Arch. Rational Mech. Anal. (2023) 247:15
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-023-01848-x

Geometric Hydrodynamics in Open Problems

Boris Khesin, Gerard Misiołek & Alexander Shnirelman

Communicated by V. Šverák

Abstract

Geometric Hydrodynamics has flourished ever since the celebrated 1966 paper of V. Arnold. In this paper we present a collection of open problems along with several new constructions in fluid dynamics and a concise survey of recent developments and achievements in this area. The topics discussed include variational settings for different types of fluids, models for invariant metrics, the Cauchy and boundary value problems, partial analyticity of solutions to the Euler equations, their steady and singular vorticity solutions, differential and Hamiltonian geometry of diffeomorphism groups, long-time behaviour of fluids, as well as mechanical models of direct and inverse cascades.

Contents

Published online: 25 February 2023

1. Introduction and Notations

We start with the basic setup of geometric hydrodynamics. This will provide the background and motivation for various developments discussed in the sequel.

Consider an ideal (that is, incompressible and inviscid) fluid in a fixed domain M in $\mathbb{R}^n$ ($n = 2, 3$). In the Eulerian representation a fluid motion is described by the evolution of its velocity field which satisfies the *incompressible Euler equations*

$$\partial_t v + v \cdot \nabla v = -\nabla p, \quad \operatorname{div} v = 0, \tag{1.1}$$

where the velocity field $v = v(t, x)$ is assumed to be tangent to the domain's boundary, if $\partial M \neq \emptyset$. The pressure function $p = p(t, x)$ on the right-hand side is defined uniquely by these conditions, modulo an additive constant, and the divergence of the field v is computed with respect to the volume form μ in $\mathbb{R}^n$.

While some ideas can be traced back to Helmholtz and Kelvin, the modern geometric approach to hydrodynamics began with the seminal 1966 paper of ARNOLD [2]. It is based on the Lagrangian representation of fluid flows in terms of particle trajectories which can be viewed as curves in the infinite-dimensional configuration space given by the group $\mathscr{D}_\mu(M)$ of volume-preserving diffeomorphisms of M. Arnold showed that fluid motions (in analogy with the classical case of the rigid body) in fact correspond to geodesics of the right-invariant metric on $\mathscr{D}_\mu(M)$ defined by the kinetic energy. This is a direct consequence of the least action principle and the postulate that fluid particles are allowed neither to fuse nor to split. Indeed, assuming appropriate smoothness conditions, let $\gamma = \gamma(t, x)$ be the flow of the velocity field v, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t, x) = v(t, \gamma(t, x)), \quad \gamma(0, x) = x.$$

Differentiating both sides of the flow equation in t and using (1.1) leads immediately to the second order system

$$\frac{d^2\gamma}{dt^2} = -\nabla p \circ \gamma, \tag{1.2}$$

which, roughly speaking, expresses the fact that the acceleration of the fluid is L^2-orthogonal (in the kinetic energy metric) to the space of divergence-free velocity fields. The latter constitute the tangent space at the identity to the group $\mathscr{D}_\mu(M)$ and the orthogonality condition represents the fact that the particle trajectories describe a geodesic curve in $\mathscr{D}_\mu(M)$.

More generally, let the fluid domain be an n-dimensional Riemannian manifold M and let μ be the Riemannian volume form. The kinetic energy metric on $\mathscr{D}_\mu(M)$ is given at the identity e by the L^2 inner product

$$\langle v, w\rangle_{L^2} = \int_M \big(v(x), w(x)\big)_{T_xM}\,\mu \tag{1.3}$$

of divergence-free vector fields $v, w \in T_e\mathscr{D}_\mu(M)$. As before, the trajectories of fluid particles satisfy the equations (1.2) and their velocities satisfy the Euler equations (1.1) with the nonlinear term $v\cdot\nabla v$ replaced now by the covariant derivative $\nabla_v v$ on M and ∇p by the corresponding Riemannian gradient on M, see [6]. (This construction also applies if μ is an arbitrary volume form which does not coincide with the Riemannian volume form on M, provided that the divergence div v is taken with respect to μ.)

2. Ramifications of the Euler Equations

2.1. The Euler Equations with Sources and Sinks

Various interesting, physically relevant and as yet unresolved, problems can be formulated already at this stage.

Problem 1. *Find a variational (preferably geodesic) formulation describing the motion of an ideal fluid in a fixed domain M containing sources and sinks. What is the correct formulation of the variational problem: should one take into account the exterior forces and/or the "memory" of the fluid?*

For example, consider the case of a horizontal pipe with a fluid entering at one end and exiting at the other end. Such problems have a long history. On the one hand, as any mechanical system, fluid motions should obey some least action principle; see [5]. On the other hand, the energy of the fluid in the pipe may not be conserved since, depending on the boundary conditions, it could be supplied or drained at the two ends. For instance, in addition to the equations (1.1) and the initial condition $v(0)$, the full system of the Euler-type equations in the 2D setting would have to include as data two other items: the function $v \cdot n$ describing the normal component of the velocity v on the penetrable boundary, as well as the vorticity function $\omega := \operatorname{curl} v$ defined on the source part of the boundary through which the fluid is supplied. (Since vorticity is transported by the flow, this data will be sufficient to define it for all times, see [52,138,148,151,152].)

V. Yudovich used such data to formulate a stability criterion for a steady pipe flow in 2D, which may hint at the appropriate boundary conditions needed to obtain a variational formulation, see [151,152,154]. We should add that for a "dual" problem involving a fixed amount of fluid in a domain with a dynamic boundary its Hamiltonian formulation is described in [86].

2.2. *The Euler Equation for Multiphase Fluids and Groupoids*

While Arnold's approach to fluids is limited to systems whose symmetries form a Lie group, there are many problems in fluid dynamics, such as free boundary problems, a rigid body in a fluid or fluid flows with vortex sheets, whose symmetries should instead be regarded as a *Lie groupoid*. Groupoids can be thought of as groups with partially defined multiplication: for instance, fluid configurations with free boundary correspond to diffeomorphisms from one fluid domain to another; only maps for which the image of one coincides with the source of another admit composition ("multiplication").

In [59,60] Arnold's framework was extended from Lie groups to Lie groupoids to give a groupoid-theoretic description for incompressible multiphase fluids, generalized flows, and fluid flows with vortex sheets (the latter are flows whose velocity field has a jump discontinuity along a hypersurface). A multiphase fluid consists of several fractions that can freely penetrate through each other without resistance and are constrained only by the conservation of total density. Beyond the vortex sheet setting, multiphase fluids arise for example in plasma physics and chemistry. Of particular interest are multiphase fluids with continuum of phases (or generalized flows), introduced by Brenier [19]. One can think of them as flows in which every fluid particle spreads into a cloud thus moving to any other point of the manifold with certain probability [131].

The Euler equations for multiphase flows on a Riemannian manifold M have the form

$$\begin{cases} \partial_t v_j + v_j \cdot \nabla v_j = -\nabla p, \\ \partial_t \rho_j + \operatorname{div}(\rho_j v_j) = 0. \end{cases}$$

Here $\rho_1, \ldots, \rho_n \in C^\infty(M)$ are densities of n phases of the fluid subject to the total incompressibility condition $\sum_{j=1}^n \rho_j = 1$, the vector fields $v_1, \ldots, v_n \in \operatorname{Vect}(M)$ are the corresponding fluid velocities, and the pressure $p \in C^\infty(M)$ is common for all phases. For generalized flows the integer index $j = 1, \ldots, n$ enumerating the phases is replaced by a continuous parameter. In the case of vortex sheets the densities are indicator functions of different parts of the manifold.

It turned out that in all of the above cases the corresponding configuration space has a natural groupoid structure. Using the corresponding Lie groupoids of multiphase diffeomorphisms instead of the Lie group of volume-preserving transformations in Arnold's setting one can describe the corresponding Lie algebroids and obtain geometric and Hamiltonian interpretations for the motion of the corresponding multiphase fluids, "homogenized" vortex sheets, and generalized flows. Solutions of the above Euler equations were proved to be precisely the geodesics of an L^2-type right-invariant (source-wise) metric on the corresponding Lie groupoids of multiphase volume-preserving diffeomorphisms [59,60]. Another interesting domain for applications of Lie groupoids is provided by elasticity theory, cf. [65,82,98].

Problem 2. *Extend the geodesic and Hamiltonian descriptions of the Euler-Arnold equations on Lie groupoids to problems of elasticity theory.*

Many other open problems discussed below for the Euler equations related to diffeomorphism Lie groups can also be posed for the corresponding Lie groupoids, see, for example Sections 5, 7, and 13. For instance, it is natural to extend Arnold's study of the differential geometry of infinite-dimensional groups to those groupoids in view of possible applications to fluid stability problems.

Problem 3. *Describe the differential geometry (including computations of sectional and Ricci curvatures, conjugate points, etc.) for the right-invariant (source-wise) L^2-metric on the Lie groupoid of multiphase or generalized volume-preserving diffeomorphisms (analogous to Arnold's description of the differential geometry of the group $\mathscr{D}_\mu(M)$).*

3. Variational Setting for Compressible Fluids

3.1. Variational Setting for Shocks

The inviscid Burgers equation

$$\partial_t v + v \cdot \nabla v = 0$$

describes freely moving non-interacting particles in a manifold M of any dimension. It can be also viewed as a geodesic equation, which in this case is defined on the full diffeomorphism group $\mathscr{D}(M)$ equipped with a non-invariant L^2-metric [119]. Once shock waves develop, the Lagrangian representation breaks down in the sense that the equation ceases to define an evolution in $\mathscr{D}(M)$; see for example [74]. However, particles that stick inside the shocks continue to move along their own trajectories. For potential solutions with convex potentials there is a pointwise variational principle described by a "circle law" proposed by Bogaevsky, see [16] and its generalization in [69,70]. It prescribes the velocity v^* of the common point of several colliding waves with velocities v_i: the joint velocity v^* of the shock is given by the center of the smallest ball (a disk in 2D) covering all the velocities v_i of the colliding waves.

It is an interesting problem to formulate a more general variational principle for maps of M to itself (one should possibly consider Lipschitz maps to ensure differentiability almost everywhere) describing trajectories of particles, which would be valid before and after the formation of shocks and which would agree with both the non-invariant L^2-metric for the Burgers equation (before the collision) and the "circle law" for particles sticking to each other after the collision.

Problem 4. *Is it possible to extend Arnold's geodesic framework from the group of diffeomorphisms $\mathscr{D}(M)$ to the semigroup of maps $\mathcal{M}ap(M)$ that would capture both smooth solutions and their continuations beyond emergence of shock waves for Burgers-like and compressible fluid equations?*

To describe the motion of particles which are fused together inside shocks one might employ the setting of generalized solutions (see [6,19,20]) to the Euler equations and the methods of control theory, which are well-adapted to study non-uniqueness of trajectories of dynamical systems.

3.2. Variational Setting for Sticking Particles

There are various promising approaches to the variational formulation of the problem for sticking particles, see for example [20,126]. Here, we propose to look at it from yet another point of view. We begin with the simplest situation: a motion of two sticking particles of equal mass moving without friction along a line. After the collision they form a new compound particle whose total mass and momentum are conserved. The kinetic energy obviously decreases upon collision. This loss can be interpreted as a transfer of energy to new unobservable degrees of freedom. For example, we can imagine that the particles move along two very close parallel lines and that, at the moment of their near-collision, they are joined by a rigid rod. The compound particle (in the shape of a dumbbell) will remain in the state of rotation: the angular coordinate of the axis of the rotating dumbbell is the new degree of freedom. Thus, a portion of the apparently vanishing energy has been allocated to this "invisible" degree of freedom. There may be physically different realizations of such invisible degrees of freedom. However, we only need to know that they exist and we are free to use them at will.

Let x_1, x_2 be the coordinates of the particles with $x_1 \leq x_2$. The configuration space of our system is the half-plane $X = \{(x_1, x_2) |\ x_1 \leq x_2\} \subset \mathbb{R}^2$. Let $\Delta = \{(x_1, x_2) |\ x_1 = x_2\}$ be the diagonal and let $\Delta^\perp = \{(x_1, x_2) |\ x_1 + x_2 = 0\}$ be its orthogonal complement.

Extend the configuration space to the set $Z \subset \mathbb{R}^2_1 \oplus \mathbb{R}^2_2$, where

$$\begin{aligned} Z &= (X \oplus \{0\}) \cup (\Delta \oplus \Delta^\perp) \\ &= \big\{(x_1, x_2, y_1, y_2) |\ x_1 \leq x_2,\ y_1 = y_2 = 0\big\} \\ &\cup \big\{(x_1, x_2, y_1, y_2) |\ x_1 = x_2,\ y_1 + y_2 = 0\big\} \end{aligned}$$

is the union of the original space $X \subset \mathbb{R}^2_1$ and the plane $\Delta \oplus \Delta^\perp$ and where $\Delta \oplus \{0\}$ is identified with $\Delta \subset \mathbb{R}^2_1$.

Now, consider two points $z_0 \in X \subset Z$ and $z_1 \in Z$ and a trajectory $z(t)$ in Z for $0 \leq t \leq 1$ with $z(0) = z_0$, $z(1) = z_1$ such that the action $J(z(\cdot)) = \int_0^1 \frac{1}{2} |\dot{z}(t)|^2 dt$ is minimal among all trajectories in Z connecting z_0 and z_1. Let P be the projection of $\mathbb{R}^2 \oplus \mathbb{R}^2$ onto the first summand and define the trajectory $x(t) = Pz(t)$. It is easy to see that $x(t) = (x_1(t), x_2(t))$ represents the motion of two particles on the line colliding and sticking upon collision with the total momentum being constant.

Thus, we have established the variational principle in the simplest case of two particles on the line.

In a similar way we may consider a configuration of n particles $x_1, \dots, x_n$ on the line where $x_1 \le x_2 \le \dots \le x_n$. Let $X \subset \mathbb{R}^n$ be the set of such configurations. The set X is stratified: let $\Delta_{m_1,\dots,m_k}$ denote the set of $(x_1, \dots, x_n)$ such that

$$x_1 = \dots = x_{m_1},\ x_{m_1+1} = \dots = x_{m_1+m_2},\ \dots\ x_{m_1+\dots+m_{k-1}+1} = \dots = x_n,$$

where $m_1 + \ldots + m_k = n$. In $\mathbb{R}^n \oplus \mathbb{R}^n$ let

$$\begin{aligned} Y_{m_1,\dots,m_k} &= \Delta_{m_1,\dots,m_k} \oplus \Delta^{\perp}_{m_1,\dots,m_k} \\ &= \Big\{ (x_1, \dots, x_n, y_1, \dots, y_n) \mid \\ &\qquad (x_1, \dots, x_n) \in \Delta_{m_1,\dots,m_k},\ \begin{matrix} y_1 + \dots + y_{m_1} = 0, \dots \\ y_{m_1+\dots+m_{k-1}+1} + \dots + y_n = 0 \end{matrix} \Big\}, \end{aligned}$$

and define the extended configuration space

$$Z = \big(X \oplus \{0\}\big) \bigcup \left(\bigcup_{\substack{k=1,\dots,n-1 \\ m_1+\dots+m_k=n}} Y_{m_1,\dots,m_k} \right).$$

Let P be the orthogonal projection from $\mathbb{R}^n_1 \oplus \mathbb{R}^n_2$. Let $x_0 \in X,\ z_0 = x_0 \oplus \{0\}$ and $z_1 \in Z$. We can define the trajectory $z(t) \in Z,\ 0 \le t \le 1$, connecting z_0 and z_1 and whose least action $J(z(\cdot))$ is minimal. Then, the trajectory $x(t) = Pz(t)$ in X connects x_0 and $x_1 = Pz_1$ and describes the motion of sticking particles with the momentum preserved upon every collision.

