ADVANCES IN SOVIET MATHEMATICS
Volume 1, 1990

Singularities of Light Hypersurfaces and Structure of Hyperbolicity Sets for Systems of Partial Differential Equations

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§0. Introduction

Wave propagation in nonhomogeneous media (in plasma, for example) is described by the field of characteristic cones in space-time (in its cotangent bundle, to be more precise). The union of these cones defines a (singular) hypersurface in the cotangent bundle called the *light surface* of the differential equation. It turns out that singularities of generic systems of partial differential equations of arbitrary order (generic variational systems, respectively) are diffeomorphic to the singularities of the cone of degenerate matrices (degenerate symmetric matrices, resp.); see §1.

In §2 we study the hyperbolicity set in the space of systems with constant coefficients. It turns out that the sequence of simplest ("regular") singularities of this set coincide with the boundary singularities of scalar hyperbolic equations. We prove that the hyperbolicity set in the neighborhood of a nonstrictly hyperbolic system satisfying certain regularity conditions is algebraic. This result generalizes the corresponding result of F. John [J1]. We show that as opposed to the case of scalar hyperbolic polynomials [N], the set of variational hyperbolic systems contains several connected components, some of which are noncontractible.

Proofs of the main results are given in §3. A motivation of the questions considered here can be found in [A1, A2].

The author is deeply grateful to V. I. Arnold for proposing the problems and permanent help, and to A. B. Givental and B. Z. Shapiro for fruitful conversations.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35L55, 35A30, 35B99.

§1. Transversality theorems

1.1. Definitions and the transversality of symbols.

DEFINITION. The *full* and *main symbols of a system* of m partial differential equations of order $\leq l$ in space-time $\mathbb{R}^{n+1}$ are maps ρ and $\sigma: T^*\mathbb{R}^{n+1} \to \mathrm{Mat}_m(\mathbb{R})$ which are degree l polynomial and homogeneous degree l polynomial in impulses, respectively. The main symbol of a variational system is a (homogeneous and polynomial in momenta) map $\check{\sigma}: T^*\mathbb{R}^{n+1} \to \mathrm{SMat}_m(\mathbb{R})$, where $\mathrm{SMat}_m(\mathbb{R})$ is the space of symmetric $(m \times m)$–matrices.

In local coordinates, the operator $P(q, i\partial/\partial q)$ defining our system is polynomial in the variables $\partial/\partial q_j$ with matrix-valued coefficients smoothly depending on q. Its full symbol is the matrix polynomial $\rho = P(q, p)$ obtained by substituting the momentum variables p_j in place of $i\partial/\partial q_j$, and its main symbol ρ is the main homogeneous part of ρ (here q_j are coordinates in space-time $\mathbb{R}^{n+1}$ and p_j are coordinates in the dual space $T_q^*\mathbb{R}^{n+1}$).

THEOREM 1. *For any natural l in the space of all maps σ (respectively, variational maps $\check{\sigma}$) which are homogeneous degree l polynomials in impulses, almost all elements are transversal to the stratified variety of degenerate matrices in* $\mathrm{Mat}_m(\mathbb{R})$ *(in* $\mathrm{SMat}_m(\mathbb{R})$*, resp.) at all points except the zero point of* $(\mathrm{S})\mathrm{Mat}_m(\mathbb{R})$.

1.2. Singularities of light hypersurfaces of generic systems.

DEFINITION. The *light hypersurface* (i.e., the union of characteristic cones) of a system on the manifold M^{n+1} is the inverse image in T^*M of the set of all degenerate matrices of $\mathrm{Mat}_m(\mathbb{R})$ (or, respectively, $\mathrm{SMat}_m(\mathbb{R})$ in the case of variational systems) for the map σ ($\check{\sigma}$, resp.) defined by the main symbol.

It will be convenient, together with the main symbol σ, to introduce its spherization, i.e., the map

$$S\sigma: ST^*M \to (\mathrm{Mat}_m(\mathbb{R}) \setminus 0)/\mathbb{R}^+.$$

Here ST^*M is the spherization of the bundle T^*M in momenta, i.e., the bundle over M^{n+1} with the fibre $(\mathbb{R}^{n+1*} \setminus 0)/\mathbb{R}^+$. The map $S\sigma$ is well-defined since σ is homogeneous. The inverse image of classes of degenerate matrices under the map $S\sigma$ is the spherization (in momenta) of the light hypersurface in T^*M. Below we will describe the singularities of this spherization, while the singularities of the light hypersurface itself (outside the zero section in T^*M) are Cartesian products of the spherization singularities by the interval.

