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GEOMETRY OF DIFFEOMORPHISM GROUPS, COMPLETE INTEGRABILITY AND GEOMETRIC STATISTICS

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Abstract. We study the geometry of the space of densities $\mathrm{Dens}(M)$, which is the quotient space $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ of the diffeomorphism group of a compact manifold M by the subgroup of volume-preserving diffeomorphisms, endowed with a right-invariant homogeneous Sobolev $\dot{H}^1$-metric. We construct an explicit isometry from this space to (a subset of) an infinite-dimensional sphere and show that the associated Euler–Arnold equation is a completely integrable system in any space dimension whose smooth solutions break down in finite time. We also show that the $\dot{H}^1$-metric induces the Fisher–Rao metric on the space of probability distributions and its Riemannian distance is the spherical version of the Hellinger distance.

Contents

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1 Introduction

The geometric approach to hydrodynamics pioneered by Arnold [Arn66] is based on the observation that the particles of a fluid moving in a compact n-dimensional Riemannian manifold M trace out a geodesic curve in the infinite-dimensional group $\mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms (volumorphisms) of M. The general framework of Arnold turned out to include a variety of nonlinear partial differential equations of mathematical physics, now often referred to as *Euler–Arnold* equations.

Historically the metrics of most interest in infinite-dimensional Riemannian geometry have been L^2 metrics, which correspond to kinetic energy. On the other hand, in recent years there have appeared a number of interesting nonlinear evolution equations described as geodesic equations on diffeomorphism groups with respect to weak Riemannian metrics of Sobolev H^1-type; see e.g., [AK98, HMR98, KM03, You10] and their references. In this paper we focus on the H^1 metrics both from a differential-geometric and a dynamical systems perspective. Our main results concern the geometry of a subclass of such metrics, namely, degenerate right-invariant $\dot{H}^1$ Riemannian metrics on the full diffeomorphism group $\mathrm{Diff}(M)$ and the properties of solutions of the associated geodesic equations. The $\dot{H}^1$ metric is given at the identity diffeomorphism by

$$\langle\langle u, v\rangle\rangle = b \int_M \operatorname{div} u \cdot \operatorname{div} v \, d\mu \tag{1.1}$$

for some $b > 0$. It descends to a non-degenerate Riemannian metric on the homogeneous space of right cosets (densities) $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$. Furthermore, it turns out that the corresponding geometry on densities is *spherical* for any compact manifold M. More precisely, we prove that equipped with (1.1) the space $\mathrm{Dens}(M)$ is isometric to (a subset of) an infinite-dimensional sphere in a Hilbert space.

One motivation for studying this geometry is that such H^1 metrics arise naturally on (generic) orbits of diffeomorphism groups in the manifold of all Riemannian structures on M, using the natural Riemannian metric studied by Ebin [Ebi70]. The induced metric is a special case of the following general form of the right-invariant (a-b-c) Sobolev H^1 metric on $\mathrm{Diff}(M)$ given at the identity by

$$\langle\langle u, v\rangle\rangle = a \int_M \langle u, v\rangle \, d\mu + b \int_M \operatorname{div} u \cdot \operatorname{div} v \, d\mu + c \int_M \langle du^\flat, dv^\flat\rangle \, d\mu, \tag{1.2}$$

where $u, v \in T_e\mathrm{Diff}(M)$ are vector fields on M, μ is the Riemannian volume form,[1] $\flat$ is the isomorphism $TM \to T^*M$ defined by the metric on M, and a, b and c are non-negative real numbers. We derive the Euler–Arnold equations for the metric (1.2) in

[1] The volume form μ is denoted by $d\mu$ whenever it appears under the integral sign.

the Appendix, which include as special cases the n-dimensional (inviscid) Burgers equation, the Camassa–Holm equation, as well as the Euler-α equation. A detailed study of the related curvatures appears in a separate publication [KLMP11].

In the special case of the homogeneous $\dot{H}^1$-metric (1.1) the Euler–Arnold equation has the form

$$\rho_t + u \cdot \nabla \rho + \tfrac{1}{2}\rho^2 = -\frac{\int_M \rho^2 \, d\mu}{2\mu(M)}, \tag{1.3}$$

where $u = u(t, x)$ is a time-dependent vector field on M satisfying $\operatorname{div} u = \rho$.[2] This equation is a natural generalization of the completely integrable one-dimensional Hunter–Saxton equation [HS91] which is also known to yield geodesics on the homogeneous space $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ (the quotient of the diffeomorphism group of the circle by the subgroup of rotations), see [KM03].

We prove that the solutions of (1.3) describe the great circles on a sphere in a Hilbert space, and, in particular, the equation is a *completely integrable* PDE for any number n of space variables. The corresponding complete family of conserved integrals can be constructed in terms of angular momenta. Furthermore, we show that the maximum existence time for smooth solutions of (1.3) is necessarily finite for any initial conditions, with the L^∞ norm of the solution growing without bound as t approaches the critical time. On the other hand, the geometry of the problem points to a method of constructing global weak solutions.

The structure of the paper is as follows. In Section 2 we review the geometric background on Euler–Arnold equations on Lie groups and describe the space of densities, as well as reductions to homogeneous spaces, particularly as relates to $\mathrm{Diff}(M)$, its subgroup $\mathrm{Diff}_\mu(M)$, and their quotient $\mathrm{Dens}(M)$.

In Section 3 we introduce the homogeneous $\dot{H}^1$-metric on the space of densities and study its geometry. Generalizing the results of [Len07] for the case of the circle we show that for any n-dimensional manifold the space $\mathrm{Dens}(M)$ is isometric to a subset of the sphere in $L^2(M, d\mu)$ with the induced metric. The $\dot{H}^1$ metric generalizes the Fisher–Rao information metric in geometric statistics and its Riemannian distance is shown to be the spherical analogue of the Hellinger distance.

In Section 4 we study properties of solutions to the corresponding Euler–Arnold equation. Since for $M = S^1$ our equation reduces to the Hunter–Saxton equation we thus obtain an integrable generalization of the latter to any space dimension. We show that all solutions break down in finite time and indicate how to construct global weak solutions. Finally we describe the construction of an infinite family of conserved quantities.

In Section 5 we present a geometric approach which yields right-invariant metrics of the type (1.2) as induced metrics on the orbits of the diffeomorphism group from the canonical Riemannian L^2 structures on the spaces of Riemannian metrics and volume forms on the underlying manifold M.

[2] We will show that the solution ρ does not depend on the choice of u, which happens precisely because the metric descends to the quotient space.