Now consider a continuum of material points distributed on the line. To be specific, consider the following situation: let S be the segment $0 \le s \le 1$ on the s-axis where s is the label of a fluid particle. A fluid configuration is defined by the coordinate $f(s)$ for a particle with the label s, that is it is a map $f : S \to \mathbb{R}$, and we assume that f is a monotone function, that is $s_1 \le s_2 \Rightarrow f(s_1) \le f(s_2)$. The configuration space is $X = \{f \in W := L^2(S, \mathbb{R}) \mid f$ is a monotone function on $S\}$. This space is stratified in the following way. Let $f(s) \in X$. This function may be constant on at most countably many intervals σ. Let Σ be the collection of such intervals where $f(s) = \text{const}$ and define the stratum X_Σ to be the set of all functions f in X such that $f|_\sigma = \text{const}$ for every $\sigma \in \Sigma$. Let $W_\Sigma = \{f \in W \mid f = \text{const on each } \sigma \in \Sigma\}$; then $X_\Sigma = X \cup W_\Sigma$. We see that its orthogonal complement is

$$W_\Sigma^\perp = \left\{ f \in W \mid \int_\sigma f\, ds = 0 \text{ for every } \sigma \in \Sigma \text{ and } f = 0 \text{ outside } \cup_{\sigma\in\Sigma} \sigma \right\}.$$

Consider the space $W \oplus W$ and define the set $Z \subset W \oplus W$ (which is the desired extension of the space X) as follows. First, let $Y_0 = X \oplus \{0\} \subset W \oplus W$. Next, for every $\Sigma \ne \emptyset$, let $Y_\Sigma = X_\Sigma \oplus W_\Sigma^\perp$ and set

$$Z := Y_0 \bigcup \left(\bigcup_\Sigma Y_\Sigma \right).$$

We are now in the position to formulate the variational problem whose solutions describe motions of a continuous family of particles on the line that stick upon collisions. Let $f_0 \in X$ be the initial position of the particles and set $g_0 = f_0 \oplus \{0\} \in Y_0 \subset Z$. Let $g_1 \in Z$. For any trajectory $g_t \in Z$ with $0 \leq t \leq 1$ we define its action $J\{g_t\} = \int_0^1 \frac{1}{2}|\dot{g}_t|^2 dt$. Among all trajectories g_t in Z connecting g_0 and g_1 we choose the one with minimal $J\{g_t\}$ (such trajectory exists and is unique). Now, define the trajectory f_t in X by $f_t = P\, g_t$, where P is the projection from $W \oplus W$ onto the first coordinate. This is the desired trajectory of the system.

Remark 3.1. Note that the additional variables (the second term in $W \oplus W$ in the definition of the extended space Z) are necessary to define a sufficiently wide class of motions. If we simply defined $f_0, f_1 \in X$ then the action minimizing trajectory f_t from f_0 to f_1 would be a motion along the straight segment connecting these two points (since X is a convex set). For such a trajectory the particles would not collide at all (or, if you wish, they would collide only at $t = 1$).

Remark 3.2. If we attempt to use this approach to construct the motion of a continuum of particles in $\mathbb{R}^d,\ \ d > 1$ then we encounter a new difficulty: the set of admissible configurations of the particles is no longer convex. Therefore, particle collisions are not as easily controlled and parametrized as in the one-dimensional case. In particular, it is unclear how to use this method to give a variational description of the formation of "shock waves", that is, hypersurfaces where the mass is concentrated with positive density. In the theory of shock waves individual particle motions are described for potential solutions only, cf. Section 3.1.

However, there exists another class of sticking flows in $\mathbb{R}^3$, namely, flows with constant density (and decreasing energy), see [133]. Such flows are dissipative weak solutions of the Euler equations and are vaguely similar to turbulent flows. Their construction is based on different ideas (not on a variational principle) and it would be interesting to define them in a way corresponding to the above 1D systems.

We thus arrive at the following problem:

Problem 5. *Find a variational description of the system of material particles, moving in $\mathbb{R}^d$ with $d > 1$ and sticking upon collision, for both potential and non-potential fluid flows. This formulation should be sufficiently flexible to describe the formation and development of shock waves in the system described by the multi-dimensional Burgers equation.*

Remark 3.3. The above variational principle should be closely related to the models of adhesion particle dynamics studied in [126]. It seems to give the same results for a finite number of sticking particles. However, the approach in [126] is fundamentally one-dimensional, while the approach described above may be extended to higher dimensions as well.

Indeed, for a finite number of particles even in higher dimensions one can assign which particles stick together. However, on the way to this collision they may bump into other particles. For a finite number of particles there is an excuse that the probability of this happening is zero. But, in the case of a continuum, for example, given a continuous density at the initial moment particle trajectories will intersect en masse. One might allow that by envisioning a "dusty matter" with a multi-flow structure and no pressure (for example, this might be the case of stars in the universe when different streams of stars move in different directions in the same volume). Alternatively, one might confine to piecewise-smooth flows with stratified density (supported on a stratified manifold with components of different dimensions). The latter setting is close to Kantorovich's theory of optimal mass transport, cf. [19,20], where such a motion in 1D with gluing of particles and shock waves is described and the variational principle is written in terms of differential inclusions on the space of transport maps.

4. Rigid and Fluid Modelling of Invariant Metrics

Since the work of V. Arnold it is well-known that while the Euler equation of a rigid body corresponds to a *left-invariant metric* (depending on the body shape) on the group $SO(3)$, the Euler equations of ideal hydrodynamics correspond to an L^2 *right-invariant metric* on the diffeomorphism group $\mathscr{D}_\mu(M)$, see [2,5]. Left invariance of the rigid body metric is related to the fact that the body's energy depends on the angular momentum in the body and does not depend on its position in the ambient space. On the other hand, right invariance of the fluid metric is related to the norm of the velocity field in the space but does not depend on the parametrization of fluid particles. In other words, the energy metric on $SO(3)$ is left-invariant because the space $\mathbb{R}^3$ is *isotropic*, while the metric on $\mathscr{D}_\mu(M)$ is right-invariant because the fluid is *homogeneous*. We thus see that the reasons of right- and left-invariance are quite different.

A natural question arises: does there exist an interesting mechanical system with $SO(3)$ as a configuration space such that the energy metric is *right*-invariant? The answer is *yes*, and in what follows we give its description.

First, we define an object called a "hedgehog". This is a ball B whose center is a fixed point $O \in \mathbb{R}^3$ so that it can freely rotate around it. Suppose that at every point on the surface of the ball grows a "needle", that is, there is a (sufficiently long) radial segment. The whole structure rotates around the fixed center O as a solid body so that its configuration space is $SO(3)$.

Next, suppose that on every needle there is an infinitesimal point mass which is able to move freely along the needle; let us call it a bead. Let $\rho(\omega)$ be the angular mass density, so that the bead mass in the solid angle $d\omega$ is $\rho(\omega)d\omega$ (the hedgehog itself is massless). Now, suppose that there is a closed surface S surrounding the ball so that every needle pierces S. Lastly, suppose that every bead is forced to remain at all times on the surface S and at the same time is confined to its own needle. It is natural to call this system "beads on the hedgehog" (we do not consider here its practical realizations). This system depends on the angular bead density $\rho(\omega)$ and the surface S.

There are two cases of the hedgehog having some symmetry. In the first case, the surface S is a sphere concentric with the hedgehog, while the bead density $\rho(\omega)$ is arbitrary (Fig. 1). It is easy to see that this is the same as a solid body with a fixed

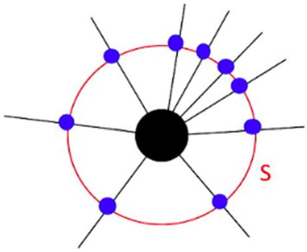

Fig. 1. Beads on a spherical "hedgehog model": the angular density of beads on a concentric sphere S is arbitrary. The corresponding metric on $SO(3)$ is left-invariant

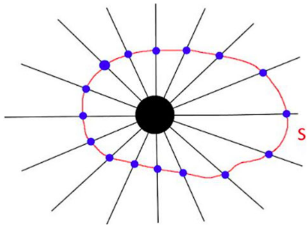

Fig. 2. Beads on a star-shaped "hedgehog model": the surface $S \subset \mathbb{R}^3$ is arbitrary, the angular density of beads $\rho(\omega)$ is constant. The corresponding metric on $SO(3)$ is right-invariant

point, the Euler top. In this case the metric on $SO(3)$ is left-invariant: the energy depends on the angular velocity of the system in the body but not in the space.

In the second case, S is an arbitrary star-shaped surface (fixed in $\mathbb{R}^3$), while the angular density $\rho(\omega)$ is constant (Fig. 2). In this case the metric on the group $SO(3)$ is right-invariant: now the energy depends on the angular velocity of the system in the space but not in the body. Note that the equations describing this dynamics will be the Euler-Arnold equations of the right-invariant metric on $SO(3)$ which differ only by a sign from the standard equations for the Euler top.

In all other cases the metric is neither left- nor right-invariant. It would be interesting to investigate this system for generic S and $\rho(\omega)$. It would also be of interest to define and study a model system where the configuration space is $SL(2)$, since this group looks more "liquid-like". A similar question about possible fluid models is very intriguing.

Problem 6. *Describe a model for an L^2 left-invariant metric on the group $\mathscr{D}_\mu(M)$. Are there any interesting physical systems exhibiting this type of invariance?*

5. The Cauchy and Boundary Value Problems

5.1. On Local Well-Posedness of the Cauchy Problem

First rigorous results on local existence and uniqueness of solutions to the Cauchy problem for the incompressible Euler equations (1.1) were obtained in the 1920s by Gunther [53] and Lichtenstein [90] in the class of Hölder $C^{1,\alpha}$ spaces. Global existence in 2D was established shortly thereafter by Wolibner [149]. Various subsequent extensions and improvements of these results in Hölder, Sobolev H^s and $W^{s,p}$ and more exotic Besov $B^s_{p,q}$ and Triebel-Lizorkin $F^s_{p,q}$ spaces can be found in the papers [12,36,66,68,151,152]; see also recent monographs and surveys [7,10,25,34,96].

Roughly speaking, a Cauchy problem is *locally well-posed* in a Banach space X (in the sense of Hadamard) if given any initial data in X there exists a $T > 0$ and a unique solution in the space $C([0,T], X)$ which depends at least continuously on the data. Otherwise, the problem is said to be *ill-posed* in X. A number of ill-posedness mechanisms have been investigated in the literature, from loss of regularity properties of the solution map, to energy decay, to nonuniqueness and finite time blowup. Although global (in time) well-posedness of the 3D Euler equations has long been seen as the major open problem in analysis and PDE, interesting questions concerning local well-posedness (in any dimension) have also remained open for a long time.

Recall that the Cauchy problem for the incompressible Euler equations is not well-posed in the standard Hölder spaces in the sense that solutions may depend discontinuously on general initial data in $C^{1,\alpha}$. However, this dependence is known to be continuous in, for example, the "little" Hölder space (essentially, the completion of smooth functions in the Hölder norm), as well as in $W^{s,p}$ Sobolev spaces with $p \geqq 2$ and $s > n/p + 2$, cf. for example, [36,67,105]. Somewhat more refined existence and uniqueness results are available in $B^1_{\infty,1}$ and $B^{n/p+1}_{p,1}$ where $1 < p < \infty$, see for example [22,120,144]. On the other hand, there are examples of 3D solutions of (1.1) which exhibit instantaneous loss of regularity. Examples in C^α with $0 < \alpha < 1$, as well as in $B^1_{\infty,\infty}$, $F^1_{\infty,2}$ and $\log \mathrm{Lip}^\alpha$ with $0 < \alpha \leqq 1$ were constructed in [11,85,104]. Many other ill-posedness results in the borderline Sobolev and Besov spaces $W^{n/p+1,p}$, $B^{n/p+1}_{p,q}$ with $1 \leqq p < \infty$ and $1 < q \leqq \infty$ and in the classical spaces C^k and $C^{k-1,1}$ can be found in [17,18,38].

One remaining case of particular interest can be formulated as follows:

Problem 7. *Are the Euler equations (1.1) ill-posed in the Besov spaces $B^1_{\infty,q}$ for $1 < q < \infty$?*

To put this problem in a functional space context, first recall that

$$H^s \subset C^{1+\alpha} \subset B^1_{\infty,1} \subset C^1 \subset \mathrm{Lip} \subset F^1_{\infty,2} \subset B^1_{\infty,\infty} \subset \mathrm{logLip} \subset C^\beta$$

for any $0 < \alpha, \beta < 1$ and $s > n/2 + 1 + \alpha$ and next observe that, in the specified range, the $B^1_{\infty,q}$ spaces interpolate between $B^1_{\infty,1}$ and the Zygmund space $B^1_{\infty,\infty}$.

5.2. A Two-Point Boundary Value Problem on Diffeomorphism Groups

Turning to the geodesic equation (1.2), consider the Sobolev completion $\mathscr{D}^s_\mu(M)$ of the diffeomorphism group $\mathscr{D}_\mu(M)$. As is well known, the Cauchy problem for (1.2) can be solved (for small values of t) by standard Banach contraction arguments provided that $s > n/2 + 1$, cf. [36]. Consequently, the L^2 metric (1.3) admits a smooth Riemannian exponential map

$$\exp_e : T_e\mathscr{D}^s_\mu(M) \to \mathscr{D}^s_\mu(M) \tag{5.1}$$

defined in a neighbourhood of the zero vector by $\exp_e tv_0 = \gamma(t)$ where γ is the unique geodesic from the identity element e with initial velocity v_0. Furthermore, as in the classical finite-dimensional Riemannian geometry, we have $d\exp_e(0) = \text{id}$, and therefore $\exp_e$ is a local diffeomorphism of Banach spaces by the inverse function theorem. In particular, this implies local well-posedness of the Euler equations in H^s for $s > n/2+1$, as well as unique solvability of the two-point boundary value problem for the geodesic equation (1.2) in any sufficiently small neighborhood of e in $\mathscr{D}^s_\mu(M)$. (In geometric language Wolibner's global existence and uniqueness result in [149] amounts to a statement about geodesic completeness of the manifold $\mathscr{D}^{1,\alpha}_\mu(M)$ of Hölder diffeomorphisms under the right-invariant L^2 metric (1.3) when $n = 2$.)

A natural question is whether the two-point problem holds in the large. We formulate this as two related problems.

Problem 8. (Two-point boundary value problem) *Let M be a compact two-dimensional Riemannian manifold and let φ be a volume-preserving diffeomorphism of M of Sobolev class H^s.*

(i) *(Surjectivity Problem) Find a divergence-free vector field $v \in H^s(M)$ such that* $\exp_e v = \varphi$.
(ii) *(Variational Problem) Find a curve $\gamma(t)$ in $\mathscr{D}^s_\mu(M)$ from e to φ which minimizes the L^2 energy functional $\mathcal{E}(\gamma) = \frac{1}{2}\int_0^1 \|\dot\gamma(t)\|^2_{L^2} dt$.*

Although two-point boundary value problems in hydrodynamics are no less fundamental than the Cauchy problem they have not received as much attention. From the geometric point of view (i) and (ii) may be regarded as infinite-dimensional versions of the classical Hopf–Rinow theorem. One strategy for (i) is to follow the classical argument of Hopf and Rinow compensating for the lack of local compactness with *a priori* estimates derived with the help of weak solutions and Lyapunov functions [132]. Another approach could use the properties of the exponential map as a nonlinear Fredholm and quasiruled map [37,129,134]. In connection with (ii) we mention a surprising result of [130,131] that there exist volume-preserving diffeomorphisms of a simply connected compact 3-manifold which cannot be joined by a shortest path in $\mathscr{D}^s_\mu(M)$. For the two dimensional case partial results can be found in [103].

5.3. Global Geometry of the Group of Volume-Preserving Diffeomorphisms

Next, we turn to questions concerning the global geometry of the group $\mathscr{D}^s_\mu$. The following problem is related to Problem 8.

Problem 9. *Does the energy (action) functional corresponding to the L^2 kinetic energy metric (1.3) in 2D hydrodynamics satisfy the Palais–Smale condition?*

Another interesting question is related to closed geodesics. Suppose that M is a compact surface (possibly with boundary) of genus at least 2 or that M is a multi-connected bounded domain in $\mathbb{R}^2$ with at least two holes.

Problem 10. *Does there exist a closed geodesic in $\mathscr{D}^s_\mu(M)$?*

Note that the Kelvin–Helmholtz theorem implies that any geodesic loop in $\mathscr{D}^s_\mu(M)$ is necessarily a closed geodesic. One may try to construct a suitable Lyapunov function (see below) to show that in this case a geodesic never returns to its initial configuration.

