The Euler non-mixing made easy

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Abstract

The non-transitivity without extra constraints in the 3D Euler equation is almost evident and can be derived, e.g. from Morse theory.

The classical Euler equation describes the motion of an incompressible fluid filling a manifold M as an evolution of its divergence-free velocity field v:

$$\partial_t v + \nabla_v v = -\nabla p.$$

Here p is the pressure function determined by the equation itself along with the divergence-free condition $\operatorname{div} v = 0$. In this note we confine to the three-dimensional setting, $\dim M = 3$. One of the main problems of hydrodynamics is the description of properties of the dynamical system defined by the Euler equation in the appropriate space of velocity fields. It is known that this is a Hamiltonian system with the Hamiltonian function given by the L^2-energy of the fluid, with the short-time existence for the corresponding flow for a sufficiently smooth initial v (for v in C^k with non-integer $k > 1$). While for finite-dimensional Hamiltonian systems on compact manifolds one always has the Poincaré recurrence, for the 2D Euler on an annulus $M = S^1 \times \mathbb{R}$ there are wandering solutions [2, 7], i.e. a neighborhood in the space of initial conditions, such that solutions starting in that neighborhood will never return to it after some time. Note that the existence of wandering solutions in 3D Euler, as well as in 2D Euler on an arbitrary M without boundary, is still an open question.

However, in 3D it is known another dynamical property of the Euler equation.

Theorem. *The 3D Euler equation on a compact M is non-transitive and hence non-mixing: there are two open neighbourhoods in the C^k, $k \geqslant 1$ phase space of velocities, so that the Euler flow image of one of them will never intersect the other (as long as the flow exists). Such neighborhoods can be chosen within (null-homologous) divergence-free fields with any fixed helicity and any nonzero energy.*

This property is based on the existence of various first integrals, and, in particular, on vorticity transport, one of remarkable properties of the 3D Euler equation: the vorticity field $w = \operatorname{curl} v$ is frozen into the flow. A unifying idea for proving non-mixing in the 3D Euler equation in [1, 5, 6] was as follows: find two neighbourhoods in the space of all velocity fields with some incompatible topological properties of their vorticities, so that the Euler solutions with initial conditions in one of them, while preserving this property, would not be able to enter the other one. In [5] those neighbourhoods contained the fields whose vorticities have many invariant tori in different isotopy classes (and then one was applying KAM, which required high regularity, $k > 4$). In [1] this was the property of vorticity to be of contact type or not, which allowed lower smoothness ($k \geqslant 1$).

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There are variations of the above formulation, see [1, 5, 6]: a) different smoothness, b) specifying bounds for helicity and energy, c) existence of a countably many neighbourhoods, d) existence in a given homotopy class for nowhere vanishing vorticity fields, e) local non-mixing close to steady solutions (the latter was the initial motivation for the non-mixing study: prove that some solutions will never get close to steady ones).

Proof. We are proving non-transitivity, from which non-mixing follows. We start with the C^2 case for velocity. Let w_0 be an initial vorticity C^1 field. Assume that it has only non-degenerate zeros (and possibly nondegenerate periodic trajectories) in M (and hence only a finite number of them). Then there is a C^1-small neighbourhood $U(w_0)$ of w_0, such that all fields from $U(w_0)$ have the same number of zeros and they all are non-degenerate. Similarly, if another initial vorticity field w_1 has a different number of non-degenerate zeros, there is a small neighborhood $V(w_1)$ of fields with the same property and the flow $\phi^t(V(w_1))$ will never intersect $U(w_0)$. Thus the Euler flow is non-transitive. Note that the argument above requires C^1-closeness for the vorticity field (and hence C^2 for the velocity field), which is the optimal smoothness for that Morse-type argument to distinguish between different number of zeros.

To lower the smoothness to C^0-regularity for vorticity (and hence C^1 for velocity) one compares vorticity without zeros with vorticity having nondegenerate zeros. Namely, C^0-small perturbations of vorticities with nondegenerate zeros have *at least* as many zeros as the unperturbed ones, while vorticities C^0-close to the ones without zeros will also have no zeros. (Note that any three-dimensional M admits a nonzero vorticity field.) Therefore the above argument still works for C^0-closeness for vorticities, thus giving the optimal smoothness for non-transitivity. Namely, a non-degenerate zero of a vector field always has index ± 1, and according to the index theorem (which is essentially the local Intermediate Value Theorem in the vector-function setting), it must persist for C^0-close perturbations (and globally sums to the Euler characteristic). Note that one can weaken the nondegeneracy assumption on the vorticity to just having a certain number of zeros of index ± 1.