We conclude in Section 6 with some applications. First we discuss gradient flow on the space of densities in the spherical metric as a heat-like equation. Next we discuss some applications to shape theory and compare with previous work, as well as to the dual connections in geometric statistics. Finally we discuss Fredholmness of the Riemannian exponential map. It also turns out that the $\dot{H}^1$-metric on the space of densities described in this paper is isometric via the Calabi-Yau map to the metric on the space of Kähler metrics introduced in the 1950s by Calabi.[3]

In the Appendix we derive the Euler–Arnold equation for the general a-b-c metric (1.2) and show that several well-known PDE of mathematical physics can be obtained as special cases.

2 Geometric Background

2.1 The Euler–Arnold equations. In this section we describe the general setup which is convenient to study geodesics on Lie groups and homogeneous spaces equipped with right-invariant metrics.

Let G be a possibly infinite-dimensional Lie group with identity element e and T_eG denoting the Lie algebra (we are primarily concerned with the case where G is a subgroup of the group of C^∞ diffeomorphisms of a compact manifold M without boundary, under the composition operation). We equip G with a right-invariant (possibly weak) Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ which is determined by its value at e. The Euler–Arnold equation on the Lie algebra for the corresponding geodesic flow has the form

$$u_t = -B(u,u) = -\mathrm{ad}_u^* u, \tag{2.1}$$

where the bilinear operator B on T_eG is defined by

$$\langle\langle B(u,v), w \rangle\rangle = \langle\langle u, \mathrm{ad}_v w \rangle\rangle, \tag{2.2}$$

see [AK98] for details. In the case where G is a diffeomorphism group, the adjoint operation is given by $\mathrm{ad}_v w = -[v,w]$, i.e., minus the Lie bracket of vector fields v and w on M.

Equation (2.1) describes the evolution in the Lie algebra of the vector $u(t)$ obtained by right-translating the velocity along the geodesic η in G starting at the identity with initial velocity $u(0)$. The geodesic itself can be obtained by solving the Cauchy problem for the flow equation

$$\frac{d\eta}{dt} = R_{\eta * e} u, \quad \eta(0) = e.$$

[3] We are grateful to B. Clarke and Y. Rubinstein for drawing our attention to this point, see more details in [CR11].

Example 2.1. Let $G = \mathrm{Diff}_\mu(M)$ be the group of volume-preserving diffeomorphisms (volumorphisms) of a closed Riemannian manifold M. Consider the right-invariant metric on $\mathrm{Diff}_\mu(M)$ generated by the L^2 inner product

$$\langle\langle u, v\rangle\rangle_{L^2} = \int_M \langle u, v\rangle \, d\mu. \tag{2.3}$$

In this case the Euler–Arnold equation (2.1) is the Euler equation of an ideal incompressible fluid in M

$$u_t + \nabla_u u = -\nabla p, \quad \operatorname{div} u = 0, \tag{2.4}$$

where u is the velocity field and p is the pressure function, see [Arn66]. In the vorticity formulation the 3D Euler equation becomes

$$\omega_t + [u, \omega] = 0, \quad \text{where} \quad \omega = \operatorname{curl} u.$$

Example 2.2. Another source of examples are right-invariant Sobolev metrics on the group $G = \mathrm{Diff}(S^1)$ of circle diffeomorphisms; see e.g., [KM03]. Of particular interest are those metrics whose Euler–Arnold equations turn out to be completely integrable. On $\mathrm{Diff}(S^1)$ with the metric defined by the L^2 product the Euler–Arnold equation (2.1) becomes the (rescaled) inviscid Burgers equation

$$u_t + 3uu_x = 0, \tag{2.5}$$

while the H^1 product yields the Camassa–Holm equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{2.6}$$

We also mention that if G is the Virasoro group, a one-dimensional central extension of $\mathrm{Diff}(S^1)$, equipped with the right-invariant L^2 metric then the Euler–Arnold equation is the periodic Korteweg-de Vries equation.

Now let H be a closed subgroup of G, and let G/H denote the homogeneous space of right cosets. The following proposition characterizes those right-invariant Riemannian metrics on G which descend to a metric on G/H.

Proposition 2.3. *A right-invariant metric $\langle\langle \cdot, \cdot\rangle\rangle$ on G descends to a right-invariant metric on the homogeneous space G/H if and only if the inner product restricted to $T_e^\perp H$ (the orthogonal complement of $T_e H$) is bi-invariant with respect to the action by the subgroup H, i.e., for any $u, v \in T_e^\perp H \subset T_e G$ and any $w \in T_e H$ one has*

$$\langle\langle v, \mathrm{ad}_w u\rangle\rangle + \langle\langle u, \mathrm{ad}_w v\rangle\rangle = 0. \tag{2.7}$$

The proof repeats with minor changes the proof for the case of a metric that is degenerate along a subgroup H; see [KM03].

Example 2.4. Let $G = \mathrm{Diff}(S^1)$ and $H = \mathrm{Rot}(S^1)$, with right-invariant metric given at the identity by

$$\langle\langle u, v \rangle\rangle_{\dot{H}^1} = \int_{S^1} u_x v_x \, dx.$$

The tangent space to the quotient $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ at the identity coset $[e]$ can be identified with the space of periodic functions of zero mean, and the corresponding Euler–Arnold equation is given by the Hunter–Saxton equation

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0, \tag{2.8}$$

see [KM03]. In [Len07] the second author constructed an explicit isometry between the quotient $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ and a subset of the unit sphere in $L^2(S^1)$ and described the corresponding solutions of equation (2.8) in terms of the geodesic flow on the infinite-dimensional sphere. Below we show that this observation is a part of a general phenomena valid for manifolds of any dimension.

2.2 The optimal transport framework. Given a volume form μ on M there is a natural fibration of the diffeomorphism group $\mathrm{Diff}(M)$ over the space of volume forms of fixed total volume $\mu(M) = 1$. More precisely, the projection onto the quotient space $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ defines a smooth ILH principal bundle[4] with fibre $\mathrm{Diff}_\mu(M)$ and whose base is diffeomorphic to the space $\mathrm{Dens}(M)$ of normalized smooth positive densities (or, volume forms)

$$\mathrm{Dens}(M) = \left\{ \nu \in \Omega^n(M) : \ \nu > 0, \int_M d\nu = 1 \right\},$$

see Moser [Mos65]. Alternatively, let $\rho = d\nu/d\mu$ denote the Radon–Nikodym derivative of ν with respect to the reference volume form μ. Then the base (as the space of constant-volume densities) can be regarded as a convex subset of the space of smooth positive functions ρ on M normalized by the condition $\int_M \rho \, d\mu = 1$. In this case the projection map π can be written explicitly as $\pi(\eta) = \mathrm{Jac}_\mu(\eta^{-1})$ where $\mathrm{Jac}_\mu(\eta)$ denotes the Jacobian of η computed with respect to μ, that is, $\eta^*\mu = \mathrm{Jac}_\mu(\eta)\mu$. The projection π satisfies $\pi(\eta \circ \xi) = \pi(\eta)$ whenever $\xi \in \mathrm{Diff}_\mu(M)$, i.e., whenever $\mathrm{Jac}_\mu(\xi) = 1$. Thus π is constant on the left cosets and descends to an isomorphism between the quotient space of left cosets to the space of densities.

