M, E \otimes E^*)$ equals the greatest common divisor (r, d) of the rank r and the degree d of E. But $(r, d) \equiv rd + r + d \bmod 2$, Q.E.D.

2) $E = E_1 \oplus E_2$. Then

$$H^0(M, E \otimes E^*) = H^0(M, E_1 \otimes E_1^*) \oplus H^0(M, E_2 \otimes E_2^*) \oplus H^0(M, E_1 \otimes E_2^* \oplus E_2 \otimes E_1^*). \tag{3.6}$$

Using the assumption of induction, congruence (3.5), and the facts that $(E_1 \otimes E_2^*)^* = E_2 \otimes E_1^*$ and $\deg(E_1 \otimes E_2^*) = r_1 d_2 + r_2 d_1$, we get the congruence

$$\dim H^0(M, E \otimes E^*) \equiv (r_1 d_1 + r_1 + d_1) + (r_2 d_2 + r_2 + d_2) + (r_1 d_2 + r_2 d_1) \bmod 2, \tag{3.7}$$

where r_i are the ranks and d_i are the degrees of E_i. But the right hand side of (3.7) equals to $(r_1 + r_2)(d_1 + d_2) + (r_1 + r_2) + (d_1 + d_2) = rd + r + d$, Q.E.D. □

4. Existence of differential operators with a prescribed monodromy

A natural question in the theory of differential equations is: given a conjugacy class in $GL_n(k)$ ($\overline{GL_n}(\mathbb{C})$), does there exist a differential operator $L \in \mathcal{L}$ whose monodromy is this conjugacy class? In other words, is the map assigning conjugacy classes to symplectic leaves of the GL_n- (affine GL_n-) Gelfand-Dickey bracket surjective? The answer to this question is positive:

Proposition 7AB. *(i) Any matrix in $GL_n(k)$ (with positive determinant if $k = \mathbb{R}$) is a monodromy matrix of an n-th order differential operator on the circle with the highest coefficient 1.*

(ii) Every holomorphic vector bundle over an elliptic curve M arises as monodromy of an n-th order operator $L = \overline{\partial}^n + \sum_{j=0}^{n-1} u_{j+1}\overline{\partial}^j$, $u_j \in C^\infty(M, \mathbb{C})$.

Proof. (i) The proof is analogous to that of (ii) given below. For $k = \mathbb{R}$, a proof of this proposition has been given in [S].

(ii) Thanks to Proposition 1B, it suffices to prove the following statement: for any monodromy matrix $R(z) \in LGL_n(\mathbb{C})_0$ there exists a smooth vector function $\mathbf{f} : \Sigma \to \mathbb{C}^n$, $\mathbf{f} = (f_1, ..., f_n)$, such that $\mathbf{f}(z + \tau) = \mathbf{f}(z)R(z)$, and the Wronskian of $\mathbf{f}$ does not vanish on Σ.

First of all, the vector bundle on M prescribed by the gluing function $R(z)$ is topologically trivial since $R(z)$ is homotopic to the identity. Therefore, it admits a smooth trivialization – a smooth function $X : \Sigma \to GL_n(\mathbb{C})$ such that $X(z + \tau) = X(z)R(z)$. Let us look for the vector function $\mathbf{f}$ in the form $\mathbf{f} = \mathbf{g}X$, $\mathbf{g} = (g_1, ..., g_n)$. Then the monodromy condition on $\mathbf{f}$ is equivalent to the condition that $\mathbf{g}$ is τ-periodic, i.e. that $\mathbf{g} \in C^\infty(M, \mathbb{C}^n)$.

Let $Y(z) = \overline{\partial}X(z) \cdot X(z)^{-1}$. This is a smooth matrix-valued function periodic with periods 1 and τ, i.e. a function on M. Consider the operator $\mathcal{D}$ on $C^\infty(M, \mathbb{C}^n)$ defined by $\mathcal{D}\mathbf{g} = \overline{\partial}\mathbf{g} + \mathbf{g}Y$.

It is easy to check that the Wronski matrix of $\mathbf{f}$ can be written in the form $W(\mathbf{f}) = W_{\mathcal{D}}(\mathbf{g})X$, where $W_{\mathcal{D}}(\mathbf{g})_{ij} = (\mathcal{D}^{i-1}\mathbf{g})_j$ (i.e. the lines of $W_{\mathcal{D}}(\mathbf{g})$ are $\mathbf{g}$, $\mathcal{D}\mathbf{g}$, $\mathcal{D}^2\mathbf{g}$,...). Therefore, our problem reduces to finding $\mathbf{g}$ such that $W_{\mathcal{D}}(\mathbf{g})$ is everywhere nondegenerate. This can be done as follows.