Consider in the space of matrices $\mathrm{Mat}_m(\mathbb{R})$ (or symmetric matrices $\mathrm{SMat}_m(\mathbb{R})$) the natural rank filtration

$$(\mathrm{S})\mathrm{Mat}_m(\mathbb{R}) \supset \Sigma_1 \supset \Sigma_2 \supset \cdots ,$$

where Σ_1 is the set of degenerate matrices, Σ_2 is the set of the corank 2 matrices, etc.

COROLLARY TO THEOREM 1. *For an arbitrary manifold* M^{n+1} *and any generic main symbol* σ ($\check{\sigma}$, *resp.*) *the singularities of the spherization of the light hypersurface are diffeomorphic to singularities of the filtration in the space of matrices*

$$\mathrm{Mat}_m(\mathbb{R}) \supset \Sigma_1 \supset \ldots \Sigma_{m-1} \supset \Sigma_m$$

(*degenerate symmetric matrices in* $\mathrm{SMat}_m(\mathbb{R})$, *resp.*) *and their trivial extensions. The set of generic maps is massive, i.e., is the intersection of a countable number of open everywhere dense sets. This set will be everywhere dense in the compact manifold* M^{n+1}; *for an arbitrary manifold it will be dense in the fine topology in the space of symbols* [AVG].

REMARK 1. Theorem 1 and its Corollary can be carried over (word for word) to the case of the space of systems with constant coefficients and (with some simplifications of the proofs) to the case of full symbols (full variational symbols), i.e., almost all the symbols are transversal to the stratified variety of degenerate matrices in $(\mathrm{S})\mathrm{Mat}_m(\mathbb{R})$ and the singularities of the light hypersurface for the generic full symbol are diffeomorphic to the singularities of the stratification of this variety.

REMARK 2. Similar arguments show that the light hypersurface of the generic symbol is in general position with respect to the natural contact structure in the space $(T^*\mathbb{R}^{n+1*} \setminus 0)/\mathbb{R}^+$. Normal forms of such a pair and the related character of light propagation in the neighborhood of the singularity, studied in [A1, A2], differ significantly from the "conic refraction" effect discovered by Hamilton. Such refraction is typical for systems with constant coefficients for which the contact structure is degenerate on the surface of the light cone.

§2. Structure of hyperbolicity sets

We will restrict ourselves to the case $M = \mathbb{R}^{n+1}$.

DEFINITION. The system σ is called *hyperbolic* if for any $q = (x, t) \in \mathbb{R}^{n+1}$ its characteristic cone (the trace of the light surface) in $T_g^*\mathbb{R}^{n+1}$ is hyperbolic with respect to the time-directed vector $(0, 1)$ (the cone $H = 0$) is hyperbolic with respect to the vector ξ if for any p the polynomial $h(\lambda) = H(p + \lambda\xi)$ in the variable λ has only real zeros).

2.1. Light surfaces of hyperbolic systems.

PROPOSITION. *For generic hyperbolic variational systems, the singularities of light surfaces are diffeomorphic to those of the stratification*

$$\mathrm{SMat}_m(\mathbb{R}) \supset \Sigma_1 \supset \cdots \supset \Sigma_{m-1} \supset \Sigma_m$$

of the variety of degenerate symmetric matrices.

Indeed, the set of degenerate symmetric matrices is hyperbolic (since

$$\{A \in \mathrm{SMat}_m(\mathbb{R}) | \det A = 0\}$$

is a hyperbolic hypersurface with respect to the direction connecting the zero and unit matrices). Thus generic singularities of light surfaces for variational systems are hyperbolic and each singularity is hyperbolic relative to its own sector of directions. Hence the requirement that the system under consideration be hyperbolic means that the time-directed vectors belong to the hyperbolicity sectors for every singularity. This condition holds, by definition, for systems lying inside the hyperbolicity domain, so that the singularities of these systems coincide with those of all generic variational symbols. q.e.d.

Following [A1, A2], note that for variational hyperbolic systems the existence of points of nonstrict hyperbolicity (i.e., singularities of the light hypersurface) occurs typically: a light surface of a generic variational system lying inside the hyperbolicity domain possesses singular points constituting a codimension 2 set on the surface.