The group $\mathrm{Diff}(M)$ carries a natural L^2-metric

$$\langle\langle u \circ \eta, v \circ \eta \rangle\rangle_{L^2} = \int_M \langle u \circ \eta, v \circ \eta \rangle \, d\mu = \int_M \langle u, v \rangle \mathrm{Jac}_\mu(\eta^{-1}) \, d\mu \tag{2.9}$$

[4] In the Sobolev category $\mathrm{Diff}^s(M) \to \mathrm{Diff}^s(M)/\mathrm{Diff}^s_\mu(M)$ is a C^0 principal bundle for any sufficiently large $s > n/2 + 1$, see [EM70].

Table 1 The geometric structures associated with L^2 and $\dot{H}^1$ optimal transport

Diff(M)	Diff$_\mu(M)$	Dens(M) = Diff(M)/Diff$_\mu(M)$
L^2-metric (non-invariant)	L^2-right invariant metric (ideal hydrodynamics)	Wasserstein distance (L^2-optimal transport)
$\dot{H}^1$-metric (right-invariant)	Degenerate (identically vanishing)	Spherical Hellinger distance ($\dot{H}^1$-optimal transport)

where $u, v \in T_e\mathrm{Diff}(M)$ and $\eta \in \mathrm{Diff}(M)$. This metric is neither left- nor right-invariant, although it becomes right-invariant when restricted to the subgroup $\mathrm{Diff}_\mu(M)$ of volumorphisms and becomes left-invariant only on the subgroup of isometries. Following Otto [Ott01] one can then introduce a metric on the base Dens(M) for which the projection π is a Riemannian submersion: vertical vectors at $T_\eta\mathrm{Diff}(M)$ are those fields $u \circ \eta$ with $\mathrm{div}\,(\rho u) = 0$, and horizontal fields are of the form $\nabla f \circ \eta$ for some $f\colon M \to \mathbb{R}$, since the differential of the projection is $\pi_*(v \circ \eta) = -\,\mathrm{div}\,(\rho v)$ where $\rho = \pi(\eta)$.[5]

The associated Riemannian distance in Diff(M)/Diff$_\mu(M)$ between two measures ν and λ has an elegant interpretation as the L^2-cost of transporting one density to the other

$$\mathrm{dist}_W^2(\nu, \lambda) = \inf_\eta \int_M \mathrm{dist}_M^2(x, \eta(x))\, d\mu \tag{2.10}$$

with the infimum taken over all diffeomorphisms η such that $\eta^*\lambda = \nu$ and where dist_M denotes the Riemannian distance on M; see [BB01] or [Ott01]. The function dist_W is called the L^2-Wasserstein (or Kantorovich-Rubinstein) distance between μ and ν in optimal transport theory.

Remark 2.5. While the non-invariant L^2 metric (2.9) on Diff(M) descends to Otto's metric on the quotient space Dens(M) = Diff(M)/Diff$_\mu(M)$, one verifies that among non-invariant H^1 metrics of this type it is the only one descending to the quotient Dens(M).

The situation is different for invariant metrics. Recall that the general condition for a right-invariant metric on a group G to descend to the quotient G/H with respect to a closed subgroup H was given in Proposition 2.3. Note that this condition is precisely what one needs in order for the projection map from G to G/H to be a Riemannian submersion, i.e., that the length of every horizontal vector is preserved under the projection.

It turns out that the degenerate right-invariant $\dot{H}^1$ metric (1.1) on Diff(M) descends to a non-degenerate metric on Dens(M). The skew symmetry condition (2.7) in this case will be verified in Theorem 4.1. On the other hand, one can check

[5] The construction in [Ott01] actually comes from $\mathrm{Jac}_\mu(\eta^{-1})$, which is important since it is left-invariant, not right-invariant.

that the right-invariant L^2-metric (2.3) does not verify (2.7) and hence does not descend. Similarly, the full H^1 metric on $\mathrm{Diff}(M)$ obtained by right-translating the a-b-c product (1.2) also fails to descend to a metric on $\mathrm{Dens}(M)$. This is summarized in Table 1.

3 The $\dot{H}^1$-Spherical Geometry of the Space of Densities

In this section we study the homogeneous space of densities $\mathrm{Dens}(M)$ on a closed n-dimensional Riemannian manifold M equipped with the right-invariant metric induced by the $\dot{H}^1$ inner product (1.1), that is

$$\langle\!\langle u \circ \eta, v \circ \eta \rangle\!\rangle_{\dot{H}^1} = \frac{1}{4} \int_M \mathrm{div}\, u \cdot \mathrm{div}\, v \, d\mu \tag{3.1}$$

for any $u, v \in T_e\mathrm{Diff}(M)$ and $\eta \in \mathrm{Diff}(M)$. It corresponds to the $a = c = 0$ term in the general (a-b-c) Sobolev H^1 metric (1.2) of the Introduction in which, to simplify calculations, we set $b = 1/4$. (We will return to the case of any $b > 0$ in Section 5 and Appendix A.)

The geometry of this metric on the space of densities turns out to be particularly remarkable. Indeed, we prove below that $\mathrm{Dens}(M)$ endowed with the metric (3.1) is isometric to a subset of a round sphere in the space of square-integrable functions on M.[6] Moreover, we show that (3.1) corresponds to the Bhattacharyya coefficient (also called the affinity) in probability and statistics and that it gives rise to a spherical variant of the Hellinger distance. Thus the right-invariant $\dot{H}^1$-metric provides good alternative notions of distance and shortest path for (smooth) probability measures on M to the ones obtained from the L^2-Wasserstein constructions used in standard optimal transport problems.

3.1 An infinite-dimensional sphere S_r^∞. We begin by constructing an isometry between the homogeneous space of densities $\mathrm{Dens}(M)$ and a subset of the sphere of radius r

$$S_r^\infty = \left\{ f \in L^2(M, d\mu) : \int_M f^2 \, d\mu = r^2 \right\}$$

in the Hilbert space $L^2(M, d\mu)$. As before, we let $\mathrm{Jac}_\mu(\eta)$ be the Jacobian of η with respect to μ and let $\mu(M)$ stand for the total volume of M.