Let $z = x + \tau y$, $x, y \in \mathbb{R}$. Set $g_m(z) = e^{2\pi imkx}$, $1 \le m \le n$, where k is an integer. If we regard k as an independent variable, then the expression $W_{\mathcal{D}}(\mathbf{g})$ is a polynomial in k and $e^{2\pi ikx}$ (with coefficients dependent of z). The highest term in k is the usual Wronskian $W(\mathbf{g})$, which equals $(\pi ik)^{n(n-1)/2}V_n e^{\pi ikn(n+1)x}$, where V_n is the Vandermonde determinant of $1, 2, ..., n$. The absolute value of this term equals $|V_n|(\pi k)^{n(n-1)/2}$, which grows as $k^{n(n-1)/2}$ as $k \to \infty$. The rate of growth of the terms with lower degrees of k is lower, so for k big enough (uniformly in x, y) the highest term will dominate. Therefore, $W_{\mathcal{D}}(\mathbf{g})$ does not vanish if k chosen to be big enough, Q.E.D. $\square$

5. Examples

Let us describe an explicit realization of vector bundles by differential operators for $n = 2$. Before we do so, let us formulate Atiyah's classification theorem for vector bundles of rank 2.

Atiyah's theorem. *(for rank 2 bundles)*[At] *Any rank 2 holomorphic vector bundle of degree zero over an elliptic curve $M = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, $\tau \in C^+$, is isomorphic to one of the following:*

1) $E(a, b, m)$ $(a, b \in \mathbb{C}^*$, $m \in \mathbb{Z}$, $m \geq 0)$ – the vector bundle corresponding to the conjugacy class of the element

$$\left(\begin{bmatrix} ae^{2\pi imz} & 0 \\ 0 & be^{-2\pi imz} \end{bmatrix}, \tau \right) \tag{5.1}$$

of $\overline{GL_n}(\mathbb{C})$. The bundles $E(a_1, b_1, m_1)$ and $E(a_2, b_2, m_2)$ are isomorphic iff $m_1 = m_2$, $a_1/a_2 = q^{k_a}$, $b_1/b_2 = q^{k_b}$, where $k_a, k_b \in \mathbb{Z}$, and $q = e^{2\pi i\tau}$.

2) $F(a)$, $a \in \mathbb{C}^*$ – the vector bundle corresponding to the conjugacy class of the element

$$\left(\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}, \tau \right) \tag{5.2}$$

of $\overline{GL_n}(\mathbb{C})$; the bundles $F(a)$ and $F(b)$ are isomorphic iff $a/b = q^k$, $k \in \mathbb{Z}$.

A bundle $E(a, b, m)$ is never isomorphic to $F(\tilde{a})$.

Let us now realize each bundle from classes 1) and 2) by a differential operator $L = \overline{\partial}^2 + u_1\overline{\partial} + u_2$.

Observe that if a bundle E is realized by a differential operator then it is easy to realize $X \otimes E$, where X is an arbitrary degree zero line bundle. Indeed, let X correspond to the conjugacy class of the element $(a, \tau) \in \overline{GL_1(\mathbb{C})}$, $a \in \mathbb{C}^*$. Let E be realized by a differential operator L. Then it is easy to see that $X \otimes E$ is realized by the differential operator $\tilde{L} = e^{\alpha(z-\bar{z})} \circ L \circ e^{-\alpha(z-\bar{z})}$, where

$$\alpha = \frac{\log a}{\tau - \bar{\tau}} \tag{5.3}$$

(any branch of log can be taken).

This observation implies that it is enough for us to realize explicitly the bundles $E(a, a^{-1}, m)$ and $F(1)$ by differential operators, since all the other bundles can be obtained by tensoring them with line bundles.

It is easy to see that the bundle $F(1)$ is realized by the operator $L = \overline{\partial}^2$; the corresponding vector $\mathbf{f}$ of solutions is $(1, y)$, where $z = x + \tau y$. The bundle $E(a, a^{-1}, 0)$ is realized by the operator $L = \overline{\partial}^2 - \alpha^2$, where α is defined by (5.3) (any nonzero value of log can be taken); the corresponding vector $\mathbf{f}$ of solutions is $(e^{\alpha(z-\bar{z})}, e^{-\alpha(z-\bar{z})})$.