On the contrary, nonstrict hyperbolicity in the space of all systems occurs only at boundary points of the hyperbolicity set (i.e., at points any neighborhood of which contains nonhyperbolic systems). Here, in contrast to the case of hyperbolic polynomials, there exist nonstrictly hyperbolic systems that can not be approximated by strictly hyperbolic ones (see [J1, J2] and Theorem 3 below). In other words, some strata of the boundary of the hyperbolicity domain lie separately from inner points of this set. Moreover, it turns out that the hyperbolicity set itself has no inner points at all for certain relations between m, n, and l (e.g., if $l = 1$, $n = 3$, $m = \pm 2, \pm 3, \pm 4 \pmod 8$; see [L]).

2.2. Singularities of the boundary of hyperbolicity domains. The description of singularities of the set of hyperbolic systems in the space of all systems or of all variational systems is related to the nontransversality of the map σ and the stratified variety

$$(\mathrm{S})\mathrm{Mat}_m(\mathbb{R}) \supset \Sigma_1 \supset \cdots \supset \Sigma_{m-1} \supset \Sigma_m.$$

Boundary singularities are numbered by the corank k of the map σ in the inverse image of the nontransversality point and the number j of the stratum Σ_j to which that point belongs. To an arbitrary pair (k, j) of this type corresponds its own hierarchy of singularities connected with the codimension of various arrangements of these manifolds. The following theorem describes the beginning of this classification.

DEFINITION. A singularity of the boundary of the hyperbolicity domain is called *regular* in the space of all systems if for the corresponding symbol σ the hypersurfaces $\sigma(T_q^*\mathbb{R}^{n+1})$ and $\Sigma_1 \setminus \Sigma_2$ are tangent to each other at a point regular for both of them, i.e., the pair (k, j) is equal to the simplest pair $(0, 1)$.

Notice that for $n+1 > m^2$ the map σ can not have any corank 0 points in its inverse image, i.e., there are no regular singularities among the singularities of the boundary.

The hierarchy of stable regular singularities is described by the following theorem.

THEOREM 2 (B. Z. Shapiro, B. A. Khesin). *Regular singularities of the boundary of the hyperbolicity domain in the space of all systems* (*or all variational systems*) *coincide stably* (*with respect to the degree of homogeneity* l) *with the boundary singularities of the hyperbolicity domain for hypersurfaces in* $\mathbb{R}^n$ (*see* [VSh]).

2.3. Algebraic properties of the hyperbolicity boundary. Now let us discuss the structure of the hyperbolicity set $\mathfrak{A}$ in the case of general systems with constant coefficients. Such systems constitute a finite-dimensional real space $\mathbb{R}^N$, which is the space of coefficients of polynomial maps σ.

THEOREM 3. (i) (Sturm, see [K]) *The set of hyperbolic systems* $\mathfrak{A} \subset \mathbb{R}^N$ *is semialgebraic and closed.*

(ii) *Let* $\sigma_0 \in \mathfrak{A}$ *be a nonstrictly hyperbolic system in* $n = 3$ *independent space variables. Suppose also that* $\mathrm{Im}\sigma_0$ *and* $\Sigma_1 \setminus \Sigma_2$ *are transversal everywhere except zero and that the intersection of the homogeneous varieties* $\mathrm{Im}\sigma_0$ *and* $(\Sigma_2 \setminus \Sigma_3)$ *consists of a finite number of rays, while any point from* $(\mathrm{Im}\sigma_0) \cap (\Sigma_2 \setminus \Sigma_3)$ *is a regular value of the map* σ_0.

Then in the neighborhood of σ_0 *the set* $\mathfrak{A}$ *is a nonsingular algebraic variety in* $\mathbb{R}^N$ *and its codimension is equal to the number of rays described above.*

COROLLARY. *Let* $m = 3$ *and* $l = 2$. *Then in the neighborhood of the map* σ_0 *with matrix entries*

$$(\sigma_0)_{i,j} = (p_0^2 + c_i(p_1^2 + p_2^2 + p_3^2))\delta_{i,j} - (1 - c_i)p_i p j$$

(*here* $0 < c_1 < c_2 < c_3 < 1$ *are arbitrary constants*) *the set* $\mathfrak{A}$ *is an algebraic variety of codimension* 4.

This statement is, in fact, a theorem of John (see [J1, J2]). Note that algebraicity is proved in [J2] by constructing an explicit rational parameterization of the set $\mathfrak{A} \subset \mathbb{R}^{81}$ which involves 77 parameters in the neighborhood of the point σ_0.

2.4. Global structure of hyperbolicity domains. In [N] the following result about the structure of hyperbolicity domains for polynomials was proved.