Theorem 3.1. *The map $\Phi : \mathrm{Diff}(M) \to L^2(M, d\mu)$ given by*

$$\Phi : \eta \mapsto f = \sqrt{\mathrm{Jac}_\mu \eta}$$

[6] This construction has an antecedent in the special case of the group of circle diffeomorphisms considered in [Len07].

defines an isometry from the space of densities $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ *equipped with the* $\dot{H}^1$*-metric* (3.1) *to a subset of the sphere* $S_r^\infty \subset L^2(M, d\mu)$ *of radius*

$$r = \sqrt{\mu(M)}$$

with the standard L^2 *metric.*

For $s > n/2+1$ *the map* Φ *is a diffeomorphism between* $\mathrm{Diff}^s(M)/\mathrm{Diff}^s_\mu(M)$ *and the convex open subset of* $S_r^\infty \cap H^{s-1}(M)$ *which consists of strictly positive functions on* M.

Proof. First, observe that the Jacobian of any orientation-preserving diffeomorphism is a strictly positive function. Next, using the change of variables formula, we find that

$$\int_M \Phi^2(\eta)\, d\mu = \int_M \mathrm{Jac}_\mu \eta\, d\mu = \int_M \eta^*\, d\mu = \int_{\eta(M)} d\mu = \mu(M)$$

which shows that Φ maps diffeomorphisms into S_r^∞. Furthermore, observe that since for any $\xi \in \mathrm{Diff}_\mu(M)$ we have

$$\mathrm{Jac}_\mu(\xi \circ \eta)\mu = (\xi \circ \eta)^*\mu = \eta^*\mu = \mathrm{Jac}_\mu(\eta)\mu;$$

it follows that Φ is well-defined as a map from $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$.

Next, suppose that for some diffeomorphisms η_1 and η_2 we have $\mathrm{Jac}_\mu(\eta_1) = \mathrm{Jac}_\mu(\eta_2)$. Then $(\eta_1 \circ \eta_2^{-1})^*\mu = \mu$ from which we deduce that Φ is injective. Moreover, differentiating the formula $\mathrm{Jac}_\mu(\eta)\mu = \eta^*\mu$ with respect to η and evaluating at $U \in T_\eta \mathrm{Diff}(M)$, we obtain

$$\mathrm{Jac}_{\mu * \eta}(U) = \mathrm{div}(U \circ \eta^{-1}) \circ \eta \; \mathrm{Jac}_\mu \eta.$$

Therefore, letting $\pi : \mathrm{Diff}(M) \to \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ denote the bundle projection, see Figure 1, we find that

$$\begin{aligned}\langle\!\langle (\Phi \circ \pi)_{*\eta}(U), (\Phi \circ \pi)_{*\eta}(V) \rangle\!\rangle_{L^2} &= \frac{1}{4}\int_M (\mathrm{div}\, u \circ \eta) \cdot (\mathrm{div}\, v \circ \eta)\, \mathrm{Jac}_\mu \eta\, d\mu \\ &= \frac{1}{4}\int_M \mathrm{div}\, u \cdot \mathrm{div}\, v\, d\mu = \langle\!\langle U, V \rangle\!\rangle_{\dot{H}^1},\end{aligned}$$

for any elements $U = u \circ \eta$ and $V = v \circ \eta$ in $T_\eta \mathrm{Diff}(M)$ where $\eta \in \mathrm{Diff}(M)$. This shows that Φ is an isometry.

When $s > n/2+1$ the above arguments extend to the category of Hilbert manifolds modelled on Sobolev H^s spaces, see Remark 3.3 below. The fact that any positive function in $S_r^\infty \cap H^{s-1}(M)$ belongs to the image of the map Φ follows from Moser's Lemma [Mos65] whose generalization to the Sobolev setting can be found for example in [EM70]. □

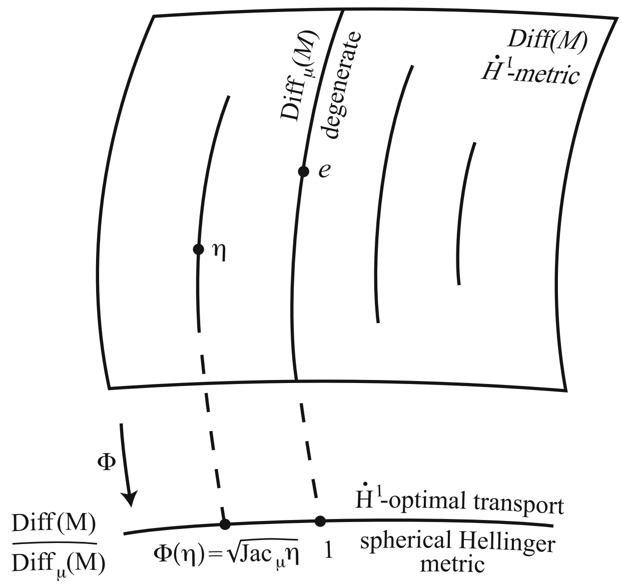

Figure 1: The fibration of $\mathrm{Diff}(M)$ with fiber $\mathrm{Diff}_\mu(M)$ determined by the reference density μ together with the $\dot{H}^1$-metric

As an immediate consequence we obtain the following result.

Corollary 3.2. *The space* $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ *equipped with the right-invariant metric* (3.1) *has strictly positive constant sectional curvature equal to* $1/\mu(M)$.

Proof. As in finite dimensions, sectional curvature of the sphere S_r^∞ equipped with the induced metric is constant and equal to $1/r^2$. The computation is straightforward using for example the Gauss–Codazzi equations. □

It is worth pointing out that the bigger the volume $\mu(M)$ of the manifold the bigger the radius of the sphere S_r^∞ and therefore, by the above corollary, the smaller the curvature of the corresponding space of densities $\mathrm{Dens}(M)$. Thus, in the case of a manifold M of infinite volume one would expect the space of densities with the $\dot{H}^1$-metric (3.1) to be “flat”. Observe also that rescaling the metric (3.1) to

$$b \int_M \mathrm{div}\, u \cdot \mathrm{div}\, v \, d\mu$$

changes the radius of the sphere to $r = 2\sqrt{b}\sqrt{\mu(M)}$.

Remark 3.3. (Hilbert manifold structures for diffeomorphism groups). As we pointed out in the Introduction, even though for our purposes it is convenient to work with C^∞ maps, the constructions of this paper can be carried out in the framework of Sobolev spaces. Now we describe this setup briefly and refer the reader to [Ebi70, EM70] or [MP10] for further details.