Finally, by using the local construction on the vorticity described in the example below one can generate pairs of new nondegenerate zeros with arbitrary helicity. An appropriate scaling also allows one to provide any energy of the velocity, as taking the vorticity (or its inverse) is a linear operator. □

Example. Rectify the vorticity field in a neighborhood of a nonsingular point and consider a short invariant cylinder inside that neighborhood. We deform the field in the following way to introduce inside the invariant topological cylinder two non-degenerate zeros (and a nondegenerate periodic orbit) while keeping the field divergence-free and barely changing it on the boundary.

First consider the 2D setting and a family of Hamiltonian fields with Hamiltonian functions $H = y(x^2 + y^2 + a)$ near the origin. For a positive value of a the Hamiltonian does not have critical points, and the field is topologically equivalent to that for the Hamiltonian $H = y$ near the origin. When the parameter a changes from 0^+ to 0^-, two saddles and two centers are born for the corresponding Hamiltonian (i.e. area-preserving) field.

Now we consider an axisymmetric 3D analog of that Hamiltonian field by adding a rotation about the x-axis. Then the above family of fields depending on the parameter a locally deforms from a 3D filed without zeros to a field with two nondegenerate zeros born from the saddles and one nondegenerate periodic orbit born from the two centers. By adjusting the volume form such a construction can be made divergence-free. Furthermore, the periodic orbit is encased in a family of invariant tori. By changing the original Hamiltonian near the centers one can control the speed of rotation and hence helicity accumulated near the periodic orbits without disturbing the surrounding orbits. By making this insertions rotating in different directions in various parts of M one can attain any prescribed total helicity. As discussed above, the total scaling allows one achieve an arbitrary energy of the velocity as well.

Remark. An advantage of the C^1-setting for vorticity is that one has plenty of locally defined continuous Casimirs – namely, multiplicators (i.e. eigenvalues of the linearization) of non-degenerate zeros of vorticity. Each nondegenerate zero of vorticity in 3D gives 2 locally defined Casimirs (the eigenvalues at each singularity sum to zero because of the divergence-free condition). These Casimirs may replace the first integrals of [5] measuring the volume of invariant tori of a given isotopy class (or maybe other more subtle invariants for nonvanishing fields). In the C^0-case one has only semicontinuous integer-valued Casimirs measuring the number of zeros or the like. For instance, in [1] one studies vorticities of contact type, which must have no zeros, so the index argument provides a weaker requirement for a neighborhood to stay away from fields with zeros, which are certain not to be of contact type. In order to prove c) one can consider fields w_k that have at least $2k$ non-degenerate zeros, thus providing a countable number of neighbourhoods.

Finally, note that for d) and e) one needs to use more subtle arguments. For instance, whenever one imposes additional constraints for a nonvanishing field, e.g. to stay in the same homotopy class, subtle invariants of contact homology are employed in [1]. The local non-mixing discussed in [6] was based on a specific fibrated structure of 3D steady solutions.

To summarize, without imposing extra constraints Problem #31 from [4] about non-mixing in low smoothness becomes rather straightforward, due to the existence of various Casimirs separating coadjoint orbits. (Actually, this is a property of general Euler-Arnold equations: the existence of a Casimir separating neighborhoods of coadjoint orbits implies the non-mixing of the corresponding equation in the dual of the corresponding Lie algebra. This property is based solely on the “kinematics” of the equation.) On the other hand, the tools proposed in [1, 5, 6] (contact type forms and KAM) might be of particular interest by themselves and should find other applications in hydrodynamics. The above also leads to the following natural question:

Problem. Is the Euler equation non-mixing on the coadjoint orbits of initial vorticity functions in 3D?

An answer to this question might involve the actual “dynamics” of the Euler equation, similarly to the study of wandering orbits in 2D fluids in [2, 7], which is counterposed to finite dimensions with Poincaré’s recurrence on all compact coadjoint orbits.

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