THEOREM ([N]). (i) *The set of* (*strictly*) *hyperbolic polynomials is the union of two connected and simply-connected components.*

(ii) *Each hyperbolic polynomial is the limit of a sequence of strictly hyperbolic polynomials.*

If we pass to the matrix case, then not only the local properties of hyperbolicity domains change (compare the theorem in [J1] and Theorem 3 with statement (ii) of the theorem in [N]), but their global properties also change significantly. All these effects occur even for second-order variational systems with one space variable $(l = 2, n = 1)$ and we further concentrate our efforts on this case.

REMARK. Systems with one space variable have the following specific feature: the inverse image $\sigma^{-1}(\Sigma)$ in $\mathbb{R}^2 = T_q^*\mathbb{R}^{1+1}$ of the set on degenerate matrices Σ is a set of lines, and hence the hyperbolicity of the set $\sigma^{-1}(\Sigma)$ with respect to some direction implies its hyperbolicity for almost all other directions (different from $\sigma_{-1}(\Sigma)$). On the set of these symbols there is an action of the group $GL_2(\mathbb{R})$ of linear transformation in the inverse image. The property mentioned above means that hyperbolicity of some symbol implies hyperbolicity of almost all symbols from its orbit.

Consider the connected components of the hyperbolicity domain in the space of GL_2-orbits of symbols. Note that the reduction to the orbit space means in fact that we pass from the space of maps $\{\sigma\}$ to the space of their images $\{\mathrm{Im}\sigma\}$, ignoring their parameterization and the hyperbolic direction (see Subsection 3.4 below).

REMARK. One can easily verify that part (i) of the theorem in [N], in these terms, means that the set of orbits for hyperbolic polynomials in two variables ($m = 1$, $n + 1 = 2$) consist of one simply-connected component (notice that all scalar polynomials are variational systems of order $m = 1$, since (1×1)–matrices are always symmetric). For hyperbolic variational systems of order $m > 1$ the situation is quite different.

THEOREM 4. *Let $\mathbb{R}^N$ be the space of variational $(m \times m)$–systems of the second order with one space variable ($l = 2$, $n = 1$, $N = N(m)$), and $\widetilde{\mathfrak{A}}$ be the space of GL_2-orbits of hyperbolic maps $\mathfrak{A} \subset \mathbb{R}^N$. Then*

(i) *for $m = 2$, the set $\widetilde{\mathfrak{A}}$ consists of two connected components, one of which is contractible and the other is homotopy equivalent to the circle S_1; moreover, the generator of its fundamental group can be realized as a symbol with variable coefficients;*

(ii) *for $m > 2$, the set $\widetilde{\mathfrak{A}}$ contains at least one contractible component and at least $[m/2]$ components homotopy equivalent to $S^{\frac{m(m+1)}{2}-2}$;*

(iii) *for $m = 3$, the set $\widetilde{\mathfrak{A}}$ contains at least 5 connected components.*

REMARK. The answer to the following question, raised in [A2], is still unknown: Can nontrivial generators of the homotopy groups of the components of $\mathfrak{A}$ or $\widetilde{\mathfrak{A}}$ be realized by (pseudo-)differential operators with constant coefficients but with a large number of space variables?

§3. Proof of the main theorems

3.1. Proof of the transversality theorem for symbols. In the proof of Theorem 1, the homogeneity in momenta hampers the use of the weak transversality theorem. Hence the proof is a simplified version of the corresponding strong theorem. We consider the case of all systems here (the case of variational systems differs only in the replacement of the space $\mathrm{Mat}_m(\mathbb{R})$ by $\mathrm{SMat}_m(\mathbb{R})$).

LEMMA 1. *Consider the smooth map of the Cartesian product of the cotangent space $\mathbb{R}^{2n+2} = T^*\mathbb{R}^{n+1}$ by the space $\mathbb{R}^s$ to the matrix space $\mathrm{Mat}_m(\mathbb{R})$ defined by the formula*

$$(p, q, \varepsilon) \mapsto \sigma(\varepsilon), \quad \text{where } \sigma(\varepsilon) := \sigma + \varepsilon_1 e_1 + \cdots + \varepsilon_s e_s,$$

the $e_1, \dots, e_s$ being all the different products of arbitrary monomials $p_{i_1}, \dots, p_{i_l}$ of degree l by the basis matrices, i.e., by the basis vectors in the space $\mathrm{Mat}_m(\mathbb{R})$. Then this map has no critical values different from zero. This means that the given map is transversal to any submanifold in $\mathrm{Mat}_m(\mathbb{R})$ at any nonzero point.