It remains to realize the bundles $E(a, a^{-1}, m)$ for $m > 0$.

Let $z_1, ..., z_m \in M$ be pairwise distinct points, and let $\psi : M \to \mathbb{C}$ be a smooth function on the elliptic curve which has the following properties:

(i) ψ vanishes at $z_1, ..., z_m$ and nowhere else;

(ii) in the neighborhood of z_i the function ψ has the form

$$\psi(z) = |z - z_i|^2. \tag{5.4}$$

Such a function is very easy to construct: set

$$\psi(z) = \psi_0(z) + \sum_{i=1}^{m} \psi_i(z)|z - z_i|^2, \tag{5.5}$$

where $\psi_0(z) = 1$ everywhere except the disks $B(z_i, r)$ centered at z_i of a small radius r, and $\psi_0(z) = 0$ in $B(z_i, r/2)$; for $i > 0$, ψ_i is a nonnegative function equal to 1 in $B(z_i, r/2)$ and to 0 outside $B(z_i, r)$ (all ψ_i have to be smooth everywhere and positive in the annuli $r/2 < |z - z_i| < r$).

From the definition of ψ it follows that the function $u = \overline{\partial}^2\psi/\psi$ defined a priori in $M \setminus \{z_1, ..., z_n\}$, can be continued to the points $z_1, ..., z_n$ (since it is simply equal to zero in their neighborhoods). This implies that ψ is a solution of the equation $L\psi = 0$, where $L = \overline{\partial}^2 - u$. Pick a vector $\mathbf{f} = (f_1, f_2)$ of solutions of this differential equation with a nondegenerate Wronski matrix. Then there exist unique holomorphic functions $c_1(z)$, $c_2(z)$ on the cylinder Σ such that $\psi = c_1 f_1 + c_2 f_2$, and the vector-function $\mathbf{c} = (c_1, c_2)$ is a global holomorphic section of the holomorphic vector bundle $\mathcal{E}_L$.

Let us show that this section vanishes at the points $z_1, ..., z_m$ and only at them, and these zeroes are simple. Indeed, the vector $\mathbf{F} = \begin{pmatrix} \psi \\ \partial\psi \end{pmatrix}$ equals $W(\mathbf{f})\mathbf{c}^t$, thus $\mathbf{c} = 0$ iff $\mathbf{F}$ vanishes, and the vanishing points of $\mathbf{F}$ are exactly $z_1, ..., z_m$. Also, in the neighborhood of z_i one has $\mathbf{F} = (z - z_i)\begin{pmatrix} \bar{z} - \bar{z}_i \\ 1 \end{pmatrix}$, which shows that z_i is a simple zero of $\mathbf{c}$.

It follows from the theory of holomorphic bundles that the presence of a section $\mathbf{c}$ with the above properties guarantees that $\mathcal{E}_L$ has a line subbundle X of degree m defined by the monodromy function $e^{2\pi i m(z-z_0)}$. The bundle $\Lambda^2\mathcal{E}_L$ is trivial since the operator L does not contain a first order term, and hence the Wronskian (which is a section of $\Lambda^2\mathcal{E}_L$) is constant. This fact together with Atiyah's classification theorem implies that $\mathcal{E}_L$ is isomorphic to $X \oplus X^*$, which is the same as $E(a, a^{-1}, m)$, where $a = e^{-2\pi i m z_0} = \prod_j e^{2\pi i z_j}$. Since the points z_i could be chosen arbitrarily, one can get any value of a.

Let us now describe the codimensions of symplectic leaves for $n = 2$ (Case 2). It follows Theorem 5AB that it is enough to do it for vector bundles.

Proposition 8B. *(i) If $\mathcal{E}_L = E(a, b, m)$ then codim($\mathcal{O}_L$) equals $2m + 2$ if $m > 0$, 2 if $m = 0$ and a/b is not an integral power of q, and 4 if $m = 0$ and $a/b = q^k$, $k \in \mathbb{Z}$.*

(ii) If $E_L = F(a)$ then codim($\mathcal{O}_L$) equals 2.

Proof. (i) $E(a, b, m) = X_{a,m} \oplus X_{b,-m}$, where $X_{a,m}$, is the line bundle described by the monodromy function $ae^{2\pi i m z}$. Therefore, $E(a, b, m) \otimes E(a, b, m)^* = X_{1,0} \oplus X_{1,0} \oplus X_{a/b,2m} \oplus X_{b/a,-2m}$. The number of linearly independent holomorphic sections of this bundle is 2 if $m = 0$ and $a/b \neq q^k$, 4 if $m = 0$ and $a/b = q^k$, and $2m + 2$ if $m \neq 0$, which proves (i).