Yet another basic problem concerns the fluid configuration space $\mathscr{D}_\mu(M)$ itself. As an infinite dimensional (Frechet) Lie group it can be viewed as a Riemannian homogeneous space equipped with a right-invariant L^2 metric (1.3). However, the group $\mathscr{D}_\mu(M)$ is not a Riemannian symmetric space. This means that the (geodesic) central symmetry about the identity in $\mathscr{D}_\mu(M)$ is not an isometry of the L^2-metric. In this respect, we have to keep in mind that the geodesic symmetry is not the same as the group-theoretic symmetry, that is, the map $g \to g^{-1}$. The latter (group-theoretical) inversion maps a right-invariant metric on the group into a left-invariant one, hence, indeed, it is not an isometry. However, on the group of finitely differentiable diffeomorphisms like $\mathscr{D}^s_\mu(M)$ the group inversion is not even differentiable, while if we use the Holder $C^{k,\alpha}$-class diffeomorphisms, it is not even continuous. In contrast, the geodesic central symmetry is a smooth map in $\mathscr{D}_\mu(M)$ for any reasonable model space (like H^s and $C^{k,\alpha}$) but the lack of its isometry property is not so immediately seen. Hence, phrasing our question somewhat informally, we may ask

Problem 11. *How "far" is $\mathscr{D}_\mu(M)$ from being an infinite-dimensional (locally) symmetric Riemannian space?*

This question is related to the following long standing (although little known) paradox. Consider the parallel sinusoidal steady fluid flow given by the stream function $\psi = \cos(k\,y)$ on the two-dimensional torus. Then well-known Arnold's theorem claims that the sectional curvature of the group of exact area-preserving torus diffeomorphisms is nonpositive in all (and negative in most) two-dimensional directions containing the direction given by ψ, see [2,6]. (There is a similar statement for a plane-parallel flow in a periodic channel.) Following Arnold's idea on an intrinsic relation between negative curvature and the flow (Lagrangian) instability, one could expect that *any* plane-parallel flow is unstable. But this does not seem to be the case, since there are (Eulerian) stable parallel flows (for instance those with convex velocity profiles on "short tori", see discussion in [6]), while Lagrangian and Eulerian instabilities are closely related, cf. [122].

The root of this misunderstanding lies in our "symmetric" intuition. In fact, this relation between the curvature sign and (in)stability of geodesics exists for symmetric spaces (say, on groups with bi-invariant metrics), see for example [6, 100]. On the other hand, the group $\mathscr{D}_\mu(M)$ is not symmetric, but rather "chiral": as a Riemannian space it is somewhat twisted in one direction, and hence we observe a discrepancy between instability of geodesics and its negative curvature.

The chirality of $\mathscr{D}_\mu(M)$ might have some other, more profound, consequences beyond the instability issues, which would be interesting to explore.

6. Partial Analyticity of Solutions in 2D

6.1. Analyticity of Particle Trajectories

The Euler equations keep bringing surprises—such as a relatively recent theorem, which we now present. Suppose that M is a compact 2D real analytic manifold or a bounded domain with analytic boundary. Let u be a solution of the Euler equations in M of Sobolev class H^s for $s > 2$, obeying the slip condition $u \parallel \partial M$ if $\partial M \neq \emptyset$. Finally, let $x(t)$ be any particle trajectory satisfying the flow equation $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = u(t, x(t))$.

Theorem 6.1. *Any particle trajectory $x(t)$ of the flow is a real-analytic function of t.*

This theorem was first established by Serfati [127], then in various forms by Inci et al. [57], Shnirelman [135], Constantin et al. [26], Zheligovsky and Frisch [157] and, in case of stationary flows, Nadirashvili [116]. The proofs by the above authors are based on two entirely different ideas. In the works [26,127,157] the function $x(t)$ was formally expanded in the Taylor series, whose convergence was proved using commutator estimates. Thus, this was a straightforward proof. On the other hand, the works [57,116,135] used the Lagrangian description of the fluid motion, that is they considered the flow as the motion g_t on the infinite-dimensional manifold $\mathscr{D}^s_\mu(M)$, equipped with the L^2-metric, along a geodesic. The manifold $\mathscr{D}^s_\mu(M)$ is a real-analytic Banach manifold and the geodesic spray (generating the geodesic flow in the tangent bundle to $\mathscr{D}^s_\mu(M)$) is an analytic vector field, since it can be extended to the complexification $\mathbb{C}\mathscr{D}^s_\mu(M)$ as a holomorphic vector field. Then the standard theorems of existence, uniqueness and analytical dependence of solution g_t on t and on the initial condition x_0 can be applied, since they hold for any analytic Banach manifold [31,56].

The work [135] was based on a similar idea: using the Kelvin–Helmholtz vorticity theorem, one can reduce the Lagrange equation to a first order equation of the form $\frac{\mathrm{d}}{\mathrm{d}t}g_t = V(g_t)$ on $\mathscr{D}^s_\mu(M)$, where V is an analytic vector field on the infinite-dimensional manifold $\mathscr{D}^s_\mu(M)$. Then the above basic existence, uniqueness, etc. theorems are applicable, and the same result follows: the flow $g_t \in \mathscr{D}^s_\mu(M)$ is an analytic curve depending analytically on g_0.

Interestingly, the latter work closely follows the original approach of Lichtenstein [89] in his proof of the local in time existence and uniqueness of solutions to the Euler equation. Lichtenstein proved that the vector field V is Lipschitz (in fact, he proved that it is C^1) and that it can be continued analytically in the complexification of $\mathscr{D}^s_\mu(M)$. Thus, in 1925 he was just one step away from proving that the flow g_t is analytic in t! However, Lichtenstein's work appeared roughly 10 years before the Banach spaces acquired their name; about 20 years before the concepts of complex analysis (like the analytic implicit function theorem) were extended to the complex Banach spaces; and about 30 years before it was acknowledged that the basic concepts of smooth and analytic topology including the theory of ODEs can be transferred to the complex Banach manifolds. So, if he made that step his discovery would be truly extraordinary.

Problem 12. *Are particle trajectories analytic in time in any dimension? For fluids on manifolds with boundary how does this analyticity depend on whether the boundary is analytic or not?*

NADIRASHVILI [116] proved analyticity of flow lines of a *stationary* solution to the 2D Euler equation. His theorem is local and holds independently of the analyticity (or the lack thereof) of the boundary ∂M. It follows the classical analyticity proof for solutions of analytic elliptic equations and uses the fact that an elliptic equation becomes hyperbolic if one of the variables, say x_1, is replaced by $\mathrm{i}x_1$. (It is worth recalling that flow lines and vorticity lines coincide for stationary 2D solutions and hence are analytic simultaneously.)

Remark 6.2. In dimensions $d = 2$ and 3 the analyticity of particle trajectories in $\mathbb{R}^d$ (that is in the case without boundary) was proved in [26]. Apparently the higher dimensional case does not present fundamentally new difficulties, and the study in [26] was confined to low dimensions because of their physical significance. The paper [55] covers several related cases, including vortex patches among other situations.

There is also the following related problem: are particle trajectories of Yudovich solutions analytic in time? (Recall that the Yudovich class consists of divergence-free vector fields u satisfying $\|\mathrm{curl}\, u\|_{L^\infty} < \infty$. This inequality implies that the field u has the Osgood property [121], which guarantees the uniqueness of trajectories.) It is currently only known that they are Gevrey regular, due to the result in [50], see also [23]. It seems to be unknown if this Gevrey regularity is sharp.[1]

The difficulty here is as follows. It is known that Sobolev vector fields are integrated to Sobolev diffeomorphisms; in other words, Sobolev vector fields form the Lie algebra of the group of Sobolev diffeomorphisms. However, it is unknown what would be the result of integration for vector fields from the Yudovich class, that is what are "Yudovich homeomorphisms". Do such homeomorphisms form a group? Does this "group" admit the structure of a real Banach manifold? Can this "manifold" be complexified? Presumably, the answers are "no" to all those questions, and one has to look for other approaches to this problem.

In regard to analyticity, LEBEAU [84] considered piecewise-continuous solutions of 2D Euler equations which are irrotational outside of a time-dependent curve and have a tangential discontinuity on the curve, cf. Section 13 on vortex sheets. Lebeau proved that the curve of discontinuity is analytic as long as such a solution exists, that is, the vortex sheets are analytic in 2D.

[1] We thank the anonymous referee for this remark.

6.2. Stationary Flows and Partially Analytic Functions

For a stationary solution u of the Euler equations particle trajectories are the same as flow lines. The stream function $\psi(x)$ of a stationary flow $u = \nabla^{\perp}\psi$ is a peculiar function: it may be an H^s function but its level lines are analytic. A function $\psi(x)$ whose level lines are real-analytic will be called a *partially analytic function*. The set of partially analytic functions is by no means a vector space: just consider a pair of such functions ψ_1 and ψ_2 whose (analytic) level lines are transversal to each other; then the level lines of $\psi_1 + \psi_2$ are not necessarily analytic. Thus, one arrives at an important problem to find a natural structure on the set of partially analytic functions.

One natural idea would be to choose level lines, rather than the values at points, as an adequate representation of a partially analytic function. For example, consider a function $\psi(x_1, x_2)$ defined on a periodic curvilinear strip $M = \{(x_1, x_2) \,|\, g(x_1) \leq x_2 \leq h(x_1)\}$, where g and h are real periodic functions with the same period ℓ, and such that every level line $\psi(x) = \text{const}$ has an equation $x_2 = a(x_1, \psi)$ with $a(x_1, 0) \equiv g(x_1)$ and $a(x_1, 1) \equiv h(x_1)$. One can assume that $x_1 \in \mathbb{T} = \mathbb{R}/\ell\,\mathbb{Z}$. Then the function $a(x_1, \psi)$ uniquely defines $\psi(x_1, x_2)$ in the flow domain M. The function $a(x_1, \psi)$ is of class H^s and it is analytic in x_1 for any fixed ψ.

To be more precise, let us define the corresponding function spaces. Let us fix $\sigma > 0$.

Definition 6.3. The space X^s_σ consists of real-analytic functions $f(x_1)$, $x_1 \in \mathbb{T}$ which can be analytically continued into the strip $|\text{Im}\, x_1| \leq \sigma$ and such that $f(\cdot \pm i\sigma) \in H^s(\mathbb{T})$, where the norm is $||f||_{X^s_\sigma} = \|f(\cdot - i\sigma)\|_{H^s(\mathbb{T})} + \|f(\cdot + i\sigma)\|_{H^s(\mathbb{T})}$.

Definition 6.4. The space Y^s_σ consists of functions $a(x_1, \psi)$ such that

(i) for any $\psi \in [0, 1]$, $a(x_1, \psi) \in X^s_\sigma$;
(ii) the functions $a(x_1 \pm i\sigma, \psi)$ belong to $H^s(\mathbb{T} \times [0, 1])$.

The norm in the space Y^s_σ is defined as follows:

$$||a||_{Y^s_\sigma} = ||a(\cdot + i\sigma, \cdot)||_{H^s(\mathbb{T}\times[0,1])} + ||a(\cdot - i\sigma, \cdot)||_{H^s(\mathbb{T}\times[0,1])}$$

Finally we define the space Z^s_σ follows:

Definition 6.5. A function $\psi(x_1, x_2)$ defined in the domain $M = \{(x_1, x_2) \,|\, g(x_1) \leq x_2 \leq h(x_1)\}$ belongs to the space Z^s_σ if its level lines $\psi = \text{const}$ can be defined by the equation $x_2 = a(x_1, \psi)$, where the function $a(\cdot, \cdot)$ belongs to the space Y^s_σ. The norm in Z^s_σ is induced from the space Y^s_σ.

An immediate application of this space is to the description of the set of stream functions of stationary solutions of the Euler equations in the periodic domain $M = \{(x_1, x_2) \,|\, g(x_1) \leq x_2 \leq h(x_1) \text{ for } x_1 \in \mathbb{T}\}$. Indeed, if we consider stationary solutions $u(x_1, x_2)$ in H^s with stream functions $\psi(x_1, x_2)$ in H^{s+1} then the set of stationary solutions does not form a smooth manifold in H^s. This difficulty was partially circumvented by Choffrut and Sverak [24] by resorting to the C^∞ Frechet spaces and the Nash–Moser–Hamilton inverse function theorem. In this

C^∞ setting the stationary solutions form a smooth manifold parametrized locally by distribution functions of the vorticity. However, those tools might be too powerful for the task in finite smoothness.

On the other hand, using the space Y^s_σ, Danielski [29] established a local description of the set of stationary flows in the periodic channel. Let u_0 be the velocity field of a parallel flow satisfying several conditions in the domain $M_0 = \mathbb{T} \times [0, 1]$. Namely, assume that $u_0(x_1, x_2) = (U_0(x_2), 0)$ satisfies (1) $U_0(x_2) > 0$, (2) $U_0'(x_2) = F_0(x_2) > 0$, (3) $U_0''(x_2) > 0$; and finally, (4) let $\psi_0(x_1, x_2) = \Psi_0(x_2) = \int_0^{x_2} U_0(t)dt$ be the stream function of the flow u_0, then ψ_0 satisfies the boundary conditions $\psi_0(x_1, 0) = 0$, $\psi_0(x_1, 1) = 1$. Its level curves $\psi_0 = \text{const}$ have the equation $x_2 = a_0(\psi)$ where $a_0(\psi)$ is the inverse function to $\Psi_0(\cdot)$.

One can regard the above periodic domain $M = \{(x_1, x_2) \,|\, g(x_1) \le x_2 \le h(x_1) \text{ for } x_1 \in \mathbb{T}\}$ as being close to the parallel strip M_0. We are looking for the stationary flows in M which are close to the parallel flow u_0 in M_0.

Theorem 6.6. *For any parallel flow u_0 possessing the properties* (1)–(4) *there exists $\varepsilon > 0$ such that the following holds. Suppose that $||g(x_1) - 0||_{X^s_\sigma} < \varepsilon$ and $||h(x_1) - 1||_{X^s_\sigma} < \varepsilon$. Then there exist stationary flows $u \in H^s$ close to u_0 in the following sense:*

(1) *for each solution u, its stream function $\psi \in Z^{s+1}_\sigma$;*
(2) *the stream function ψ satisfies relation $\Delta\psi = F(\psi)$ for a certain monotone function $F \in H^{s-1}$ close to F_0 in H^{s-1};*
(3) *the functions ψ form an analytic submanifold $\Sigma \subset Z^s_\sigma$ which is locally analytically diffeomorphic to a neighbourhood of F_0 in $H^{s-1}[0, 1]$.*

The stationary flows with stagnation points present some additional difficulties.

6.3. An Attractor of 2D Euler Equations and its Semianalytic Structure

Consider a compact analytic Riemannian 2-manifold M with or without boundary, for example, a 2-torus. Let $YU(M)$ be the Yudovich space of vector fields on M consisting of divergence-free vector fields u such that $\operatorname{curl} u$ is in $L^\infty(M)$. The Euler equations define a perfect dynamics on the space $YU(M)$, that is a one-parameter group S of transformations $S_t : YU(M) \to YU(M)$ continuous in the H^1 topology (that is weakly continuous in $YU(M)$). For any $u \in YU(M)$ let $\mathscr{O}(u)$ be the *orbit* of u. Let $\bar{\mathscr{O}}(u)$ be the weak closure of $\mathscr{O}(u)$ in $H^1(M)$. Note that for any $v \in \mathscr{O}(u)$, one has $||\operatorname{curl} v||_{L^2} = ||\operatorname{curl} u||_{L^2}$ and for any $w \in \bar{\mathscr{O}}(u)$, $||\operatorname{curl} w||_{L^2} \le ||\operatorname{curl} u||_{L^2}$.

Definition 6.7. A field $u \in YU(M)$ is called a generalized minimal flow (or a GM-flow) if for any $w \in \bar{\mathscr{O}}(u)$ we have $\|\operatorname{curl} w\|_{L^2} = \|\operatorname{curl} u\|_{L^2}$. The set of all generalized minimal flows is denoted by $\mathscr{GM}$.