PROOF OF LEMMA 1. Choosing appropriate values of $\varepsilon_1, \dots, \varepsilon_s$, we can require that the matrix polynomial $\varepsilon_1 e_1 + \dots + \varepsilon_s e_s$ assume any chosen value at any chosen point (p, q), where $p \neq 0$. In other words, any point from $\mathrm{Mat}_m(\mathbb{R})$ belongs to the image of our map, and the space $\mathbb{R}^s$ (and thus $\mathbb{R}^{2n+2} \times \mathbb{R}^s$) maps onto the tangent space to $\mathrm{Mat}_m(\mathbb{R})$ at any nonzero point, q.e.d.

PROOF OF THEOREM 1. For any submanifold in $\mathrm{Mat}_m(\mathbb{R})$ and almost all ε, the map $\sigma(\varepsilon)$ is transversal to this submanifold at a nonzero point of $\mathrm{Mat}_m(\mathbb{R})$. This follows from Lemma 1 and Sard's theorem (see the lemma in the proof of the transversality theorem in [AVG]). In particular, $\sigma(\varepsilon)$ is transversal to the rank stratification of degenerate matrices

$$\mathrm{Mat}_m(\mathbb{R}) \supset \Sigma_1 \supset \dots \supset \Sigma_{i-1} \supset \Sigma_i \supset \dots .$$

Choosing a sufficiently small ε, we obtain a map arbitrarily close to σ and transversal to this stratification everywhere except zero.

PROOF OF THE COROLLARY. The following result is quite well known (see [A1, AVG]).

THEOREM. *The smooth manifold*

$$Q = ((\mathrm{S})\mathrm{Mat}_m(\mathbb{R}) \setminus 0)/\mathbb{R}^+$$

has an algebraic stratification $Q \supset Q_1 \supset Q_2 \supset \cdots$, *where* $Q_r = (\Sigma_r \setminus O)/\mathbf{R}^+$ *is the set of classes of (symmetric) matrices whose corank exceeds* $r - 1$. *The codimension of* Q_r *in* Q *equals* r^2 *in the space of all matrices and* $r(r+1)/2$ *in the space of symmetric matrices. The trace of* Q_1 *on any* r^2*-dimensional (respectively* $(r(r-1)/2)$*-dimensional) manifold transversal to* $Q_r \setminus Q_{r+1}$ *in* Q *is locally diffeomorphic to the manifold of degenerate matrices (degenerate symmetric matrices, resp.) of order* r, *i.e., to the stratification*

$$(\mathrm{S})\mathrm{Mat}_r(\mathbb{R}) \supset \Sigma_1 \supset \dots \supset \Sigma_{r-1} \supset \Sigma_r.$$

The Corollary follows from this theorem and Theorem 1 claiming that the spherization of a generic symbol is transversal to the stratification $Q_1 \supset \dots \supset Q_{m-1}$. Indeed, the inverse image of a submanifold under a map transversal to it is the cylinder over the intersection of the submanifold with the image of the map (see [AVG]).

3.2. Regular singularities of the boundary of the domain of hyperbolic systems. In this subsection we prove Theorem 2 about the coincidence of stable regular singularities of the boundary of hyperbolicity domains for systems and hypersurfaces.

Studying stable (in degree l) singularities, we can ignore restrictions connected with the degree of homogeneity and consider $S\sigma$ as an arbitrary smooth map from $ST^*\mathbb{R}^{n+1}$ in $\mathrm{Mat}_m(\mathbb{R}) \setminus 0)/\mathbb{R}^+$.

The fact that the symbol belongs to the boundary of the hyperbolic domain means we have nontransversality at some point A of the spherizations $S\mathrm{Im}\sigma_0$ and $S\Sigma$ of the surfaces $\mathrm{Im}\sigma_0$ and Σ. By definition, a singularity of the boundary is regular if the surfaces $S\mathrm{Im}\sigma_0$ and $S\Sigma$ are smooth in the neighborhood of the point A. It is convenient to consider the germ of the smooth surface $S\Sigma_1$ near the point A as the level of some function with nonvanishing gradient. Restrict this function to the surface $S\mathrm{Im}\sigma_0$. Deformations (in particular, versal deformations) of the mutual positions of the surfaces $\mathrm{Im}\sigma$ and Σ_1 are (versal) deformations of the restricted function [Kh]. Now among the deformations of the given function, we must choose those whose 0-level surface is hyperbolic (i.e., the preimage of Σ_1 in $ST^*\mathbb{R}^{n+1}$ must be hyperbolic). Since we study the 0-level surface only (the trace of $S\Sigma_1$ in $S\mathrm{Im}\sigma$), the nonuniqueness of the choice of function is unimportant.