For a compact Riemannian manifold M, the set $H^s(M,M)$ consists of maps $f: M \to M$ such that for any $p \in M$ and for any local chart (U,ϕ) at p and any local chart (V,ϕ) at $f(p)$, the composition $\phi \circ f \circ \phi^{-1}$ belongs to $H^s(\phi(U),\mathbb{R}^n)$. Using the Sobolev Lemma, one shows that if $s > n/2$, then this definition is independent of the choice of charts on M. The tangent space at $f \in H^s(M,M)$ is defined as the set of all H^s-sections of the pull-back bundle $T_f H^s(M,M) = H^s(f^{-1}TM)$. A differentiable atlas for $H^s(M,M)$ is constructed using the Riemannian exponential map on M. For example, to find a chart at the identity map $f = e$ consider $\mathrm{Exp}: TM \to M \times M$ given by $\mathrm{Exp}(v) = \big(\pi(v), \exp_{\pi(v)} v_{\pi(v)}\big)$ where $\pi: TM \to M$ is the tangent bundle projection. Since Exp is a diffeomorphism from an open subset U containing the zero section in TM onto a neighbourhood of the diagonal in $M \times M$, one can define a bijection from the set

$$\mathcal{U}_e = \{v \in H^s(TM):\ v(M) \subset U\}$$

onto a neighbourhood of the identity map in $H^s(M,M)$ by

$$\Phi : \mathcal{U}_e \subset T_e H^s(M,M) \to H^s(M,M), \quad v \to \Phi(v) = \mathrm{Exp} \circ v.$$

The pair $(\mathcal{U}_e, \Phi)$ gives a chart in $H^s(M,M)$ around $f = e$. Compactness, properties of exp and standard facts about compositions of Sobolev maps ensure that the charts are well-defined and independent of the Riemannian metric on M, with smooth transition functions on the overlaps.

For any $s > n/2 + 1$ the group of H^s diffeomorphisms can be now defined as

$$\mathrm{Diff}^s(M) = C^1\mathrm{Diff}(M) \cap H^s(M,M),$$

where $C^1\mathrm{Diff}(M)$ is the set of C^1 diffeomorphisms of M. Since $C^1\mathrm{Diff}(M)$ forms an open set in $C^1(M,M)$, it follows by the Sobolev Lemma that $\mathrm{Diff}^s(M)$ is also open as a subset of the Hilbert manifold $H^s(M,M)$ and hence itself a smooth manifold. Furthermore, it is a topological group under composition of diffeomorphisms. In fact, right multiplications $R_\eta(\xi) = \xi \circ \eta$ are smooth in the H^s topology, whereas left multiplications $L_\eta(\xi) = \eta \circ \xi$ and inversions $\eta \to \eta^{-1}$ are continuous but not Lipschitz continuous. The subgroup of volume-preserving diffeomorphisms

$$\mathrm{Diff}^s_\mu(M) = \{\eta \in \mathrm{Diff}(M) : \eta^*\mu = \mu\}$$

is a closed C^∞ submanifold of $\mathrm{Diff}^s(M)$. This follows essentially from the implicit function theorem for Banach manifolds and the Hodge decomposition.

3.2 The metric space structure of $\mathbf{Diff}(M)/\mathbf{Diff}_\mu(M)$. The *right invariant* metric (3.1) induces a distance function between densities (measures) of fixed total volume on M that is analogous to the Wasserstein distance (2.10) induced by the *non-invariant* L^2 metric used in the standard optimal transport. It turns out that the isometry Φ constructed in Theorem 3.1 makes the computations of distances in $\mathrm{Dens}(M)$ with respect to (3.1) simpler than one would expect by comparison with the Wasserstein case.

Consider two (smooth) measures λ and ν on M of the same total volume $\mu(M)$ which are absolutely continuous with respect to the reference measure μ. Let $d\lambda/d\mu$ and $d\nu/d\mu$ be the corresponding Radon–Nikodym derivatives of λ and ν with respect to μ.

Theorem 3.4. *The Riemannian distance defined by the $\dot{H}^1$-metric (3.1) between measures λ and ν in the density space $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ is*

$$\mathrm{dist}_{\dot{H}^1}(\lambda,\nu) = \sqrt{\mu(M)}\arccos\left(\frac{1}{\mu(M)}\int_M \sqrt{\frac{d\lambda}{d\mu}\frac{d\nu}{d\mu}}\,d\mu\right). \tag{3.2}$$

Equivalently, if η and ζ are two diffeomorphisms mapping the volume form μ to λ and ν, respectively, then the $\dot{H}^1$-distance between η and ζ is

$$\mathrm{dist}_{\dot{H}^1}(\eta,\zeta) = \mathrm{dist}_{\dot{H}^1}(\lambda,\nu) = \sqrt{\mu(M)}\arccos\left(\frac{1}{\mu(M)}\int_M \sqrt{\mathrm{Jac}_\mu\eta\cdot\mathrm{Jac}_\mu\zeta}\,d\mu\right).$$

Proof. Let $f^2 = d\lambda/d\mu$ and $g^2 = d\nu/d\mu$. If $\lambda = \eta^*\mu$ and $\nu = \zeta^*\mu$ then using the explicit isometry Φ constructed in Theorem 3.1 it is sufficient to compute the distance between the functions $\Phi(\eta) = f$ and $\Phi(\zeta) = g$ considered as points on the sphere S_r^∞ with the induced metric from $L^2(M, d\mu)$. Since geodesics of this metric are the great circles on S_r^∞ it follows that the length of the corresponding arc joining f and g is given by

$$r\arccos\left(\frac{1}{r^2}\int_M fg\,d\mu\right),$$

which is precisely formula (3.2). □

We can now compute precisely the diameter of the space of densities using standard formula

$$\mathrm{diam}_{\dot{H}^1}\mathrm{Dens}(M) := \sup\left\{\mathrm{dist}_{\dot{H}^1}(\lambda,\nu):\ \lambda,\nu\in\mathrm{Dens}(M)\right\}.$$

Corollary 3.5. *The diameter of the space $\mathrm{Dens}(M)$ equipped with the $\dot{H}^1$-metric (3.1) equals $\frac{\pi}{2}\sqrt{\mu(M)}$, or one quarter the circumference of S_r^∞.*

Proof. The upper bound follows easily from formula (3.2), since the argument of the arccosine is always between 0 and 1. To prove it is arbitrarily close to 0, we choose the positive functions f and g as in the proof of Theorem 3.4 with supports concentrated in disjoint areas. □

The Riemannian distance function $\mathrm{dist}_{\dot{H}^1}$ on the space of densities $\mathrm{Dens}(M)$ introduced in Theorem 3.4 is very closely related to the Hellinger distance in probability and statistics. Recall that given two probability measures λ and ν on M that are absolutely continuous with respect to a reference probability μ the *Hellinger distance* between λ and ν is defined as

$$\mathrm{dist}^2_{Hel}(\lambda,\nu) = \int_M \left(\sqrt{\frac{d\lambda}{d\mu}} - \sqrt{\frac{d\nu}{d\mu}}\right)^2 d\mu.$$

As in the case of $\mathrm{dist}_{\dot{H}^1}$ one checks that $\mathrm{dist}_{Hel}(\lambda,\nu) = \sqrt{2}$ when λ and ν are mutually singular and that $\mathrm{dist}_H(\lambda,\nu) = 0$ when the two measures coincide. It can also be expressed by the formula $\mathrm{dist}^2_{Hel}(\lambda,\nu) = 2\,(1 - BC(\lambda,\nu))$, where $BC(\lambda,\nu)$ is the so-called Bhattacharyya coefficient (affinity) used to measure the "overlap" between statistical samples; see e.g., [Che82] for more details.