This definition has the following meaning. For any fluid flow its vorticity is transported by the fluid (the Kelvin–Helmholtz theorem). Thus, the vorticity field is deformed by the flow and, as $t \to \infty$, this deformation effectively leads to *mixing*. The mixing operator K has the form $Kf(x) = \int_M K(x, y) f(y)\, dy$, where the

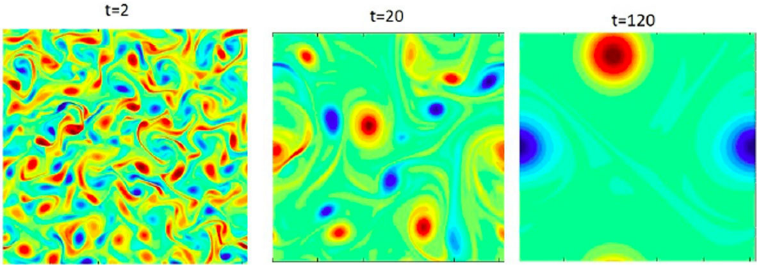

Fig. 3. The appearance of vorticity blobs in the flows on the two-torus

kernel $K(x, y)$ is a non-negative measure in $M \times M$ such that $\int_M K(x, y)\, dx \equiv 1$ and $\int_M K(x, y)\, dy \equiv 1$ (that is K is a bistochastic operator). Any mixing operator is a contraction in $L^2(M)$. Thus, for any $w \in \bar{\mathcal{O}}(u)$, $\operatorname{curl} w = K(\operatorname{curl} u)$ for some mixing operator K. If $u \in \mathcal{GM}$ then for any $w \in \bar{\mathcal{O}}$, $\operatorname{curl} w$ is equimeasurable with $\operatorname{curl} u$. In other words, $\operatorname{curl} u$ is not mixed by the Euler flow; the measure $\mu = (\operatorname{curl} u)\, dx$ can be disintegrated into components μ_α which are permuted by the flow and keep their individuality even as $t \to \infty$.

In [136] it was proved that the set $\mathcal{GM} \subset YU(M)$ is nonempty and it is a weak attractor for the Euler flow (see also [32]). (In fact, it is an attractor in a specific sense and it has not been proved that it is an attractor in the usual sense.) For some domains (including the periodic strip) Bedrossian and Masmoudi [13] proved that this subset is nontrivial, that is, there exists $u \in YU(M)$ such that $u \notin \mathcal{GM}$.

Any stationary flow is by definition a GM-flow. However, numerical examples show that there exist nonstationary GM flows (at least, on the torus). Such a flow comprises several large vortices (blobs) gracefully moving around and permanently changing their shapes. These flows appear to be time-periodic and quasiperiodic; it is unclear whether they can be more complex (say, chaotic).

If $u(x)$ is a stationary flow (that is a fixed point of the group $\{S_t\}$) then the level lines of vorticity are at the same time the flow lines and hence are real-analytic. Futhermore, if $u \in \mathcal{GM}$ is, say, periodic or quasiperiodic, our conjecture is that the level lines of vorticity $\operatorname{curl} u(x, t) = \text{const}$ are analytic as well. At least this property is preserved by the Euler evolution. Hence, we propose several problems/conjectures.

Problem 13. *Prove that for any two-dimensional compact analytic Riemannian manifold M with analytic boundary the set $\mathcal{GM}(M)$ of generalized minimal flows is a nonempty and proper subset of the Yudovich space $YU(M)$.*

Problem 14. *Prove that for any GM-flow $u(x, t)$ the level lines of vorticity* $\operatorname{curl} u(x, t) = \textit{const}$ *are real-analytic.*

One other property of GM-flows is observed in the numerical simulations, cf. Figure 3. The flow domain M contains some number of vorticity blobs $B_1, \ldots, B_N$ and a background B_0. Accurate numerical results hint at the following conjecture.

Problem 15. *For any GM-flow* $\operatorname{curl} u = \textit{const}$ *in* B_0.

Thus, the level sets of curl u are of two sorts: real-analytic curves and entire domain B_0.

6.4. Are Anisotropic Function Spaces the Future of Hydrodynamics?

The function spaces used in the theory of linear PDEs are less suitable for the nonlinear ones and they seem completely inadequate for the Euler equations. One of the most prominent features of the Euler solutions is the built-in analyticity of the particle trajectories, flow lines for stationary flows, and (conjectured) analyticity of level lines of vorticity of GM-flows.

Problem 16. *Define anisotropic spaces more appropriate for hydrodynamics with larger smoothness in the direction of flow lines. Prove existence and uniqueness theorems for ideal hydrodynamics in those function spaces in any dimension.*

The concept of GM-flows appears closely related to the "ancient flows" for the Navier–Stokes equations but, in fact, it is different. Both concepts express the general idea that, in reality, we observe flows that have existed since long time ago (one may imagine a river flow). In this regard, it is not natural to consider flows starting at some definite time t_0 with some (arbitrary) initial velocity u_0. GM-flows have natural, intrinsic structure (analytic level lines of vorticity) and some natural, finite regularity in the transversal direction. The question of which functional classes are suitable for their study is an intriguing open problem.

Problem 17. *Determine the true regularity of GM-flows in the framework of anisotropic function spaces.*

7. More Differential Geometry: the L^2 Exponential Map and its Singularities

In classical finite-dimensional Riemannian geometry singular values of the exponential map are the conjugate points. The question concerning existence of conjugate points in diffeomorphism groups $\mathscr{D}^s_\mu(M)$ and their role in hydrodynamics was posed by Arnold [2]. Examples for the two-torus $\mathbb{T}^2$ and the spheres S^n were constructed in [101,102] with further examples in [15,33,88,123,124,131,140,147].

In infinite dimensions conjugate points are of two types depending on whether the derivative of the exponential map fails to be one-to-one (mono-conjugate points) or onto (epi-conjugate points). Moreover, they can accumulate along finite geodesic segments or have infinite order. Such pathological situations can be ruled out in 2D but not in 3D hydrodynamics. More precisely, we have the following result from [37,103]:

Theorem 7.1. *Let M be an n-dimensional Riemannian manifold (possibly with boundary).*

(i) *If $n = 2$ then the L^2 exponential map is a nonlinear Fredholm map of index zero.*

(ii) *If $n \geqq 3$ the the Fredholm property of the L^2 exponential map fails in general.*

Thus the structure of singularities of the exponential map in 2D hydrodynamics resembles that of a smooth map between finite-dimensional manifolds. This leads to the following natural question.

Problem 18. *Quantify the failure of the Fredholm property of the L^2 exponential map in $\mathscr{D}^s_\mu(M)$ for a compact Riemannian 3-manifold M.*

Examples of three-dimensional manifolds for which the exponential map (5.1) is not Fredholm can be found for example in [37,103,123]. It is reasonable to expect that this failure is borderline in the following sense. Explicit formulas for the derivative of the exponential map of a general right-invariant Sobolev H^r metric derived in [103] decompose it as $A + K_r$ where A is an invertible operator and

$$w \mapsto K_r w = (\mathbf{1} - \Delta)^{-r/2} P_e(\iota_w d(\mathbf{1} - \Delta)^{r/2} v_0^\flat)^\sharp, \qquad v_0 \in T_e\mathscr{D}^s_\mu$$

where $P_e = \mathbf{1} - \nabla\Delta^{-1}\mathrm{div}$ is the usual Helmholtz–Weyl (or Leray–Hopf) projector onto divergence-free vector fields, ι_w denotes the interior multiplication by a vector field w, while $^\flat$ and $^\sharp$ stand for the standard isomorphisms of the Riemannian metric on M. When $n \geqq 3$ and $r > 0$ then the operator K_r turns out to be compact and the associated H^r exponential map is Fredholm. In the case of the 3D fluids K_0 is no longer compact. In order to measure the deviation of K_0 from being a compact operator one can, for example, examine its essential spectrum. Explicit examples like the rotating solid cylinder or the Taylor–Green vortex, as well as careful numerical experiments, may provide some valuable insight.

Here are several other interesting questions concerning the structure and the role of conjugate points in 2D hydrodynamics.

Problem 19. (a) *Determine the order of the first conjugate point along any L^2 geodesic starting from the identity in $\mathscr{D}^s_\mu(M)$.*
(b) *Is there a relation between existence of conjugate points in $\mathscr{D}^s_\mu(M)$ and Arnold stability criterion for stationary flows in M?*

The answer to (*a*) may have some bearing on Problem 8. For example, if the order of any conjugate point turns out to be always greater than one then the fact that the exponential map is Fredholm of index zero would indicate that there is only one connected component of the identity over which $\exp_e$ is a covering map, see [103].

Regarding (*b*) recall that according to Arnold's criterion [2,5,6] a stationary flow of an ideal fluid is Lyapunov stable if the quadratic form given by the second derivative of the kinetic energy restricted to the coadjoint orbits is positive or negative definite. It can be shown that for simple domains such as the disk, the annulus and the straight channel no steady flows satisfying Arnold's stability possess conjugate points. (We assume here that the two-dimensional fluid domain M has a nonempty boundary.) It is therefore tempting to expect that this is true more generally. See [33] for additional background and [141] for recent results in this direction.

8. Long Time Behaviour of 2D Flows

Since existence, uniqueness and regularity of 2D solutions of (1.1) on the infinite time interval is quite well established we can proceed to ask questions concerning the long time behaviour of fluid flows.

8.1. Complexity Growth for 2D Flows

Let M be a two-dimensional compact manifold possibly with boundary. Recall that the L^2 exponential map (5.1) is a local diffeomorphism near the identity e in $\mathscr{D}^s_\mu(M)$. Fix $\varepsilon > 0$ and let $\mathcal{U}_\varepsilon = \exp_e(B_\varepsilon)$ where $B_\varepsilon = \{v \in T_e\mathscr{D}^s_\mu \mid \|v\|_{H^s} < \varepsilon\}$ is an open H^s ball of radius ε. Any diffeomorphism in $\mathscr{D}^s_\mu(M)$ can be represented as a product $\eta = \eta_1 \circ \cdots \circ \eta_N$ of a finite number of elements from $\mathcal{U}_\varepsilon$, see [92–94]. Let $\mathcal{C}_\varepsilon(\eta)$ denote the minimal number of factors in this representation and let $\mathcal{C}(\eta) = \limsup_{\varepsilon\to 0} \big(\varepsilon\, \mathcal{C}_\varepsilon(\eta)\big)$ be the *absolute complexity* of η.

Problem 20. *Show that for any geodesic $\gamma(t)$ in $\mathscr{D}^s_\mu(M)$ its absolute complexity is exponentially bounded above:*

$$\mathcal{C}(\gamma(t)) \lesssim e^{t\|\dot\gamma_0\|_{H^s}}.$$

This estimate would imply the well known double exponential estimate for solutions of the Euler equations (1.1), namely $\|v(t)\|_{H^s} \lesssim e^{C_1 e^{tC_2}}$. However complexity of a flow as defined above has a broader sense than its regularity.

Problem 21. *Show that for a typical geodesic $\gamma(t)$ in $\mathscr{D}^s_\mu(M)$ we have $\mathcal{C}(\gamma(t)) \simeq t$.*

Roughly, we say that a family of geodesics starting from the identity in $\mathscr{D}^s_\mu(M)$ is *typical* if the complement of the corresponding set of initial velocities in $T_e\mathscr{D}^s_\mu$ has infinite codimension. Examples of "typical" L^2 geodesics whose complexity grows linearly in time are the one-parameter subgroups of $\mathscr{D}^s_\mu(M)$ and (possibly) the quasi-periodic solutions of (1.1).

8.2. Aging of the Fluid, Irreversibility and Lyapunov Functions

Consider an arbitrary solution $u = u(t, x)$ of the Euler equations in M. Given any two time instants can one decide which velocity field $u(t_1, x)$ or $u(t_2, x)$ corresponds to an earlier time instant? In other words, is it possible to determine the aging of the fluid from its velocity field? Numerical experiments suggest that fluids "age" with time: starting with a smooth initial velocity $u_0(x)$ the corresponding solution becomes "wrinkled" in that its derivatives generally grow.[2]

Problem 22. *(Aging problem) Is it possible to quantify the "aging" property of the fluid?*

[2] V. Yudovich referred to this phenomenon as "regularity deterioration".

Perhaps the best way is to find a Lyapunov function L defined and continuous on the space of fluid velocities in $T_e\mathscr{D}^s_\mu$ such that $\frac{d}{dt}L(u(t,\cdot)) \geqq 0$ where u is a solution of (1.1) with equality holding on a "slim" subset of $T_e\mathscr{D}^s_\mu$ (say, of infinite codimension).

The first Lyapunov function in this context was constructed by Yudovich in 1970s for flows with a rectilinear streamline (for example, flows in domains whose boundary contains a straight line segment), see [153,155]. The construction was subsequently generalized for arbitrary bounded domains in [114]. It implies "regularity deterioration" at least on the boundary.

It is natural to expect that there are also Lyapunov functions which are supported inside the fluid domain. Examples describing evolution of weak singularities in the Lagrangian flow were found in [132]. The fact that these singularities become gradually "sharper" as the fluid evolves suggests the same deterioration phenomenon stressed by Yudovich.

Consider an ideal fluid in a periodic channel $M = \mathbb{T} \times (0, a)$. Assume that its initial velocity u_0 is C^1 close to that of a plane-parallel flow whose velocity profile $v = v(x_2)$ satisfies $v' > 0$ and $v'' > 0$ in $(0, a)$. Suppose that the level lines of $\omega_0 = \operatorname{curl} u_0$ satisfy $\nabla\omega_0 \neq 0$ in M and that one of the lines $\omega_0 = \text{const}$ has a "kink".

Problem 23. *Show that the "kink" does not disappear as the fluid evolves in time.*

9. Entropy and Fluids

9.1. Entropy of a Set and Entropy of a Measure

In general terms, entropy is a measure of diversity of some ensemble. For a finite set S with N elements of equal "weights" (that is, equivalent in some respect) the entropy $H(S)$ is equal to $\log_2 N$. If the elements $s_i \in S$ $(i = 1, \dots N)$ have different weights w_i then we define the entropy of the weighted finite set S to be $H(S) = -\sum_{i=1}^N w_i \log_2 w_i$. For example, if the whole mass (assumed to be 1) is concentrated at some s_k then $H(S) = 0$; otherwise $H(S) > 0$ with maximum value $\log_2 N$ if $w_i = 1/N$ for all $i = 1, \dots N$.

If S is an infinite set then the definition of $H(S)$ is not so clear. It is based on approximations of the set S by finite sets and of the weight (that is, the probability measure) μ on S by some discrete weights.

Suppose that S is a compact subset of a complete metric space X. In the absence of a measure on S we can define the Kolmogorov ε-entropy of the set S as $H_\varepsilon(S) = \log_2 N_\varepsilon$ where N_ε is the cardinality of the minimal ε-net, that is, the minimal number of ε-balls in X covering S. (In fact, $H_\varepsilon(S)$ is usually defined as an equivalence class of such functions as $\varepsilon \to 0$.) If S is equipped with a measure then we define an analogue of a weighted finite set as above and we can try to define a suitable analogue of the ε-entropy.[3] One option is to define it as the ε-entropy of the support

[3] It is not clear if such an object has anything in common with the entropy of an invariant measure as typically defined in the theory of dynamical systems.

of the given probability measure μ. This, however, would give only an upper bound, just like for weighted finite sets. Another option is to define

$$H_{\varepsilon,\delta}(\mu) = \inf \left\{ H_\varepsilon(Y) \mid Y \subset X \text{ is compact and } \mu(X \setminus Y) \leqq \delta \right\}.$$

What is the relation of $H_{\varepsilon,\delta}(\mu)$ to the entropy of a weighted finite set?

The definition of entropy given above depends on a pre-existing measure on X. If X is a finite set then we can take μ to be a counting measure; if X is a phase space of classical mechanics then μ could be a Liouville measure. But on a general metric space no such choices are available. On the other hand, if X is also a vector space then we can cover its compact subsets by congruent sets other than metric balls. For example, we can use cylindrical domains with finite-dimensional base. Suppose that X is a Hilbert space with coordinates $x_1, x_2, \dots$ and let X_n be the subspace defined by $x_i = 0, i > n$. Subdivide X_n into cubes C_j of side length $\varepsilon > 0$ and let $K_j = \pi_n^{-1}(C_j)$ where π_n is the orthogonal projection onto X_n. Given a compact set $S \subset X$ for any n let $\tilde{N}(n, \varepsilon)$ be the number of cubes $C_j \subset X_n$ having nonempty intersection with $\pi_n(S)$. Let $\tilde{N}(\varepsilon) = \sup_n \tilde{N}(n, \varepsilon) < \infty$ and now define $\tilde{H}_\varepsilon(S) = \log_2 \tilde{N}(\varepsilon)$. Note the similarity of this function to $H_\varepsilon(S)$ defined previously.