Thus the study of stable (in l) regular boundary singularities in the space of systems is reduced to the study of the boundary of the hyperbolic domain in the space of hypersurfaces in $\mathbb{R}^n$ (or in the space of functions $\mathbb{R}^n \to \mathbb{R}$). Therefore the list of singularities in the two problems is the same. Boundary singularities of the hyperbolic domain in the space of hypersurfaces are described in detail in the paper [VSh].

This proof can be carried over word for word to the case of variational systems, since in our study of regular singularities we only use the smoothness of the surface $S\Sigma$ at the point A of the space $((\mathrm{S})\mathrm{Mat}_m(\mathbb{R}) \setminus 0)/\mathbb{R}^+$.

3.3. Proof of the theorem on local structure and algebraic properties of the set of hyperbolic systems. In order to prove that $\mathfrak{A}$ is algebraic in the neighborhood of σ_0 (Theorem 3(ii)) it suffices to check smoothness (or nonsingularity) of this set in the neighborhood of the given point. But the semialgebraicity of $\mathfrak{A}$ proved in (i) means that this set is determined by a finite number of algebraic equalities and inequalities, while the smoothness of this set in the neighborhood of σ_0 means that only the equalities are significant in this neighborhood.

The nonsingularity of $\mathfrak{A}$ follows from the regularity of the intersection $(\mathrm{Im}\sigma_0) \cap \Sigma_2$ and dimensional considerations. Indeed

$$\dim(\mathrm{Im}\sigma_0) = \mathrm{codim}\Sigma_2 = 4.$$

Let the point B lie on the ray along which $\mathrm{Im}\sigma_0$ and Σ_2 intersect, and let

V denote the germ of the (4-dimensional) manifold transversal to Σ_2 at the point B. The trace W_0 of the 4-dimensional surface $\mathrm{Im}\sigma_0$ on V is 3-dimensional because $\mathrm{Im}\,\sigma_0$ is homogeneous. The regularity of $(\mathrm{Im}\sigma_0)\cap\Sigma_2$ implies the smoothness of W_0 at the point B.

FIGURE 1. Sections of the cone of signature $(2, 2)$.

The trace K of the homogeneous surface Σ_1 on V^4 is diffeomorphic to the cone $\{ab - cd = 0\} \in \mathbb{R}^4$ (see Subsection 3.1). The cone K (which has signature (2,2)) is not hyperbolic for any direction. Its 3-dimensional sections by 3-planes (or surfaces) not passing through the vertex are also nonhyperbolic (these sections are hyperboloids of one sheet; see Figure 1,a)). However, for sections of the cone K passing through the vertex, there exists an open sector of hyperbolic directions ξ (see Figure 1,b)). By assumption, the 3-dimensional trace W_0 of the surface $\mathrm{Im}\sigma_0$ on the transversal V passes through the vertex of the cone K (i.e., intersects Σ_2) and therefore the intersection $W_0 \cap K$ is a hyperbolic cone of signature (2,1) (see Figure 1,b)).

A symbol σ close to σ_0 has an image $\mathrm{Im}\,\sigma$ close to $\mathrm{Im}\,\sigma_0$. So the trace W of this image on the transversal manifold V is smooth and close to W_0. The hyperbolicity condition for σ implies that the germs of all intersections of $\mathrm{Im}\sigma$ and Σ are hyperbolic. In particular, the intersection of W and K must be hyperbolic. As we showed above, this is possible only if W passes through the vertex of the cone K.

The existence condition of the intersection ray $\mathrm{Im}\sigma \cap \Sigma_2$ (or that of the intersection point of the smooth 3-dimensional trace W with the vertex of the cone K on the 4-dimensional transversal manifold V) defines a smooth hypersurface in the space of all systems $\{\sigma\} = \mathbb{R}^N$ in the neighborhood of σ_0. The intersection of such hypersurfaces corresponding to different rays is smooth (since the preimages of these rays in $T_q^*\mathbb{R}^{n+1}$ do not coincide) and has codimension equal to the number of these rays. q.e.d.