In order to compare the Hellinger distance dist_{Hel} with the Riemannian distance $\mathrm{dist}_{\dot{H}^1}$ defined in (3.2) recall that probability measures λ and ν are normalized by the condition $\lambda(M) = \nu(M) = \mu(M) = 1$. As before, we shall consider the square roots of the respective Radon–Nikodym derivatives as points on the (unit) sphere in $L^2(M, d\mu)$. One can immediately verify the following two corollaries of Theorem 3.1.

COROLLARY 3.6. *The Hellinger distance* $\mathrm{dist}_{Hel}(\lambda,\nu)$ *between the normalized densities* $d\lambda = f^2 d\mu$ *and* $d\nu = g^2 d\mu$ *is equal to the distance in* $L^2(M, d\mu)$ *between the points on the unit sphere* $f, g \in S^\infty_1 \subset L^2(M, d\mu)$.

COROLLARY 3.7. *The Bhattacharyya coefficient* $BC(\lambda,\nu)$ *for two normalized densities* $d\lambda = f^2 d\mu$ *and* $d\nu = g^2 d\lambda$ *is equal to the inner product of the corresponding positive functions* f *and* g *in* $L^2(M, d\mu)$*:*

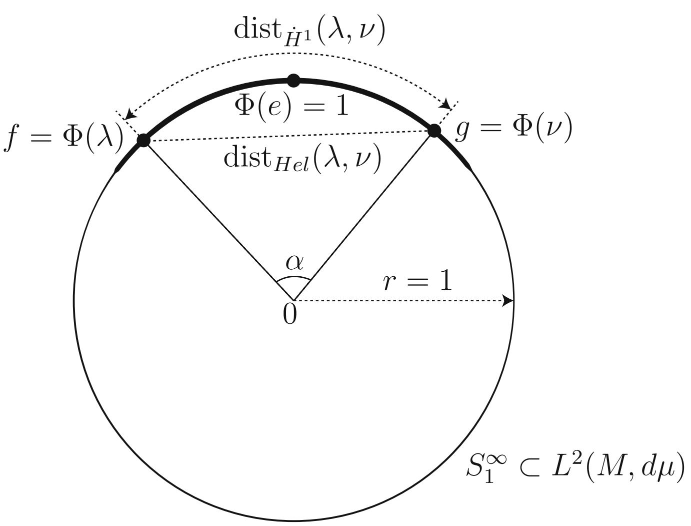

Figure 2: The Hellinger distance $\mathrm{dist}_{Hel}(\lambda,\nu)$ and the spherical Hellinger distance $\mathrm{dist}_{\dot{H}^1}(\lambda,\nu)$ between two points $f = \Phi(\lambda)$ and $g = \Phi(\nu)$ in S^∞_1. The thick arc represents the image of $\mathrm{Diff}(M)$ under the map Φ

$$BC(\lambda, \nu) = \int_M \sqrt{\frac{d\lambda}{d\mu}\,\frac{d\nu}{d\mu}}\, d\mu = \int_M f g\, d\mu.$$

Let $0 < \alpha < \pi/2$ denote the angle between f and g viewed as unit vectors in $L^2(M, d\mu)$. Then we have

$$\mathrm{dist}_{Hel}(\lambda, \nu) = 2\sin(\alpha/2) \quad \text{and} \quad BC(\lambda, \nu) = \cos\alpha,$$

while

$$\mathrm{dist}_{\dot{H}^1}(\lambda, \nu) = \alpha = \arccos BC(\lambda, \nu).$$

Thus, we can refer to the Riemannian distance $\mathrm{dist}_{\dot{H}^1}(\lambda, \nu)$ on $\mathrm{Dens}(M)$ as the *spherical Hellinger distance* between λ and ν, see Figure 2.

3.3 The Fisher–Rao metric in infinite dimensions. It is remarkable that the right-invariant $\dot{H}^1$ metric (3.1) provides an appropriate geometric framework for an *infinite-dimensional* Riemannian approach to mathematical statistics. Efforts directed toward finding suitable differential geometric approaches to statistics go back to the work of Fisher, Rao [Rao93] and Kolmogorov.

In the classical approach one considers finite-dimensional families of probability distributions on M whose elements are parameterized by subsets E of the Euclidean space $\mathbb{R}^k$,

$$\mathcal{S} = \left\{ \nu = \nu_{s_1,\ldots,s_k} \in \mathrm{Dens}(M) : (s_1, \ldots, s_k) \in E \subset \mathbb{R}^k \right\}.$$

When equipped with a structure of a smooth k-dimensional manifold such a family is referred to as a statistical model. Rao [Rao93] showed that any $\mathcal{S}$ carries a natural structure given by a $k \times k$ positive definite matrix

$$I_{ij} = \int_M \frac{\partial \log \nu}{\partial s_i} \frac{\partial \log \nu}{\partial s_j} \nu\, d\mu \qquad (i, j = 1, \ldots, k), \tag{3.3}$$

called the Fisher–Rao (information) metric.[7]

In our approach we shall regard a statistical model $\mathcal{S}$ as a *k-dimensional Riemannian submanifold* of the *infinite-dimensional Riemannian manifold* of probability densities $\mathrm{Dens}(M)$ defined on the underlying n-dimensional compact manifold M. The following theorem shows that the Fisher–Rao metric (3.3) is (up to a constant multiple) the metric induced on the submanifold $\mathcal{S} \subset \mathrm{Dens}(M)$ by the (degenerate) right-invariant Sobolev $\dot{H}^1$-metric (1.1) we introduced originally on the full diffeomorphism group $\mathrm{Diff}(M)$.

[7] The significance of this metric for statistics was also noted by Chentsov [Che82]. An infinite-dimensional version was perhaps first mentioned by Dawid in a commentary [Daw77] on the paper of Efron [Efr75].

Theorem 3.8. *The right-invariant Sobolev $\dot{H}^1$-metric (3.1) on the quotient space* $\mathrm{Dens}(M)$ *of probability densities on M coincides with the Fisher–Rao metric on any k-dimensional statistical submanifold of* $\mathrm{Dens}(M)$.