Finally, consider an analogue of the ε, δ-entropy for a probability measure μ. Let X_n be a finite-dimensional subspace of X as before subdivided into ε-cubes C_j and let $K_j = \pi_n^{-1}(C_j)$. Set $H_{\varepsilon,n}(\mu) = \sum_j \mu(K_j) \log_2 \mu(K_j)$ and observe that for any fixed ε this quantity is bounded uniformly in n. Define the ε-entropy of μ to be $H_\varepsilon(\mu) = \sup_n H_{\varepsilon,n}(\mu)$.

Problem 24. *Investigate properties of the entropy functions H_ε, $H_{\varepsilon,\delta}$, $\tilde{H}_\varepsilon$ and $\tilde{H}_{\varepsilon,\delta}$ in this section and explain relations between them.*

As an example, if $M = Q^d$ is the unit cube in $\mathbb{R}^d$ and $\mu = \mu^d$ is the d-dimensional Lebesgue measure then $H_\varepsilon(\mu) = \varepsilon^{-d} \varepsilon^d \log_2 \varepsilon^d = d \log_2 \varepsilon$. The result is the same if we view the cube as a subset $Q^d \subset X_d$ supporting μ^d and compute its ε-entropy in the whole space X. But what will happen if we consider Q^d as a subset of X_n for some $n > d$ and rotate it so that it is no longer a coordinate cube? In this case the sum becomes $\sum_j \mu^d(C_j) \log_2 \mu^d(C_j) \sim \log_2 \varepsilon^d + o(\log_2 \varepsilon)$ so that the principal asymptotic does not change under rotations of M. The same can be said about other deformations: the ε-entropy behaves like $d \log_2 \varepsilon \cdot \mu^d(X)$.

9.2. Entropy Decrease for the Euler Flow

Computer experiments show that the velocity field of an ideal 2D fluid behaves similarly for all initial conditions. The outcome is a small collection of moving vortices forming a hierarchical structure of "islands", "lakes", "satellites", "archipelagoes", etc. that are not mixing but instead preserve their individuality. This scenario looks quite strange from the physical viewpoint. It is obvious that the diversity of the initial conditions is much higher than that of the outcomes. The quantitative measure of the diversity in this case is the ε-entropy.

The natural (physical) phase space here is $V^0 = \{u \in L^2(M, \mathbb{R}^2) \mid \operatorname{div} u = 0,\ u \parallel \partial M\}$. Consider an initial velocity ensemble, that is, a probability measure

μ_0 in V^0. In fact, the initial velocity should be more regular than merely L^2 so that the Cauchy problem is correctly posed. A good example is the Yudovich space $Y = \{u \in V^0 \mid \operatorname{curl} u \in L^\infty\}$, but we can also start with an initial velocity in $V^s = T_e \mathscr{D}^s_\mu$ for some $s > 2$. The exact value of the Sobolev index s is not essential because in the long run the flow will approach some asymptotic regime possessing a "natural" though presently unknown regularity. This asymptotic flow will belong to the Yudovich space and may be more regular but it is not clear if this regularity can be captured by some nice function space (for example, it may be an element of a Frechet space and there may be no reasonable choice of a Banach space for this purpose).

Suppose that μ_0 is a compactly supported measure in V^0. We can define its ε-entropy $H_\varepsilon(\mu_0)$ (or at least its principal asymptotic as $\varepsilon \to 0$). Let μ_t be the measure at time t transported by the Euler flow. The phenomenon discussed here can be described by the inequality $\liminf_{t\to\infty} H_\varepsilon(\mu_t) \preceq H_\varepsilon(\mu_0)$ in the sense that for a limiting measure μ_∞ we have $H_\varepsilon(\mu_\infty) = \mathcal{O}(H_\varepsilon(\mu_0))$ as $t \to \infty$ and for some measures μ_0 the "big Oh" is replaced by the "little oh".

Such a result would make a physicist uneasy because it looks like a violation of the Liouville theorem. However, this is not a real violation since the true phase space of the fluid includes not only velocities of all the fluid particles but also their positions. Therefore, the elements of the space V represent merely "half" of the phase space coordinates with the other "half" represented by the flow map in $\mathscr{D}_\mu$.

Problem 25. *Explain the phenomenon of entropy decrease for solutions of the Euler equations* (1.1).

10. Hamiltonian Properties of the Euler Equation

The Riemannian geometric approach to hydrodynamics has a Hamiltonian reformulation, see for example [3,6]. Namely, consider again the group of smooth volume-preserving diffeomorphisms $\mathscr{D}_\mu(M)$ and denote its Lie algebra of divergence-free vector fields by $\mathfrak{g} = \mathrm{Vect}_\mu(M)$. The regular dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ can be naturally identified with the space of 1-forms modulo differentials of functions on M, that is with the space of cosets $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$. The inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ relies on the choice of metric on M and to a divergence-free vector field v it associates the coset $[v^\flat]$ of the 1-form $u = v^\flat$ related by means of the metric. Then the hydrodynamic Euler equation (1.1) can be written as an evolution of 1-forms

$$\partial_t u + L_v u = -df$$

for a certain time-dependent function f on M, or as an evolution of cosets of 1-forms:

$$\partial_t [u] = -L_v [u],$$

where $u = v^\flat \in \Omega^1(M)$ and $[u] \in \Omega^1(M)/d\Omega^0(M)$. This is a Hamiltonian equation with respect to the Lie-Poisson structure on $\mathfrak{g}^*$ and with the Hamiltonian function given by the fluid's kinetic energy $E(v) = \frac{1}{2}\langle Av, v\rangle = \frac{1}{2}\|v\|^2_{L^2(M)}$. This way, the equations of an ideal fluid dynamics in any dimension form a Hamiltonian system.

10.1. Nonintegrability

In the two-dimensional case, besides the kinetic energy, this system has infinitely many enstrophy invariants. These invariants are Casimir functionals: they do not depend on the metric on M, but specify a coadjoint orbit (that is a particular set of isovortical vector fields) on which the Euler evolution takes place but say nothing about the dynamics on the orbit itself.

Problem 26. *Prove non-integrability of the 2D Euler equation.*

It is worth to emphasize that, while there is a number of papers related to infinitely many conserved quantities or the Lax form of the Euler equation in 2D (see, for example, [47,87]), these features cannot be regarded as good indicators of integrability. The existence of a Lax pair is a property of all Euler–Poincaré equations, since the latter are Hamiltonian on the dual space to a Lie algebra with respect to the Lie–Poisson structure and hence are given by the coadjoint operator, "mimicking the commutator" in the Lie algebra. Typically, in order to prove algebraic-geometric integrability of a Lax equation one needs to present a Lax form nontrivially depending on a spectral parameter. However, no such form has been found for the 2D Euler equations.

On the other hand, one could try to prove non-integrability of this infinite-dimensional system with the help of the methods used to show non-integrability of finite-dimensional Hamiltonian systems. This could invoke, for instance, the methods related to nontrivial monodromies for periodic orbits [113,159], or Melnikov integrals for bifurcations of saddle separatrices, or one might be able to proceed by means of a "local" analysis in the vicinity of a steady solution. Since there are several (not necessarily equivalent) definitions of integrability in infinite dimensions, the above problem would be to show that the 2D Euler equations fail to satisfy at least one of these integrability definitions.

Note that the set of all Casimirs has been recently fully described in [58,61] for two-dimensional surfaces M without boundary for the groups of symplectic and Hamiltonian diffeomorphisms. The two-dimensional boundary case was settled in [79].

10.2. Finite-Dimensional Approximations

It is also worth pointing out that there are various approximations of the 2D Euler equations on the plane, the 2-torus or the 2-sphere by finite-dimensional Hamiltonian systems. Two of the best known approximations are (i) by $\mathrm{SU}(N)$-algebras, whose structure constants converge to those of the algebra $\mathrm{Vect}_\mu(\mathbb{T}^2)$ [156] and (ii) by a system of N point vortices on $\mathbb{R}^2$, as $N \to \infty$ [97]. In a recent paper [108] it was shown how both of these approximations can be unified within a quantization approach to the 2D hydrodynamics.

Dynamics of N point vortices on the plane has been studied since the time of Helmholtz and Kirchhoff [97] and is of particular interest; see for example [118]. For instance, on the plane or the sphere the corresponding finite-dimensional Hamiltonian systems turn out to be integrable for $N \leq 3$, while for the torus the system is integrable only for $N \leq 2$. This is related to the fact that the corresponding Kirchhoff equations for point vortices on $\mathbb{R}^2$ and S^2 are invariant with respect to the three-dimensional isometry groups $E(3)$ and $SO(3)$, while for $\mathbb{T}^2$ this group of isometries is isomorphic to $\mathbb{R}^2$ and hence two-dimensional. While integrability still holds for 4 vortices at zero total vorticity on the sphere S^2 and at zero total vorticity and momentum on the plane $\mathbb{R}^2$, for $N \geq 4$ point vortices of generic strengths the motion becomes non-integrable, see [80,81,158].

In [106,108,136] the authors observed the following intriguing phenomenon: on the two-dimensional sphere and torus the 2D Euler motion for a smooth and sufficiently general initial vorticity after some time (and with a very small numerical viscosity) leads to merging of smaller vortex formations of the same sign into larger blobs, cf. Figure 3. Surprisingly, in those numerical simulations it was recovered that this clustering continues until the blob dynamics, approximated by point vortex motion, "becomes integrable". Namely, such clustering leads to an integrable dynamics of 2 vortex blobs on the torus, 3 vortex blobs for general angular momentum on the sphere, and 4 vortex blobs on the sphere provided that the initial angular momentum was zero, thus exactly recovering integrable cases for $N = 2$, 3, and 4 point vortices on $\mathbb{T}^2$, $\mathbb{R}^2$ and S^2 discussed above see [106,108].

Problem 27. *Justify the phenomenon observed in* [106,108] *that integrable cases of point vortices seem to be attractors for 2D Euler flows with generic smooth initial vorticity in 2D (for small numerical viscosity). Design a model of dissipation in the 2D Euler equation which would produce such an integrable dynamics at large times.*

Historically, studies of point vortices include constructions of explicit solutions, conditions for the presence and absence of a collapse, description of relative equilibria, bifurcation of solutions for $N = 2$ and $N = 3$ point vortices, see [1,118]. A nice collection of integrable motions on the sphere and torus can be found in [107,118]. For manifolds with boundary there is a broader variety of motions. For instance, the cusp motion of a pair of point vortices on a half-plane is related to the golden ratio [78]. The motion of 3 point vortices on a half-plane is already non-integrable [150], as well as, apparently, the motion of two point vortices in the quadrant. The motion of point vortices on the half-plane and the quadrant for the lake equation is related to the motion of vortex rings, membranes and to the more general binormal equation [62,63,150].

Problem 28. *Study in detail the motion and bifurcations of a small number of vortices on various manifolds: half- and quarter-plane, hemi- and quarter-sphere, disk, torus, cylinder and half-cylinder, etc.*

Of particular interest is the study of point vortices on *non-orientable manifolds*, which started only recently, see for example [8,9,137,143] for a description of the motion and bifurcations of a small number of vortices on such non-orientable surfaces as the Möbius band, projective plane and the Klein bottle.

Problem 29. *Study the first non-integrable cases for a small number of point vortices on non-orientable surfaces. Describe the full set of Casimirs and finite-*

dimensional approximations for the group of area-preserving diffeomorphisms of non-orientable surfaces, analogous to the ones in the orientable cases.

11. Dynamical Properties of the Euler Equation: Wandering Solutions, Chaos, Non-mixing and KAM

Consider a finite-dimensional model of Euler hydrodynamics: the Euler–Poincaré equation on a finite-dimensional Lie group corresponding to a positive-definite energy form. This system is Hamiltonian on a coadjoint orbit and satisfies the condition of the Poincaré recurrence theorem. Indeed, by fixing the energy level one confines the dynamics to the compact set (even for a non-compact group) which is the intersection of the energy level and the orbit. The dynamics preserves the volume form on this intersection (see [6]) and hence yields to Poincaré's recurrence. Therefore every point of the orbit in the course of evolution returns arbitrarily close to its initial position after arbitrarily large time.

This is not the case for a general infinite-dimensional dynamical system and, in particular, for the Eulerian hydrodynamics. Nadirashvili showed that the Euler equation of a 2D fluid has wandering solutions: there is an initial condition of a fluid in a 2D annulus whose neighbourhood never returns sufficiently closely to the initial condition after a certain time [117]. A 3D analogue of that result is unknown.

Problem 30. *Prove that the 3D Euler equation has wandering solutions.*

The only type of results in this direction are the non-transitivity and non-mixing properties of the 3D Euler equations proved using the ideas of the KAM theory; see [72]. It turns out that the dynamical system defined by the hydrodynamical Euler equation on any closed Riemannian 3-manifold M is not mixing in the C^k topology (for $k > 4$ and non-integer) for any prescribed value of helicity and sufficiently large energy. Furthermore, this non-mixing property of the flow of the 3D Euler equation has a local nature: in any neighbourhood of a "typical" steady solution on $\mathbb{S}^3$ there is a generic set of initial conditions such that the corresponding Euler flows will never enter a vicinity (in the C^k norm for any non-integer $k > 10$) of the original steady flow; see [73].

Along the way one constructs a family of functionals on the space of divergence-free C^1 vector fields on M which are integrals of motion of the 3D Euler equation: given a vector field these functionals measure the part of the manifold M foliated by ergodic invariant tori of fixed isotopy types. The KAM theory allows one to establish certain continuity properties of these functionals in the C^k-topology and to get a lower bound on the C^k-distance between a divergence-free field (in particular, a steady solution) and a trajectory of the Euler flow. This way one obtains an obstruction for the mixing under the Euler flow of C^k-neighbourhoods of divergence-free vector fields on M. The local version of non-mixing is based on a similar KAM-type argument to generate knotted invariant tori from elliptic orbits in nondegenerate steady Euler flows.

Problem 31. *Relax the restrictions on the smoothness index k of the C^k spaces (related to application of the KAM) and prove non-transitivity and non-mixing properties of the 3D Euler in full generality.*

It turns out that the Euler equations on higher-dimensional Riemannian manifolds possess a kind of universal embedding property, somewhat similar to the theorems of Whitney and Nash on embeddings of manifolds as submanifolds in higher-dimensional Euclidean spaces.

Namely, it was shown in [139] that a certain large class of finite-dimensional quadratic dynamical systems in $\mathbb{R}^d$ can be realized as subsystems of the hydrodynamical Euler equation on the manifold $SO(d) \times \mathbb{T}^{d+1}$ with a certain metric depending on the original system. Subsequently, Torres de Lizaur [142] proved that such a realization is possible for any dynamical system of a finite-dimensional manifold or for its approximation. This way essentially any finite-dimensional dynamical system or its approximation to an arbitrary degree can be embedded as an invariant (tiny) subsystem in a higher-dimensional Euler equation for a certain metric.

The construction in [142], which has already found other applications, goes as follows: for a given finite-dimensional dynamical system one first embeds it via Whitney to a system on a submanifold inside a higher-dimensional torus, extends it hyperbolically to a dynamical system in the torus, and then writes it via the smooth vector field (represented by a Fourier series) in that torus. Then one truncates it to a Fourier polynomial (this is where the approximation with an arbitrary precision takes place). Finally, one observes that a Fourier polynomial vector field $v(x) = \sum_l c_l\, e^{ilx}\, \partial/\partial x$ rewritten in trigonometric coordinates $p_l := e^{ilx}$ becomes quadratic: $v(p) = \mathrm{i} \sum_{k,l} k\, c_l\, p_k p_l\, \partial/\partial p_k$. After that one employs Tao's embedding [139] of quadratic systems to the higher-dimensional Euler equations. The examples include such structurally stable systems exhibiting chaos as the ABC flows inside the higher Euler phase space.

One should note that the dynamics of the Euler equation outside of this tiny submanifold is not controlled and, in principle, could be rather regular. For instance, one might have dynamical systems with a very regular behaviour almost everywhere, but with some chaotic behaviour on a very tiny submanifold.