3.4. Connected components of the hyperbolic domain in the space of variational systems. Before proving Theorem 4 about the global structure of the hyperbolic domain, let us make the following

REMARK. Systems with $n = 1$, $l = 2$ (besides being hyperbolic with respect to almost all directions, as we mentioned above) have the following important feature: for generic $\check{\sigma}$, the set $\mathrm{Im}\check{\sigma}$ is the double cover of the quadratic half cone. Indeed, the image of a line not passing through the origin in $\mathbb{R}^2 = T_q^*\mathbb{R}^{l+1}$ under a quadratic ($l = 2$) map is a parabola in the space of matrices and therefore the surface $\mathrm{Im}\,\check{\sigma}$, by the homogeneity of $\check{\sigma}$, is half of the quadratic cone. Since l is even, the map $\check{\sigma}$ is a double covering. For exceptional values of $\check{\sigma}$, this half-cone degenerates into a 4-fold covered flat sector with vertex at the zero of the space of symmetric matrices.

LEMMA 2. *Main symbols* $\check{\sigma}\colon \mathbb{R}^2 \to \mathrm{SMat}_m(\mathbb{R})$ *with coinciding images (half-cones) in the space of matrices lie in one orbit of the* GL_2*-action by linear changes of independent space variables.*

PROOF. If symbols have the same image, then they map any line that does not pass through $O \in \mathbb{R}^2$ into a parabola generating the same half-cone (see the Remark above). For any two parabolas, an appropriate rotation of the plane $\mathbb{R}^2$ will identify their points at infinity, i.e., one and the same unique ruling of the cone will not intersect either of the parabolas. An appropriate homothetic transformation of the source plane will identify parabolas with the same point at infinity. Thus symbols with coinciding image differ only by a linear change of variables. q.e.d.

So the space of GL_2-orbits on the set of all symbols is the space of all positions of the half-cone (and its degenerate versions) in the space $\mathrm{SMat}_m(\mathbb{R})$.

The hyperbolicity of the map $\check{\sigma}$ means that $\mathrm{Im}\check{\sigma} \cap \Sigma_1$ consists of $2m$ rays (multiplicity taken into account)—the maximal possible value allowed by the order of the system.

(I) LEMMA 3. *For* $m = 2$, *the set of variational hyperbolic symbols consists of two connected components one of which is contractible and the other is homotopy equivalent to the circle.*

PROOF. For $m = 2$, we have $\dim \mathrm{SMat}_2(\mathbb{R}) = 3$ and the cone of degenerate matrices Σ is quadratic (as well as $\mathrm{Im}\check{\sigma}$) so that the maximal possible number of intersection rays is 4. Consider the traces of these cones on the spherization of the space of symmetric matrices $\mathbb{R}^3 = \mathrm{SMat}_2(\mathbb{R})$ (or on any 2-plane lying in $\mathbb{R}^3 \setminus 0$ and intersecting both halves of the cone Σ). Two classes of mutual positions of the hyperbola (the section of 2-plane by the cone Σ) and the ellipse (section by the half-cone $\operatorname{Im}\check{\sigma}$) with maximal number of intersections (see Figure 2,a),b)) complete the description of the connected components in this dimension.

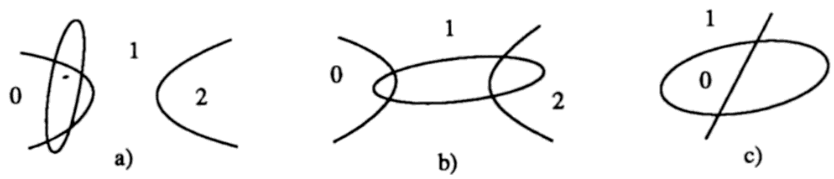

FIGURE 2

So in contrast to scalar polynomials ($m = 1$), for which the hyperbolicity domain has a unique connected component (see the remark preceeding Theorem 4), for systems such that $m > 1$, there are several different components.

Systems σ similar to those shown in Figure 2,a) can be homotoped to a system whose image $\operatorname{Im}\check{\sigma}$ is a plane sector (see the 2-dimensional section in Figure 2,c)), while systems like those in Figure 2,b) contract to a system whose image is a line joining the origin $O \in \mathbb{R}^3$ with the unit matrix. So systems of type 2,b) constitute the contractible component while the set of systems of type 2,a) are homotopy equivalent to the circle, i.e., the line segment (section of the sector) on Figure 2,c) can be rotated. q.e.d.