It is also easy to see that $F(a) \otimes F(a)^* = X_{1,0} \oplus F_3(1)$, where $F_3(1)$ is the vector bundle of rank 3 whose monodromy matrix is the 3×3 Jordan cell with eigenvalue 1. Therefore, the number of linearly independent holomorphic sections of $F(a) \otimes F(a)^*$ is 2. This settles (ii).

Remark. Thus, in Case 2, unlike Case 1, the codimensions of symplectic leaves can be arbitrarily large, even for $n = 2$. However, the conjugacy classes labeling all symplectic leaves of codimension $> n^2$ stay away from the $(\mathrm{Id}, \tau) \in \overline{GL_n}(\mathbb{C})$, by virtue of Remark 4 at the end of Section 3.

Appendix: Classification of symplectic leaves with a given monodromy.

In conclusion, let us discuss the problem of finding discrete invariants of symplectic leaves.

In Case 1 (for $k = \mathbb{R}$), this problem was studied in [OK], and it was shown that it is equivalent to the problem of homotopy classification of quasiperiodic nondegenerate curves, i.e. curves in $\mathbb{R}^n$ with prescribed monodromy and nonvanishing Wronskian. This problem, in turn, is equivalent to homotopy classification of quasiperiodic nonflattening curves – smooth curves $x(t)$ in $\mathbb{R}P^{n-1}$ with prescribed monodromy such that the vectors $x', x'', ..., x^{(n-1)}$ are linearly independent at each t (the equivalence is established by replacing the original curve in $\mathbb{R}^n$ by its projection to $\mathbb{R}P^{n-1}$).

For general n, this topological problem turns out to be difficult. It is solved only for $n = 2$ (where this problem is equivalent to classification of projective structures on the circle [Ku], of Hill's operators [LP], or coadjoint orbits of the Virasoro algebra [K1],[Se]), for $n = 3$ [KS],and for any n in case $R = \mathrm{Id}$ [S].

In the case of elliptic curve the geometric notion corresponding to the problem of finding discrete invariants of symplectic leaves is the notion of a *quasiperiodic nondegenerate tube* – a function $\gamma : \Sigma \to \mathbb{C}^n$ with prescribed monodromy $R(z)$ ($\gamma(z+\tau) = \gamma(z)R(z)$) and nonvanishing $\overline{\partial}$-Wronskian. Then we have

Proposition. *Symplectic leaves of the affine GL_n-GD bracket whose monodromy is the conjugacy class of $R(z)$ are in one-to one correspondence with homotopy classes of quasiperiodic nondegenerate tubes with monodromy $R(z)$.*

Let us consider the case of trivial monodromy ($R(z) = \mathrm{Id}$). Then a quasiperiodic nondegenerate tube is just a smooth map $\gamma : M \to \mathbb{C}^n$ with nonvanishing Wronskian. An obvious homotopy invariant of such a map is the winding invariant – the homotopy class of the map $M \to \mathbb{C}^*$ realized by the Wronski determinant $\det W(\gamma)$. This invariant takes values in $\mathbb{Z}^2$, and can take any prescribed value.

Therefore, classification of symplectic leaves with trivial monodromy depends on the answer to

Question 1. *Is it true that two quasiperiodic nondegenerate tubes are homotopic in the class of such tubes if and only if their winding invariants are the same?*

In the case $n = 2$, by projecting $\mathbb{C}^2$ to $\mathbb{C}P^1$, we can reduce this question to the problem of homotopy classification of nowhere holomorphic maps. A nowhere holomorphic map is a map $f : M \to \mathbb{C}P^1$ such that $\overline{\partial} f$ is not equal to zero at any point of M. An obvious homotopy invariant of nowhere holomorphic maps is the winding invariant – the homotopy class of the map $\sigma_f : M \to T_u\mathbb{C}P^1$ from M to the space of unit tangent vectors to $\mathbb{C}P^1$ (this space is diffeomorphic to $\mathbb{R}P^3$) given by the formula $\sigma_f(z) = \overline{\partial} f(z)/|\overline{\partial} f(z)|$. It takes value in $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Translating Question 1 into the language of nowhere holomorphic maps, we come to

Question 2. *Is it true that two nowhere holomorphic maps are homotopic in the class of such maps if and only if their winding invariants are the same?*

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