Problem 32. *Consider an integrable system on a compact $2n$-dimensional manifold, which has n first integrals in involution, functionally independent almost everywhere. How wild could such a system be on a singular submanifold, where the integrals become dependent? What are constraints dictated by the integrability? Could one observe a chaos on some tiny submanifold of an integrable system?*

12. Steady Euler Flows

Steady Euler flows in a domain $M \subset \mathbb{R}^d$ are defined by the equation $v \cdot \nabla v = -\nabla p$ along with the divergence-free restriction $\operatorname{div} v = 0$ and the condition $v || \partial M$ of tangency to the boundary. While it is easy to construct a steady 2D flow with compact support in $\mathbb{R}^2$ (take a radial stream function with compact support) it is a notoriously difficult task to perform this feat in 3D.

An explicit recent example of a smooth steady incompressible Euler flow in $\mathbb{R}^3$ with compact support was given in [51], see also a more general approach of [27]. In this type of solutions the pressure and Bernoulli function are dependent.

Arnold pointed out the remarkable topology of steady 3D fields: for an analytic steady field, not everywhere collinear with its curl, the flow domain is almost everywhere fibered into invariant tori and annuli, see [2,6]. If the steady field is everywhere collinear with its vorticity but the proportionality coefficient is a generic function, then the domain is still fibered in a similar way. If the steady field is Beltrami, that is, it is an eigenfield for the curl operator, $\operatorname{curl} v = \lambda\, v$, then its topology can be very intricate. Hence a paradox arises: a generic steady field has a very regular topology, while a sufficiently chaotic field must necessarily be an eigenfield for the curl operator.

Problem 33. *Explain this paradox: what is a typical steady field and what genericity notion is natural for steady fields?*

It is worth mentioning that the notion of a typical object in fluid dynamics might be quite different from the standard one. For instance, as discussed in Section 6.1 stream functions of steady 2D flows always have analytical levels even if they have only finite smoothness across the levels.

In [115] the authors considered slightly compressible 3D vector fields and their incompressible limit to explain the above paradox. The paper [42] also sheds more light on this problem: the levels of the Bernoulli function cannot be spheres.

It turned out that the topology of Beltrami fields can be arbitrarily complicated. In [40] it was established that for any finite link $L \subset \mathbb{R}^3$ and any nonzero real number λ one can deform the link L by a C^∞ diffeomorphism of $\mathbb{R}^3$, arbitrarily close to the identity in any C^m norm, such that the image of the link becomes a set of vortex lines of a Beltrami field v with the eigenvalue λ in $\mathbb{R}^3$, $\operatorname{curl} v = \lambda v$ in $\mathbb{R}^3$ and, moreover, v falls off at infinity as $|x|^{-1}$. In [41] a similar result was proved for the existence of a finite collection of toroidal knotted or linked stream/vortex tubes in $\mathbb{R}^3$. The boundaries of such tubes are structurally stable invariant tori for a Beltrami field with a quasiperiodic flow on them. Furthermore, there are Beltrami fields with invariant tori of arbitrary topology that enclose regions with any prescribed number of hyperbolic periodic orbits, see [39,43]. In this series of papers Encisco and Peralta-Salas with coauthors established other interesting topological properties of Beltrami fields on manifolds. We refer to these papers for various open problems related to this topic.

Another active direction of research is related to the interaction of topological and metric properties of divergence-free vector fields. By *topological* we mean those properties that are defined using the volume form only, for example average linking of the field trajectories, which is given by the field's helicity. For an exact divergence-free vector field u on a three-dimensional manifold M with a volume form μ its helicity (or asymptoptic Hopf) invariant is

$$H(u) = \int_M \omega \wedge d^{-1}\omega = \int_M (u, \operatorname{curl}^{-1} u)\, \mu,$$

where $\omega := \iota_u \mu$ (interior product) is the 2-form on M whose kernel field is u, see [109–112]. (Here the second expression for helicity, convenient in explicit computations, relies on a choice of a Riemannian metric on M, while the first one shows that helicity does not depend on that choice.) Actually, the helicity was shown to be the only topological invariant in a large class of functionals under the action of the group of volume-preserving diffeomorphisms [44]. (One should also mention that for velocity fields v that are solutions of the Euler equation, their helicity is defined as the helicity invariant of the corresponding vorticity field $u := \operatorname{curl} v$, hence in terms of the velocity the corresponding expression is $H(\operatorname{curl} v) = \int_M (\operatorname{curl} v, v)\, \mu$.)

While topological properties of the fields require only fixing a volume form, their *geometric* properties require a Riemannian metric to define them. An example of the latter is the L^2-energy of the field $E(u) := \frac{1}{2}\|u\|^2_{L^2(M)} = \frac{1}{2}\int_M (u, u)\, \mu$.

The inequality "helicity bounds energy",

$$E(u) \geq \text{const} \cdot H(u),$$

means that nontrivial average linking of the (magnetic) field's trajectories prevents its energy from complete dissipation via volume-preserving diffeomorphisms, see [4,110,111]. (This process is often called magnetic relaxation.) This inequality can be proven by noticing that the operator curl^{-1} on a compact manifold or domain M has bounded spectrum and by applying the Poincaré inequality; see [4,6]. Geometrically one can visualize this inequality for a vector field confined to a pair of simply linked solid tori. To minimize the energy of this linkage one needs to shorten trajectories of the field. On the other hand, due to the incompressibility property the shrinking of trajectories in one of the tori leads to stretching of the trajectories in the other. It is a particularly challenging problem to describe and analyze the process of magnetic relaxation to an equilibrium, possibly nonsmooth; see [14,110–112].

Note that the "helicity–energy" inequality is far from being sharp: helicity $H(u)$ could be zero, while the field u could possess nontrivially linked tori with opposite linkings or a higher order nontrivial linking. For knots there is a hierarchy by the Milnor and Massey numbers: once the preceding invariants are equal to zero, the invariants of the next level are well defined and distinguish the corresponding knots and links.

Problem 34. ([6]) *Find a sequence of higher helicity invariants for vector fields so that, given a field, if all the previous invariants are equal to zero for it, then the first nonzero invariant bounds the field's energy from below.*

We refer to the book [6] and its second edition for the discussion of open problems and a large bibliography on the subject, see also [30,46,83,112].

13. Singular Vorticities in the Euler Equation

The localized induction approximation (LIA) procedure applied to the 3D Euler equation in vorticity form gives the vortex filament (or binormal) equation:

$$\partial_t \gamma = \gamma' \times \gamma''$$

for the vorticity δ_γ supported on a curve $\gamma \subset \mathbb{R}^3$. Similarly, for the vorticity 2-form δ_P supported on a vortex membrane, a submanifold $P^{n-2} \subset \mathbb{R}^n$ of codimension 2, the LIA equation turns out to be

$$\partial_t q = J(\mathbf{MC}_P(q)),$$

where $\mathbf{MC}_P(q)$ is the vector of the mean curvature of the membrane P at the point $q \in P$ and J rotates by $\pi/2$ this vector in the normal plane to $N_q P$ to P, see [54,62,64,71,128].

The binormal equation is known to be equivalent to a 1D compressible fluid equation and to the nonlinear Schrödinger equation (NLS) in 1D via the Hasimoto transform. It also gives singular solutions of the Gross–Pitaevsky (or the 3D NLS) equation [62,63].

Problem 35. *Find a direct link from 3D NLS equation to 1D NLS equation as a reduction to singular solutions, rather than going through the LIA procedure. A similar question arises for the compressible Euler equations as a reduction from 3D to singular solutions supported on 1D submanifolds.*

Another interesting case of singular solutions is that of the vorticity supported on a hypersurface, called a vortex sheet. One can introduce a symplectic structure on vortex sheets similar to the Marsden-Weinstein symplectic structure on membranes; see [71] for the corresponding Hamiltonian formalism.

However, it is more natural to describe the motion of vortex sheets by means of a variational principle *à la* Arnold, albeit for a different object, as geodesics on an infinite-dimensional Lie groupoid, see Section 2.2. Using the corresponding vortex sheet groupoid instead of the group of volume-preserving diffeomorphisms in Arnold's framework, one obtains a geometric interpretation for discontinuous fluid flows, as well as their Hamiltonian description on the corresponding dual Lie algebroid, see [59]. It turns out that vortex sheet type solutions of the Euler equation are precisely the geodesics of an L^2-type right-invariant (source-wise) metric on the Lie groupoid of discontinuous volume-preserving diffeomorphisms. The geodesics on the groupoid turn out to be weak solutions of the Euler equation with vortex sheet initial data [59].

Geometric description of vortex sheets leads to an interesting non-local metric of hydrodynamical pedigree on shape spaces: it is an $H^{-1/2}$-metric on closed hypersurfaces bounding the same volume (or equivalently, on constant densities inside those hypersurfaces), see [59]. Such a metric is constructed with the help of the Neumann-to-Dirichlet operators and it is always nondegenerate since it is bounded below by the Kantorovich–Wasserstein distance. Geodesics with respect to this metric describe motions of potential fluid flows with vortex sheets and fluids with free dynamic boundary, cf. [59,86,91].

Problem 36. *Describe the differential geometry of shape spaces equipped with such $H^{-1/2}$-metrics obtained as metrics on dynamic boundaries or vortex sheets.*

14. The Compressible Euler Equation and the NLS Equation

There is a well-known relation between the NLS equation and (quantum) compressible fluids in any dimension. In 1927 Madelung [95] gave a hydrodynamic formulation of the Schrödinger equation. For a pair of real-valued functions ρ and θ on an n-dimensional manifold M (with $\rho > 0$) the Madelung transform is the mapping $\Phi : (\rho, \theta) \mapsto \psi$ given by $\psi = \sqrt{\rho e^{i\theta}} : M \to \mathbb{C}$.

The Madelung transform maps the system of equations for a barotropic-type fluid to the Schrödinger equation. Namely, let the function (or density) ρ and the potential velocity field $v = \nabla\theta$ satisfy the barotropic-type fluid equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + \nabla_v v + \nabla\Big(2V - 2f(\rho) - \dfrac{2\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big) = 0 \end{cases} \tag{14.1}$$

for some functions $V : M \to \mathbb{R}$ and $f : (0, \infty) \to \mathbb{R}$. Then, the (time-dependent) complex-valued wave function $\psi = \sqrt{\rho e^{i\theta}}$ given by the Madelung transform satisfies the nonlinear Schrödinger equation on M:

$$i\partial_t \psi = -\Delta\psi + V\psi - f(|\psi|^2)\psi. \tag{14.2}$$

The 1D Madelung transform, when interpreted in terms of curvature and torsion of the curve γ, reduces to the Hasimoto transform.

It turns out that the Madelung transform not only maps one Hamiltonian equation to another, but it also preserves the symplectic structures related to the equations [75,76,125]. More precisely, let $PC^\infty(M, \mathbb{C})$ denote the complex projective space of smooth complex-valued functions ψ on M: its elements are cosets $[\psi]$ of the unit L^2-sphere of wave functions. The Madelung transform induces a symplectomorphism between $PC^\infty(M, \mathbb{C}\backslash\{0\})$, the projective space of non-vanishing complex functions, and the cotangent bundle of probability densities $T^*\mathrm{Dens}(M)$ equipped with the canonical symplectic structure [75,76]. Furthermore, the Madelung transform is an isometry and a Kähler map between the spaces $T^*\mathrm{Dens}(M)$ equipped with the Sasaki-Fisher-Rao metric, which is the cotangent lift of the Fisher-Rao metric on the space of densities $\mathrm{Dens}(M)$, and $PC^\infty(M, \mathbb{C}\backslash\{0\})$ equipped with the Fubini-Study metric and the natural symplectic structures defined above, see [75–77]. Finally, in [49] it was shown that the Madelung transform can be regarded as the momentum mapping for the space of wave functions regarded as half-densities on M and acted upon by the semi-direct product group of diffeomorphisms and smooth functions.

Problem 37. *Extend the above results on symplectomorphism and the Kähler map to the wave functions with zeros on M. Explain the quantization condition controversy* [48,145,146] *in the language of the momentum map for the above semi-direct product group.*

The connection between equations of quantum mechanics and hydrodynamics might shed some light on the hydrodynamic quantum analogues studied, for example, in [21,28]: the motion of bouncing droplets in certain vibrating liquids manifests many properties of quantum mechanical particles.

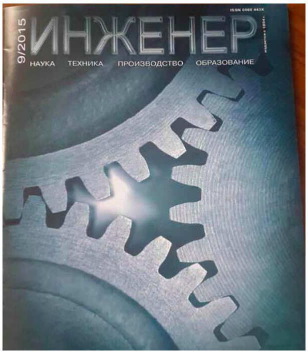

Fig. 4. The 2D direct and inverse cascades together. (Illustration from the cover of Russian magazine "Engineer", where the gears are Education, Science, and Enterprise. No comment)

Problem 38. *Explain the droplet-quantum particle correspondence by a combination of averaging and the Madelung transform.*

15. Mechanical Models of Direct and Inverse Cascades

15.1. Three Models of Energy Propagation

The inverse cascade phenomenon looks especially striking if we think of its mechanical model. Imagine a mechanism consisting of a countable number of wheels connected with gears, chains, springs and other joints which are assumed to be weightless and frictionless. Suppose that at the moment $t = 0$ some wheels are set into rotation (Fig. 4). In the course of motion the energy is redistributed among the wheels. The standard idea of statistical mechanics is that the energy tends to the uniform distribution between the wheels so that it will spread further and further. However, there are some other types of behaviour of such infinite mechanisms.

On the one hand, the energy can spread so fast that at least part of it escapes to infinity in finite time and the total energy in the system decreases. On the other hand, the energy may become "trapped", that is it does not spread at all, and, moreover, it is concentrated in the first few wheels and its distribution does not depend on the initial energy profile, provided the energy is initially contained in any *finite number* of wheels.

There is also a softer regime, where the energy does not spread to all the wheels but its distribution depends on the initial profile (the Fermi–Pasta–Ulam regime). This might seem implausible but the fluid presents us with examples of such behaviour. In fact, consider a fluid flow $u(x, t)$ on the torus $\mathbb{T}^n$, $n = 2$ or $n = 3$. We can regard the Fourier coefficients $u_k(t)$ as the analogue of the kth wheel angular velocity. The time evolution of u_k is described by a certain quadratic system of ODEs which can be regarded as a description of connections between the wheels (we could even design a realistically looking "mechanism" made of weightless and frictionless parts realizing these connections). Then, for $n = 3$, we can expect a breakdown of a regular solution of the Euler equation and a transition to a turbulent motion with decreasing energy (which will be discussed below in more detail). If $n = 2$, we observe the inverse energy cascade with the formation of a few large vortices so that the energy is concentrated in a few lower harmonics, while the energy spectrum is decreasing over frequencies.

Thus, we have at least three types of behaviour of an "infinite mechanism": the tendency to the energy equidistribution, the inverse cascade (the energy tends to concentrate in the first few modes) and the direct cascade (the energy escapes to infinity in finite time). It is important to find out which properties of the mechanisms are responsible for so different behaviour. As a first step in this direction we can try to design some models, that is some simpler devices displaying similar behaviour. The original mechanism, such as a fluid in the Fourier representation, is too complicated to yield to the statistical theory with a lot of unrelated features. It would be interesting to find simpler (though infinite) mechanical systems which display the same statistical behaviour and which can be regarded as models of the fluid in this respect.

15.2. A Model of Energy Equidistribution

A classical example of a mechanical system with a well established tendency to the energy equidistribution is the gas of a large (but finite) number of solid balls moving inside a bounded domain. It appears possible to modify this system to obtain a system with the direct energy cascade. Consider a system of a countable number of balls B_j moving inside a bounded domain ("box") D. Suppose these balls fall into a countable number of "families" $F_1, F_2, \dots$ such that the family F_i includes n_i equal balls of radius r_i and mass m_i. For simplicity suppose that the balls of a family F_i "feel" only balls of the neighbouring families F_{i-1} and F_{i+1} and can penetrate through the balls from other families without any resistance. This means that the Hamiltonian of the system has the form

$$H(p, q) = \sum_i \sum_{F_i} \frac{p_j^2}{2m_i} + \sum_i \sum_{\substack{B_j \in F_i \\ B_k \in F_{i+1}}} U_{i,i+1}(|q_j - q_k|).$$

Suppose that the total mass is $M = \sum_i n_i m_i < \infty$ and the sequences of masses m_i and radii r_i are decreasing fast enough. Then each ball $B_j \in F_i$ is moving through a "gas" formed by the balls $B_k \in F_{i+1}$. The "gas" particles $B_k \in F_{i+1}$ are feeling the resistance of the "gas" formed by the balls $B_k \in F_{i+2}$, etc. The question is whether one can define the sequences n_i, m_i, r_i in such a way that the direct energy cascade in the direction of growing i would occur, and the energy would dissipate from the system? If the answer to the first question is affirmative, could one point at the details of an actual 3D fluid which play the role of the balls of different families?