LEMMA 4. *The generator of the fundamental group π_1 of the nontrivial component of the set of hyperbolic systems for $m = 2$ can be realized by a hyperbolic symbol on $M^{1+1} = S^1 \times \mathbb{R}$ with nonconstant coefficients.*

PROOF. A system of type $\check{\sigma}_0$ whose image is the 4-fold cover of a sector may be presented, for example, as follows. Let (x, y, z) be coordinates in the space

$$\mathbb{R}^3 = \mathrm{SMat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} \right\};$$

put

$$\Sigma = \{(a, b, c) | a^2 - b^2 - c^2 = 0\}.$$

Then $\check{\sigma}_0 : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$\check{\sigma}_0(p, \lambda) = (p^2 + \lambda^2, 2(p^2 - \lambda^2), 0).$$

In this case the generator of the nontrivial connected component of the

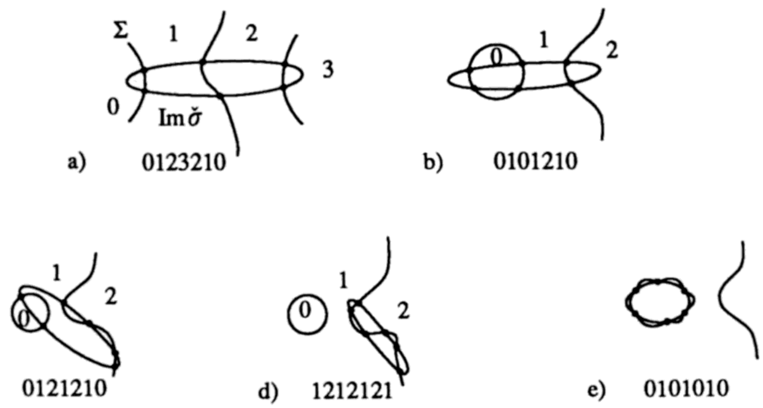

FIGURE 3

hyperbolic domain can be realized on $S^1 \times \mathbb{R}$ by the symbol

$$\check{\sigma}: T^*(\mathbb{R} \times S^1) \to \mathrm{SMat}_2(\mathbb{R}),$$

where

$$\check{\sigma}(p, \lambda, x, t) = (p^2+\lambda^2, 2(p^2-\lambda^2)\cos x, 2(p^2-\lambda^2)\sin x), \quad x \in S^1, \ t \in \mathbb{R},$$

thus finishing the proofs of Lemma 4 and statement (i) of Theorem 4.

(II) In the case of arbitrary m, notice that the cone Σ of degenerate matrices divides the space $\mathrm{SMat}_m(\mathbb{R})$ into $m+1$ components enumerated by the index of the matrices (see Figure 2). The hyperbolic embedding of the half-cone $\mathrm{Im}\,\check{\sigma}$ defines a cyclic sequence of $2m$ numbers from 0 to $2m$ (i.e., a sequence of $2m+1$ numbers with coinciding first and last numbers), which reflects the order of intersection of these components by $\mathrm{Im}\check{\sigma}$. For example, on Figure 2,a) this sequence is 01010, for Figure 2,b) it is 01210. Different sequences obviously correspond to different components of the hyperbolic domain.

For arbitrary m, systems with sequences like

$$k, (k-1), \dots, 1, 0, 1, \dots, (m-k), \dots, 0, \dots, k; \quad k = 0, 1, \dots, \left[\frac{m}{2}\right]$$

(the case $k = 0$ corresponds to the sequence $0, \dots, m, \dots, 0$) can be realized by sectors of type $\check{\sigma}_0$ as in the proof of Lemma 4. These systems are the representatives of $\left[\frac{m}{2}\right] + 1$ different connected components of the hyperbolic domain. Arguments similar to those used above in the case $m = 2$ show that the $k = 0$ component is contractible, while the others can be retracted on the spheres

$$S^{m(m-1)/2-2}, \quad \text{where } m(m+1)/2 = \dim \mathrm{SMat}_m(\mathbb{R}).$$

This estimate for the number of components is rather rough:

(III) PROPOSITION. *For $m = 3$, the hyperbolic domain has at least five connected components.*

PROOF. For $m = 3$, the cone Σ is a cubic homogeneous surface, while $\mathrm{Im}\check{\sigma}$ is still the half-cone. The Figures 3,a)–3,e) show the two-dimensional sections of hyperbolic embeddings of the half-cone in the space $\mathrm{SMat}_3(\mathbb{R})$ by Σ. These sections are different dispositions of an ellipse (the section of $\mathrm{Im}\check{\sigma}$) and an elliptic curve (the section of Σ) with the maximal possible number of intersections, equal to 6. Such embeddings realize 5 different cyclic sequences of indices of symmetric matrices (namely 0123210, 0101210, 0121210, 1212121, 0101010) and therefore represent different connected components of the hyperbolic domain. q.e.d.

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Translated by B. Z. SHAPIRO