Proof. We carry out the calculations directly in $\mathrm{Diff}(M)$. Given any v and w in $T_e\mathrm{Diff}(M)$, consider a two-parameter family of diffeomorphisms $(s_1, s_2) \to \eta(s_1, s_2)$ in $\mathrm{Diff}(M)$ starting from the identity $\eta(0,0) = e$ with $\frac{\partial}{\partial s_1}\eta(0,0) = v$, $\frac{\partial}{\partial s_2}\eta(0,0) = w$. Let

$$v(s_1, s_2) \circ \eta(s_1, s_2) = \tfrac{\partial}{\partial s_1}\eta(s_1, s_2) \quad \text{and} \quad w(s_1, s_2) \circ \eta(s_1, s_2) = \tfrac{\partial}{\partial s_2}\eta(s_1, s_2)$$

be the corresponding variation vector fields along $\eta(t, s)$.

If ρ is the Jacobian of $\eta(s_1, s_2)$ computed with respect to the fixed measure μ, then (3.3) takes the form

$$I_{vw} = \int_M \frac{\partial}{\partial s_1}\left(\log \mathrm{Jac}_\mu \eta(s_1, s_2)\right) \frac{\partial}{\partial s_2}\left(\log \mathrm{Jac}_\mu \eta(s_1, s_2)\right) \mathrm{Jac}_\mu \eta(s_1, s_2)\, d\mu.$$

Recall that

$$\frac{\partial}{\partial s_1}\mathrm{Jac}_\mu \eta(s_1, s_2) = \mathrm{div}\, v(s_1, s_2) \circ \eta(s_1, s_2) \cdot \mathrm{Jac}_\mu \eta(s_1, s_2)$$

and similarly for the partial derivative in s_2. Using these formulas and changing variables in the integral, we now find

$$\begin{aligned} I_{vw} &= \int_M \frac{\frac{\partial}{\partial s_1}\mathrm{Jac}_\mu \eta(s_1, s_2)\frac{\partial}{\partial s_2}\mathrm{Jac}_\mu \eta(s_1, s_2)}{\mathrm{Jac}_\mu \eta(s_1, s_2)}\Big|_{s_1 = s_2 = 0}\, d\mu \\ &= \int_M \big(\mathrm{div}\, v \circ \eta\big) \cdot \big(\mathrm{div}\, w \circ \eta\big)\, \mathrm{Jac}_\mu \eta\, d\mu \\ &= \int_M \mathrm{div}\, v \cdot \mathrm{div}\, w\, d\mu = 4\langle\!\langle v, w \rangle\!\rangle_{\dot{H}^1}, \end{aligned}$$

from which the theorem follows. □

Theorem 3.8 suggests that the $\dot{H}^1$ counterpart of optimal transport with its associated spherical Hellinger distance is the infinite-dimensional version of geometric statistics sought in [AN00] and [Che82].

4 The Geodesic Equation: Solutions and Integrability

In the preceding sections we studied the geometry of the $\dot{H}^1$-metric (3.1) on the space of densities $\mathrm{Dens}(M)$. In this section we shall focus on obtaining explicit formulas for solutions of the Cauchy problem for the associated Euler–Arnold equation and prove that they necessarily break down in finite time.

4.1 Local smooth solutions and explicit formulas. First we derive the geodesic equation induced on the quotient $\mathrm{Dens}(M)$ by the Riemannian metric (1.1).

Theorem 4.1. *If $a = c = 0$ then the a-b-c metric (1.2) satisfies condition (2.7) and therefore descends to a metric on the space of densities $\mathrm{Dens}(M)$. The corresponding Euler–Arnold equation is*

$$\nabla \operatorname{div} u_t + \operatorname{div} u \, \nabla \operatorname{div} u + \nabla \langle u, \nabla \operatorname{div} u \rangle = 0 \tag{4.1}$$

or, in the integrated form,

$$\rho_t + \langle u, \nabla \rho \rangle + \tfrac{1}{2}\rho^2 = -\frac{\int_M \rho^2 \, d\mu}{2\mu(M)} \tag{4.2}$$

where $\rho = \operatorname{div} u$.

Proof. We verify (2.7) for $G = \mathrm{Diff}(M)$, $H = \mathrm{Diff}_\mu(M)$ and $\mathrm{ad}_w v = -[w, v]$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields on M. Given any vector fields u, v and w with $\operatorname{div} w = 0$, we have

$$\begin{aligned}
\langle\!\langle \mathrm{ad}_w v, u \rangle\!\rangle_{\dot H^1} + \langle\!\langle v, \mathrm{ad}_w u \rangle\!\rangle_{\dot H^1} &= -b \int_M \big(\operatorname{div}[w, v] \operatorname{div} u + \operatorname{div}[w, u] \operatorname{div} v \big)\, d\mu \\
&= -b \int_M \Big(\big(\langle w, \nabla \operatorname{div} v \rangle - \langle v, \nabla \operatorname{div} w \rangle\big) \operatorname{div} u \\
&\qquad + \big(\langle w, \nabla \operatorname{div} u \rangle - \langle u, \nabla \operatorname{div} w \rangle\big) \operatorname{div} v \Big)\, d\mu \\
&= b \int_M \operatorname{div} w \cdot \operatorname{div} v \cdot \operatorname{div} u \, d\mu = 0,
\end{aligned}$$

which shows that (1.2) descends to $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$.

The Euler–Arnold equation on the quotient can be now obtained from (A.4) in the form (4.1). In integrated form it reads

$$\operatorname{div} u_t + \langle u, \operatorname{div} u \rangle + \tfrac{1}{2}(\operatorname{div} u)^2 = C(t)$$

where $C(t)$ may in general depend on time. Integrating this equation over M determines the value of $C(t)$. □

Note that in the special case $M = S^1$ differentiating equation (4.2) with respect to the space variable gives the Hunter–Saxton equation (2.8). The gradient of (4.2), augmented by terms arising from an additional L^2 term in (1.1), was derived as a 2D water wave equation in [KSD01], thus our equation represents a limiting case.

Remark 4.2. The right-hand side of equation (4.2) is independent of time for any initial condition ρ_0 because the integral $\int_M \rho^2 \, d\mu$ corresponds to the energy (the squared length of the velocity) in the $\dot H^1$-metric on $\mathrm{Dens}(M)$ and is constant along a geodesic. This invariance will also be verified by a direct computation in the proof below.

Consider an initial condition in the form

$$\rho(0,x) = \operatorname{div} u_0(x). \tag{4.3}$$

We already have an indirect method for solving the initial value problem for equation (4.2) by means of Theorem 3.1. We now proceed to give explicit formulas for the corresponding solutions.