Now, the next natural question concerns what happens with the energy that has escaped from the system? One can introduce the "limit absorption principle" for our system. Namely, one can introduce a friction for the balls of the family F_N which absorbs the energy and then let N go to infinity. But this solution is not

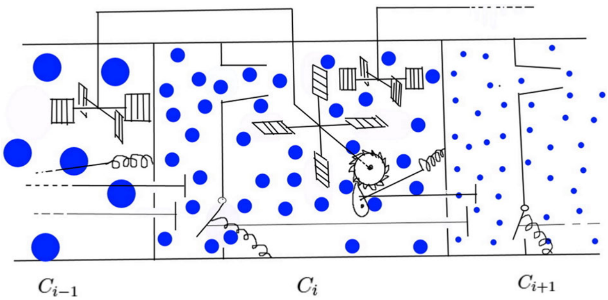

Fig. 5. A mechanical apparatus to mimic the inverse cascade

completely legitimate as here the energy sink is included from the beginning and it is only moved farther and farther away. So, this explanation of the energy escape is circular.

Problem 39. *Is there a more natural way to introduce energy dissipation without explicit introduction of a friction mechanism?*

It would be intriguing to find a relation of the above model to the mechanism of the energy dissipation in the weak solutions of the Euler equations whose rate is defined by the Duchon–Robert formula [35]. This would follow the footsteps of Maxwell's molecular vortex model for electromagnetic waves [99].

15.3. A Model of the Inverse Cascade

A mechanical model of the inverse cascade requires a more sophisticated design. First of all, the system will consist of a countable number of chambers $C_1, C_2, \ldots$ with rigid walls, see Fig. 5. Each chamber C_i is separated by a vertical wall into two parts, call them the left room and the right room. The wall contains two doors, the upper and the lower door. Each chamber C_i contains n_i equal elastic balls B^i_j of radius r_i and mass m_i moving inside C_i and interacting with the walls, with the details of the mechanism and with other balls according to the laws of elastic collision. There is a shutter at the lower door equipped with a spring which, when open, permits the balls to enter from the right room into the left one and, when shut, prevents the balls from going back. The shutter is connected to a damper which is interacting with the balls in the next chamber C_{i+1}. The upper door has no shutter and the balls can move freely through this door in both directions.

The balls enter the left room from the right one through the lower and the upper doors and exit only through the upper door, provided the shutter at the lower door works properly (this is, of course, a true Maxwell's demon). The last condition can be satisfied provided that the shutter, being a part of the system, is permanently cooled, that is its energy being transferred to the balls in the chamber C_{i+1} (see the analysis by FEYNMAN [45]). To this end, the balls in C_{i+1} should be much smaller and much more numerous than in C_i, that is $m_{i+1} \ll m_i$, $r_{i+1} \ll r_i$ and $n_{i+1} \gg n_i$. Then the balls in C_{i+1} form a "gas" which is effectively viscous and absorbs the energy of the shutter. If "Maxwell's demon" works properly then the balls in C_i are entrained into the circular motion: on average, they enter from the right room into the left room through the lower door and leave from the right room mostly through the upper door. Thus, there appears a stream of balls from the upper door. Let us put a "turbine" which is rotated by this (possibly weak) stream. In order to ensure its rotation we put a ratchet-and-pawl which would prevent the reverse rotation of the turbine. To make this device work we should attach it to the second damper using the balls in C_{i+1} to dissipate the energy. Let this turbine drive through a system of connecting parts a "stirrer" which transfers energy from the turbine to the balls in the chamber C_{i-1}; these balls should be much larger than the ones in C_i, that is $m_{i-1} \gg m_i$, $r_{i-1} \gg r_i$, and $n_{i-1} \ll n_i$.

The system should work as follows. The balls in the chamber C_i are, on the average, taking part in the circular motion entering from the right room into the left one through the lower door and from the left room into the right one through the upper door (equipped with a nozzle), while the shutter at the lower door is damped by the damper. The latter is braked by the "gas" of the balls in C_{i+1} which are much smaller than the balls in C_i. The energy of the stream of the balls in C_i is transferred through the turbine to the much larger balls in the chamber C_{i-1}. The ratchet-and-pawl pair is cooled by a similar damper (see the above analysis of this pair by Feynman). As a result, on the average the energy is transferred to the balls in the first few chambers.

This mechanism looks like a sort of *perpetuum mobile*. However, it is neither a perpetuum mobile of the first nor of the second kind. In fact, it is not a perpetuum mobile at all but, rather, it is a chain of heat engines: the balls inside each chamber C_i are the "working body" of the engine. The balls in the next chamber C_{i+1} play the role of a cooler (they are cooling the shutter, the "Maxwell's demon" and the ratchet-and-pawl), while the balls in the previous chamber C_{i-1} are playing the role of the load. The engines are quite primitive and their efficiency is very low; however, there is an infinite number of them so that their overall efficiency is 100%. Hence, here is the problem.

Problem 40. *Is there a similar mechanism which describes a cascade in real hydrodynamics? Is it possible to define the parameters n_i, m_i, and r_i in such a way that the above apparatus works as intended?*

It would be interesting and important to find a link between this device and a more convenient heat engine; in particular, to find some analogues of Maxwell's demon or, perhaps even more importantly, to show that a similar mechanism works in a 2D ideal fluid, thus ensuring inverse energy cascade in it.

Acknowledgements. We are indebted to the organizers of the IAS Symplectic Dynamics Program, where this paper was conceived, and in particular to Helmut Hofer for his constant encouragement. We are also grateful to Theodore Drivas and to the anonymous referee for many useful suggestions. B.K. was partially supported by a Simons Fellowship and an NSERC Discovery Grant. G.M. gratefully acknowledges support from the SCGP, Stony Brook University, where part of the work on this paper was done. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

1. Aref, H., Rott, N., Thomann, H.: Gröbli's solution of the three-vortex problem. *Annu. Rev. Fluid Mech.* **24**, 1–21, 1992
2. Arnold, V.: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier* **16**, 316–361, 1966
3. Arnold, V.I.: The Hamiltonian nature of the Euler equation in the dynamics of rigid body and of an ideal fluid. *Uspekhi Matem. Nauk* **24**(3), 225–226, 1969
4. Arnold, V.I.: The asymptotic Hopf invariant and its applications. *Proceedings of Summer School in Diff. Equations at Dilizhan, 1973* (1974), Erevan; English transl.: *Sel. Math. Sov.* **5**, 327–345, 1986
5. Arnold, V.: Mathematical Methods of Classical Mechanics. Springer, New York (1989)
6. Arnold, V., Khesin, B.: *Topological Methods in Hydrodynamics, Applied Mathematical Sciences*, vol. 125. Springer, New York, pp. xv+374, 1998; second extended edition: Springer-Nature Switzerland 2021
7. Bahouri, H., Chemin, J., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Springer, New York (2011)
8. Balabanova, N.A.: A Hamiltonian approach for point vortices on non-orientable surfaces II: the Klein bottle, 2022. Preprint arXiv:2202.06175
9. Balabanova, N.A., Montaldi, J.: Hamiltonian approach for point vortices on non-orientable surfaces I: the Mobius band, 2022. Preprint arXiv:2202.06160
10. Bardos, C., Titi, E.: Euler equations for an ideal incompressible fluid. *Uspekhi Mat. Nauk* **62**, 5–46, 2007
11. Bardos, C., Titi, E.: Loss of smoothness and energy conserving rough weak solutions for the 3D Euler equations. *Discrete Contin. Dyn. Syst.* **3**, 185–197, 2010
12. Beale, T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3D Euler equations. *Commun. Math. Phys.* **94**, 61–66, 1984
13. Bedrossian, J., Masmoudi, N.: Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publications mathématiques de l'IHÉS* **122**, 195–300, 2015
14. Beekie, R., Friedlander, S., Vicol, V.: On Moffatt's magnetic relaxation equations. *Commun. Math. Phys.* **39**(3), 1311–1339, 2022
15. Benn, J.: The L^2 geometry of the symplectomorphism group. Ph.D. thesis, the University of Notre Dame, 2015
16. Bogaevski, I.A.: Perestroikas of shock waves in optimal control. *J. Math. Sci.* **126**(4), 1229–1242, 2005
17. Bourgain, J., Li, D.: Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces. *Invent. Math.* **201**, 97–157, 2015
18. Bourgain, J., Li, D.: Strong ill-posedness of the incompressible Euler equation in integer C^m spaces. *Geom. Funct. Anal.* **25**, 1–86, 2015
19. Brenier, Y.: Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Commun. Pure Appl. Math.* **52**(4), 411–452, 1999

20. Brenier, Y., Gangbo, W., Savaré, G., Westdickenberg, M.: Sticky particle dynamics with interactions. *J. Math. Pures Appl.* **99**(9), 577–617, 2013
21. Bush, J.W.: Quantum mechanics writ large. *Proc. Natl. Acad. Sci. USA* **107**, 17455–17456, 2010
22. Chae, D.: Local existence and blowup criterion for the Euler equations in the Besov spaces. *Asympt. Anal.* **38**, 339–358, 2004
23. Chemin, J.: Perfect Incompressible Fluids. Oxford University Press, New York (1998)
24. Choffrut, A., Sverak, V.: Local structure of the set of steady-state solutions to the 2D incompressible Euler equations. *Geom. Funct. Anal.* **22**, 136–201, 2012
25. Constantin, P.: On the Euler equations of incompressible fluids. *Bull. Am. Math. Soc. (N.S.)* **44**, 603–621, 2007
26. Constantin, P., Vicol, V., Wu, J.: Analyticity of Lagrangian trajectories for well-posed inviscid incompressible fluid models. *Adv. Math.* **285**, 352–393, 2015
27. Constantin, P., La, J., Vicol, V.: Remarks on a paper by Gavrilov: Grad-Shafranov equations, steady solutions of the three dimensional incompressible Euler equations with compactly supported velocities, and applications. *Geom. Funct. Anal.* **29**, 1773–1793, 2019
28. Couder, Y., Protiere, S., Fort, E., Boudaoud, A.: Dynamical phenomena: walking and orbiting droplets. *Nature* **437**, 208, 2005
29. Danielski, A.: Analytical structure of stationary flows of an ideal incompressible fluid. Masters thesis, Concordia University, 2017
30. DeTurck, D., Gluck, H.: Linking, twisting, writhing and helicity on the 3-sphere and in hyperbolic 3-space. *J. Differ. Geom.* **94**, 87–128, 2013
31. Dieudonne, J.: Foundations of Modern Analysis. Academic Press, New York (1969)
32. Dolce, M., Drivas, T.: On maximally mixed equilibria of two-dimensional perfect fluids. *Arch. Ration. Mech. Anal. (ARMA)* **246**, 735–770, 2022
33. Drivas, T., Misiołek, G., Shi, B., Yoneda, T.: Conjugate and cut points in ideal fluid motion. *Ann. Math. Qué.* **46**, 207–225, 2022
34. Drivas, T., Elgindi, T.: Singularity formation in the incompressible Euler equation in finite and infinite time, 2022. Preprint arXiv:2203.17221. To appear in EMS Surveys in Mathematical Sciences
35. Duchon, J., Robert, R.: Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations. *Nonlinearity* **13**(1), 249–255, 2000
36. Ebin, D., Marsden, J.: Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math.* **92**, 102–163, 1970
37. Ebin, D., Misiołek, G., Preston, S.: Singularities of the exponential map on the volume-preserving diffeomorphism group. *Geom. Funct. Anal.* **16**, 850–868, 2006
38. Elgindi, T., Masmoudi, N.: L^∞ ill-posedness for a class of equations arising in hydrodynamics. *Arch. Ration. Mech. Anal.* **235**, 1979–2025, 2020
39. Enciso, A., Luque, A., Peralta-Salas, D.: Beltrami fields with hyperbolic periodic orbits enclosed by knotted invariant tori. *Adv. Math.* **373**, 107328, 2020
40. Enciso, A., Peralta-Salas, D.: Knots and links in steady solutions of the Euler equation. *Ann. Math.* **175**, 345–367, 2012
41. Enciso, A., Peralta-Salas, D.: Existence of knotted vortex tubes in steady Euler flows. *Acta Math.* **214**, 61–134, 2015
42. Enciso, A., Peralta-Salas, D.: Beltrami fields with a nonconstant proportionality factor are rare. *Arch. Ration. Mech. Anal.* **220**, 243–260, 2016
43. Enciso, A., Peralta-Salas, D., Torres de Lizaur, F.: Knotted structures in high-energy Beltrami fields on the torus and the sphere. *Ann. Sci. Éc. Norm. Sup.* **50**, 995–1016, 2017
44. Enciso, A., Peralta-Salas, D., Torres de Lizaur, F.: Helicity is the only integral invariant of volume-preserving transformations. *Proc. Natl. Acad. Sci. USA* **113**, 2035–2040, 2016
45. Feynman, R., Leighton, R.B., Sands, M.: *The Feynman Lectures on Physics*, vol. 1, pp. 46.1–46.9. Addison-Wesley (1977)

46. Freedman, M.H., He, Z.-X.: Divergence-free fields: energy and asymptotic crossing number. *Ann. Math.* **134**(1), 189–229, 1991
47. Friedlander, S., Vishik, M.: Lax pair formulation for the Euler equation. *Phys. Lett. A* **148**(6–7), 313–319, 1990
48. Fritsche, L., Haugk, M.: Stochastic foundation of quantum mechanics and the origin of particle spin, 2009. Preprint arXiv:0912.3442
49. Fusca, D.: The Madelung transform as a momentum map. *J. Geom. Mech.* **9**, 157–165, 2017
50. Gamblin, P.: Système d'Euler incompressible et régularité microlocale analytique. *Annales de l'Institut Fourier* **44**(5), 1449–1475, 1994
51. Gavrilov, A.V.: A steady Euler flow with compact support. *Geom. Funct. Anal.* **29**, 190–197, 2019
52. Gie, G.-M., Kelliher, J.P., Mazzucato, A.L.: The 3D Euler equations with inflow, outflow and vorticity boundary conditions, 2022. arXiv:2203.15180
53. Gunther, N.: On the motion of a fluid contained in a given moving vessel (Russian). *Izvestia Akad. Nauk USSR Ser. Fiz. Mat.* **20**, 1926, **21**, 1927, **22**, 1928
54. Haller, S., Vizman, C.: Nonlinear Grassmannians as coadjoint orbits, preprint arXiv:math.DG/0305089, 13pp, extended version of *Math. Ann.* **329**(4), 771–785, 2003
55. Hernandez, M.: Mechanisms of Lagrangian analyticity in fluids. *Arch. Ration. Mech. Anal. (ARMA)* **233**, 513–598, 2019
56. Hille, E., Phillips, R.: Functional Analysis and Semigroups. AMS Colloquium Publ, Providence (1957)
57. Inci, H., Kappeler, T., Topalov, P.: *On the Regularity of the Composition of Diffeomorphisms*. Memoirs of the American Mathematical Society 2013
58. Izosimov, A., Khesin, B.: Classification of Casimirs in 2D hydrodynamics. *Moscow Math J.* **17**(4), 699–716, 2017
59. Izosimov, A., Khesin, B.: Vortex sheets and diffeomorphism groupoids. *Adv. Math.* **338**, 447–501, 2018
60. Izosimov, A., Khesin, B.: Geometry of generalized fluid flows, 2022. Preprint arXiv:2206.01434
61. Izosimov, A., Khesin, B., Mousavi, M.: Coadjoint orbits of symplectic diffeomorphisms of surfaces and ideal hydrodynamics. *Annales de l'Institut Fourier* **66**(6), 2385–2433, 2016
62. Jerrard, R.L.: Vortex filament dynamics for Gross-Pitaevsky type equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **1**(4), 733–768, 2002
63. Jerrard, R.L., Smets, D.: Vortex dynamics for the two-dimensional non-homogeneous Gross-Pitaevskii equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **XIV**(5), 1–38, 2015
64. Jerrard, R.L., Smets, D.: On the motion of a curve by its binormal curvature. *J. Eur. Math. Soc.* **017**(6), 1487–1515, 2015
65. Jiménez, V.M., De León, M., Epstein, M.: Lie groupoids and algebroids applied to the study of uniformity and homogeneity of material bodies. *J. Geom. Mech.* **11**(3), 301–324, 2019
66. Kato, T.: On the classical solutions of the two dimensional non stationary Euler equation. *Arch. Ration. Mech. Anal.* **25**, 188–200, 1967
67. Kato, T., Ponce, G.: On nonstationary flows of viscous and ideal fluids in $L_s^p(\mathbb{R}^2)$. *Duke Math. J.* **55**, 487–499, 1987
68. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. *Commun. Pure Appl. Math.* **41**, 891–907, 1988
69. Khanin, K., Sobolevski, A.: Particle dynamics inside shocks in Hamilton-Jacobi equations. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys.* **368**(1916), 1579–1593, 2010
70. Khanin, K., Sobolevski, A.: On dynamics of Lagrangian trajectories for Hamilton–Jacobi equations. *Arch. Ration. Mech. Anal.* **219**, 861–885, 2016
71. Khesin, B.: Symplectic structures and dynamics on vortex membranes. *Moscow Math. J.* **12**(2), 413–434, 2012

72. Khesin, B., Kuksin, S., Peralta-Salas, D.: KAM theory and the 3D Euler equation. *Adv. Math.* **267**, 498–522, 2014
73. Khesin, B., Kuksin, S., Peralta-Salas, D.: Global, local and dense non-mixing of the 3D Euler equation. *Arch. Ration. Mech. Anal. (ARMA)* **238**, 1087–1112, 2020
74. Khesin, B., Misiołek, G.: Shock waves for the Burgers equation and curvatures of diffeomorphism groups. *Proc. Steklov Math. Inst.* **259**, 73–81, 2007
75. Khesin, B., Misiołek, G., Modin, K.: Geometric hydrodynamics via Madelung transform. *Proc. Nat. Acad. Sci.* **115**(24), 6165–6170, 2018
76. Khesin, B., Misiołek, G., Modin, K.: Geometry of the Madelung transform. *Arch. Ration. Mech. Anal.* **234**, 549–573, 2019
77. Khesin, B., Misiołek, G., Modin, K.: Geometric hydrodynamics and infinite-dimensional Newton's equations. *Bull. Am. Math. Soc.* **58**, 377–442, 2021
78. Khesin, B., Wang, H.: The golden ratio and hydrodynamics. *The Math. Intell. (TMIN)* **44**(1), 22–27, 2022
79. Kirillov, I.: Classification of coadjoint orbits for symplectomorphism groups of surfaces. *Int. Math. Res. Notices IMRN*, 2022. https://doi.org/10.1093/imrn/rnac041
80. Kozlov, V.V., Treschev, D.V.: Nonintegrability of the general problem of rotation of a dynamically symmetric heavy rigid body with a fixed point I. *Vestn. Mosk. Univ. Ser. 1. Matem. Mekh.* **6**, 73–81, 1985
81. Kozlov, V.V., Treschev, D.V.: Nonintegrability of the general problem of rotation of a dynamically symmetric heavy rigid body with a fixed point II. *Vestn. Mosk. Univ. Ser. 1. Matem. Mekh.* **1**, 39–44, 1986
82. Kupferman, R., Olami, E., Segev, R.: Continuum dynamics on manifolds: application to elasticity of residually-stressed bodies. *J. Elast.* **128**, 61–84, 2017
83. Laurence, P., Stredulinsky, E.: Asymptotic Massey products, induced currents and Borromeantorus links. *J. Math. Phys.* **41**(5), 3170–3191, 2000
84. Lebeau, G.: Régularité du probléme de Kelvin-Helmholtz pour l'équation d'Euler 2d. *ESAIM Control Optim. Calculus Var.* **8**, 801–825, 2002
85. Lemarie-Rieusset, P.: Espaces limites pour le contre-exemple de Bardos et Titi sur le shear flow. Personal communication of C. Bardos.
86. Lewis, D., Marsden, J., Montgomery, R., Ratiu, T.: The Hamiltonian structure for dynamic free boundary problems. *Physica D* **18**, 391–404, 1986
87. Li, Y.C., Yurov, A.V.: Lax pairs and Darboux transformations for Euler equations. *Studies in Applied Math*. 2003. https://doi.org/10.1111/1467-9590.t01-1-00229, arXiv:math/0101214
88. Lichtenfelz, L., Tauchi, T., Yoneda,T.: Existence of a conjugate point in the incompressible Euler flow on the three-dimensional ellipsoid,2022. Preprint arXiv:2204.00732
89. Lichtenstein, L.: Über einige Hilfssätze der Potentialtheorie I. *Math. Zeit.* **23**, 72–78, 1925
90. Lichtenstein, L.: Uber einige Existenzprobleme der Hydrodynamik unzusamendruckbarer, reibunglosiger Flussigkeiten und die Helmholtzischen Wirbelsatze. *Math. Zeit.* **23**, 1925, **26**, 1927, **28**, 1928, **32**, 1930
91. Loeschcke, C.: On the relaxation of a variational principle for the motion of a vortex sheet in perfect fluid. Ph.D. thesis, Rheinische Friedrich-Wilhelms-Universitaet Bonn, 2012
92. Lukatskii, A.: Homogeneous vector bundles and the diffeomorphism groups of compact homogeneous spaces (Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **39**, 1274–1283, 1437, 1975
93. Lukatskii, A.: Finite generation of groups of diffeomorphisms. *Uspekhi Mat. Nauk* **199**, 219–220, 1978
94. Lukatskii, A.: Finite generation of groups of diffeomorphisms. *Russ. Math. Surv.* **33**, 207–208, 1978
95. Madelung, E.: Quantentheorie in hydrodynamischer form. *Z. Phys.* **40**, 322–326, 1927
96. Majda, A., Bertozzi, A.: Vorticity and Incompressible Flow. Cambridge University Press, Cambridge (2002)

97. Marchioro, C., Pulvirenti, M.: *Mathematical Theory of Incompressible Nonviscous Fluids, Appl. Math. Sci.*, vol. 96. Springer-Verlag, 1994
98. Marsden, J.E., Hughes, T.J.R.: *Mathematical Foundations of Elasticity*. Dover Publ. 1994
99. Maxwell, J.C.: *On physical lines of force. Part 1. The theory of molecular vortices applied to magnetic phenomena. Philos. Mag.* **XXI**, 161–175, 1861; see Wikipedia article on “History of Maxwell’s equations"
100. Milnor, J.W.: Curvatures of left invariant metrics on Lie groups. *Adv. Math.* **21**(3), 293–329, 1976
101. Misiołek, G.: Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms. *Indiana Univ. Math. J.* **42**, 215–235, 1993
102. Misiołek, G.: Conjugate points in $\mathcal{D}_\mu(\mathbb{T}^2)$. *Proc. Am. Math. Soc.* **124**, 977–982, 1996
103. Misiołek, G., Preston, S.: Fredholm properties of Riemannian exponential maps on diffeomorphism groups. *Invent. Math.* **179**, 191–227, 2010
104. Misiołek, G., Yoneda, T.: Ill-posedness examples for the quasi-geostrophic and the Euler equations. *Contemp. Math.* **584**, 251–258, 2012
105. Misiołek, G., Yoneda, T.: Continuity of the solution map of the Euler equations in Hölder spaces and weak norm inflation in Besov spaces. *Trans. Am. Math. Soc.* **370**, 4709–4730, 2018
106. Modin, K., Viviani, M.: A Casimir preserving scheme for long-time simulation of spherical ideal hydrodynamics. *J. Fluid Mech.* **884**, A22, 2020
107. Modin, K., Viviani, M.: Integrability of point-vortex dynamics via symplectic reduction: a survey. *Arnold Math. J.* **7**(3), 357–385, 2021
108. Modin, K., Viviani, M.: Canonical scale separation in two-dimensional incompressible hydrodynamics. *J. Fluid Mech.* **943**, A36, 2022
109. Moffatt, H.K.: The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117–129, 1969
110. Moffatt, H.K.: Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology I. *J. Fluid Mech.* **159**, 1985
111. Moffatt, H.K.: Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology II. *J. Fluid Mech.* **166**, 359–378, 1986
112. Moffatt, H.K.: Some topological aspects of fluid dynamics. *J. Fluid Mech.* **914**, Paper No. P1, 2021
113. Morales-Ruiz, J.J., Ramis, J.P.: Galoisian obstructions to integrability of hamiltonian systems I, II. *Methods Appl. Anal.* **8**(1), 33–95, 97–111, 2001
114. Morgulis, A., Shnirelman, A., Yudovich, V.: Loss of smoothness and inherent istability of 2D inviscid fluid flows. *Commun. PDE* **33**, 943–968, 2008
115. Morgulis, A., Yudovich, V.I., Zaslavsky, G.M.: Compressible helical flows. *Commun. Pure Appl. Math.* **48**(5), 571–582, 1995
116. Nadirashvili, N.S.: Wandering solutions of the Euler 2D equation. *Funct. Anal. Appl.* **25**(3), 220–221, 1991
117. Nadirashvili, N.: On stationary solutions of two-dimensional Euler equations. *Arch. Ration. Mech. Anal.* **209**, 729–745, 2013
118. Newton, P.K.: The N-vortex Problem: Analytic Techniques. Springer, New York (2001)
119. Otto, F.: The geometry of dissipative evolution equations: the porous medium equation. *Commun. PDE* **26**(1–2), 101–174, 2001
120. Pak, H.C., Park, Y.J.: Existence of solution for the Euler equations in a critical Besov space. *Commun. PDE* **29**, 1149–1166, 2004
121. Petrovsky, I.G.: *Ordinary Differential Equations*. Prentice-Hill, 1966
122. Preston, S.: For ideal fluids, Eulerian and Lagrangian instabilities are equivalent. *Geom. Funct. Anal.* **14**(5), 1044–1062, 2004
123. Preston, S.: On the volumorphism group, the first conjugate point is always the hardest. *Commun. Math. Phys.* **267**, 493–513, 2006

124. Preston, S.: The WKB method for conjugate points in the volumorphism group. *Indiana Univ. Math. J.* **57**, 3303–3327, 2008
125. von Renesse, M.-K.: An optimal transport view of Schrödinger's equation. *Can. Math. Bull.* **55**, 858–869, 2012
126. Rykov, Y.G., Sinai, Y.G., Weinan, E.: Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. *Commun. Math. Phys.* **177**(2), 349–380, 1996
127. Serfati, P.: Structures holomorphes a faible regularite spatiale en mechanique de fluides. *J. Math. Pures Appl.* **74**, 95–104, 1995
128. Shashikanth, B.N.: Vortex dynamics in $\mathbb{R}^4$. *J. Math. Phys.* **53**(1), 013103, 2012
129. Shnirelman, A.: The degree of a quasiruled mapping, and the nonlinear Hilbert problem. *Mat. Sb. (N.S.)* **89**, 366–389, 1972
130. Shnirelman, A.: The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid. *Mat. Sb. (N.S.)* **128**, 82–109, 1985
131. Shnirelman, A.: Generalized fluid flows, their approximation and applications. *Geom. Funct. Anal.* **4**, 586–620, 1994
132. Shnirelman, A.: Evolution of singularities, generalized Liapunov function and generalized integral for an ideal incompressible fluid. *Am. J. Math.* **119**, 579–608, 1997
133. Shnirelman, A.: Weak solutions with decreasing energy of incompressible Euler equations. *Commun. Math. Phys.* **210**, 541–603, 2000
134. Shnirelman, A.: Microglobal analysis of the Euler equations. *J. Math. Fluid Mech.* **7**, S387–S396, 2005
135. Shnirelman, A.: On the analyticity of particle trajectories in the ideal incompressible fluid. *Glob. Stoch. Anal.* **2**(2), 149–157, 2015 arXiv:1205.5837
136. Shnirelman, A.: On the long time behavior of fluid flows. *Procedia IUTAM* **7**, 151–160, 2013
137. Simó, E.: The *N-vortex problem on the projective plane*. Undergraduate Thesis, Polyt. Univ. Catalonia and Univ. Toronto, 2022
138. Sueur, F.: 2D incompressible Euler system in presence of sources and sinks. CAM Colloquium, PennState, March 15, 2021
139. Tao, T.: On the universality of the incompressible Euler equation on compact manifolds. *Discrete Contin. Dyn. Syst.* **38**(3), 1553–1565, 2018
140. Tauchi, T., Yoneda, T.: Existence of a conjugate point in the incompressible Euler flow on an ellipsoid. *J. Math. Soc. Jpn.* **74**(2), 629–653, 2022 arXiv:1907.08365
141. Tauchi, T., Yoneda, T.: Arnold stability and Misiołek curvature. *Monatshefte für Mathematik* **199**(2), 1–19, 2022 arXiv:2110.04680
142. Torres de Lizaur, F.: Chaos in the incompressible Euler equation on manifolds of high dimension. *Invent. Math.* **228**(4), 1–29, 2022 arXiv:2104.00647
143. Vanneste, J.: Vortex dynamics on a Möbius strip. *J. Fluid Mech.* **923**, A12, 2021 arXiv:2102.07697
144. Vishik, M.: Hydrodynamics in Besov spaces. *Arch. Ration. Mech. Anal. (ARMA)* **145**, 197–214, 1998
145. Wallstrom, T.C.: Inequivalence between the Schrödinger equation and the Madelung hydrodynamic equations. *Phys. Rev. A* **49**, 1613–1617, 1994
146. Wallstrom, T.C.: On the initial-value problem for the Madelung hydrodynamic equations. *Phys. Lett. A* **184**, 229, 1994
147. Washabaugh, P., Preston, S.: The geometry of axisymmetric ideal fluid flows with swirl. *Arnold Math. J.* **3**, 175–185, 2016
148. Weigant, W.A., Papin, A.A.: On the uniqueness of the solution of the flow problem with a given vortex. *Math. Notes* **96**, 871–877, 2014
149. Wolibner, W.: Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long. *Math. Zeit.* **37**, 698–726, 1933
150. Yang, C.: Vortex motion of the Euler and Lake equations. *J. Nonlinear Sci.* **31**(3), 48, 2021 arXiv:2009.12004

151. YUDOVICH, V.: Non-stationary flows of an ideal incompressible fluid. *Zhur. Vysch. Mat. Fiz.* **3**, 1032–1066, 1963 trans: Am. Math. Soc. Transl. (2) 56, 1966
152. YUDOVICH, V.: Some bounds for solutions of elliptic equations. *Mat. Sb.* **59**, 229–244, 1962
153. YUDOVICH, V.: On the loss of smoothness of solutions of the Euler equations with time (Russian). *Dinam. Splosh. Sredy* **16**, 71–78, 1974
154. YUDOVICH, V.: Uniqueness theorem for the nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.* **2**(1), 27–38, 1995
155. YUDOVICH, V.: On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid. *Chaos* **10**, 705–719, 2000
156. ZEITLIN, V.: Finite-mode analogs of 2D ideal hydrodynamics: coadjoint orbits and local canonical structure. *Physica D* **49**, 353–362, 1991
157. ZHELIGOVSKY, V., FRISCH, U.: Time-analyticity of Lagrangian particle trajectories in ideal fluid flow. *J. Fluid Mech.* **749**, 404–430, 2014
158. ZIGLIN, S.L.: The nonintegrability of the problem on the motion of four vortices of finite strengths, appendix to K. Khanin. *Physica D* **4**(2), 268–269, 1982
159. ZIGLIN, S.L.: Dichotomy of the separatrices and the nonexistence of first integrals in systems of differential equations of Hamiltonian type with two degrees of freedom. *Math. USSR Izv.* **31**(2), 407–421, 1988

BORIS KHESIN · GERARD MISIOŁEK · ALEXANDER SHNIRELMAN
Department of Mathematics,
University of Toronto,
Toronto
Canada. e-mail: khesin@math.toronto.edu

and

GERARD MISIOŁEK
Department of Mathematics,
University of Notre Dame,
Notre Dame
IN
46556 USA. e-mail: gmisiole@nd.edu

and

ALEXANDER SHNIRELMAN
Department of Mathematics and Statistics,
Concordia University,
Montreal
Canada. e-mail: alexander.shnirelman@concordia.ca

(*Received May 9, 2022 / Accepted January 30, 2023*)