Theorem 4.3. *Let $\rho = \rho(t,x)$ be the solution of the Cauchy problem (4.2)–(4.3) and suppose that $t \mapsto \eta(t)$ is the flow of the velocity field $u = u(t,x)$, i.e., $\frac{\partial}{\partial t}\eta(t,x) = u(t,\eta(t,x))$ where $\eta(0,x) = x$. Then*

$$\rho\big(t,\eta(t,x)\big) = 2\kappa \tan\left(\arctan\frac{\operatorname{div} u_0(x)}{2\kappa} - \kappa t\right), \tag{4.4}$$

where

$$\kappa^2 = \frac{1}{4\mu(M)}\int_M (\operatorname{div} u_0)^2\, d\mu. \tag{4.5}$$

Furthermore, the Jacobian of the flow is

$$\operatorname{Jac}_\mu\big(\eta(t,x)\big) = \left(\cos\kappa t + \frac{\operatorname{div} u_0(x)}{2\kappa}\sin\kappa t\right)^2. \tag{4.6}$$

Proof. For any smooth real-valued function $f(t,x)$ the chain rule gives

$$\frac{d}{dt}\left(f(t,\eta(t,x))\right) = \frac{\partial f}{\partial t}(t,\eta(t,x)) + \langle u\left(t,\eta(t,x)\right), \nabla f\left(t,\eta(t,x)\right)\rangle.$$

Using this we obtain from (4.2) an equation for $f = \rho\circ\eta$

$$\frac{df}{dt} + \tfrac{1}{2}f^2 = -C(t), \tag{4.7}$$

where $C(t) = (2\mu(M))^{-1}\int_M \rho^2 d\mu$, as remarked above, is in fact independent of time. Indeed, direct verification gives

$$\begin{aligned}\mu(M)\frac{dC(t)}{dt} &= \int_M \rho\rho_t\, d\mu = \int_M \operatorname{div} u \operatorname{div} u_t\, d\mu \\ &= -\int_M \langle u, \nabla \operatorname{div} u\rangle \operatorname{div} u\, d\mu - \frac{1}{2}\int_M (\operatorname{div} u)^3\, d\mu = 0,\end{aligned}$$

where the last cancellation follows from integration by parts.

Set $C = 2\kappa^2$. Then, for a fixed $x \in M$ the solution of the resulting ODE in (4.7) with initial condition $f(0)$ has the form

$$f(t) = 2\kappa\tan\left(\arctan\left(f(0)/2\kappa\right) - \kappa t\right),$$

which is precisely (4.4).

In order to find an explicit formula for the Jacobian we first compute the time derivative of $\mathrm{Jac}_\mu(\eta)\mu$ to obtain

$$\frac{d}{dt}\left(\mathrm{Jac}_\mu(\eta)\mu\right) = \frac{d}{dt}(\eta^*\mu) = \eta^*(\mathcal{L}_u\mu) = \eta^*(\mathrm{div}\, u\, \mu) = (\rho\circ\eta)\,\mathrm{Jac}_\mu(\eta)\mu.$$

This gives a differential equation for $\mathrm{Jac}_\mu\eta$, which we can now solve with the help of (4.4) to get the solution in the form of (4.6). □

Note that (4.6) completely determines the Jacobian regardless of any "ambiguity" in the velocity field u satisfying $\mathrm{div}\, u = \rho$ in equation (4.2). The reason is that the Jacobians can be considered as elements of the quotient space $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ (a convenient way to resolve the ambiguity is by choosing velocity as the gradient field $u = \nabla\Delta^{-1}\rho$).

Remark 4.4 (Great circles on S^∞_r). We emphasize that formula (4.6) for the Jacobian $\mathrm{Jac}_\mu\eta$ of the flow is best understood in light of the correspondence between geodesics in $\mathrm{Dens}(M)$ and those on the infinite-dimensional sphere S^∞_r established in Theorem 3.1. Indeed, the map

$$t \to \sqrt{\mathrm{Jac}_\mu\left(\eta(t,x)\right)} = \cos\kappa t + \frac{\mathrm{div}\, u_0(x)}{2\kappa}\sin\kappa t$$

describes the great circle on the unit sphere $S^\infty_1 \subset L^2(M, d\mu)$ passing through the point 1 with initial velocity $\frac{1}{2}\,\mathrm{div}\, u_0$.

4.2 Global properties of solutions. The explicit formulas of Theorem 4.3 make it possible to give a fairly complete picture of the global behavior of solutions to the $\dot{H}^1$ Euler–Arnold equation on $\mathrm{Dens}(M)$ for any manifold M. It turns out for example that any smooth solution of equation (4.2) has finite lifespan and the blowup mechanism can be precisely described.

By the result of Moser [Mos65], the function on the right side of formula (4.6) will be the Jacobian of some diffeomorphism as long as it is nowhere zero. Hence up to the blowup time we have a smooth path in the space of densities, which lifts to a smooth path in the diffeomorphism group; see Proposition 4.6. Geodesics leave the set of positive densities and hit the boundary corresponding to the boundary of the diffeomorphism group. The latter consists of H^s maps from M to M, which are degenerations of diffeomorphisms. To make sense of weak solutions of (4.2), one would need a way of lifting the curve (4.6) to a smooth curve in $H^s(M, M)$.

First, we note that there can be no global smooth (classical) solutions of the Euler–Arnold equation (4.2). As in the case of the one-dimensional Hunter–Saxton equation all solutions break down in finite time.

Proposition 4.5. *The maximal existence time of a (smooth) solution of the Cauchy problem (4.2)–(4.3) constructed in Theorem 4.3 is*

$$0 < T_{\max} = \frac{\pi}{2\kappa} + \frac{1}{\kappa}\arctan\left(\frac{1}{2\kappa}\inf_{x\in M}\mathrm{div}\, u_0(x)\right). \tag{4.8}$$

Furthermore, as $t \nearrow T_{\max}$ we have $\|u(t)\|_{C^1} \nearrow \infty$.

Proof. This follows at once from formula (4.4) using the fact that $\operatorname{div} u = \rho$. Alternatively, from formula (4.6) we observe that the flow of $u(t,x)$ ceases to be a diffeomorphism at $t = T_{\max}$. □

Observe that before a solution reaches the blow-up time it is always possible to lift the corresponding geodesic to a smooth flow of diffeomorphisms using a slight variation of the classical construction of Moser [Mos65].

PROPOSITION 4.6. *If $\operatorname{div} u_0$ is smooth, then there exists a family of smooth diffeomorphisms $\eta(t)$ in $\mathrm{Diff}(M)$ satisfying (4.6), i.e., such that $\mathrm{Jac}_\mu(\eta(t)) = \varphi(t)$ where*

$$\varphi(t,x) = \left(\cos \kappa t + \frac{\operatorname{div} u_0(x)}{2\kappa} \sin \kappa t\right)^2, \tag{4.9}$$

provided that $0 \le t < T_{\max}$. Furthermore η is smooth in time as a curve in $\mathrm{Diff}(M)$. If u_0 is in H^s for $s > n/2 + 1$, the curve $\eta(t)$ is in $\mathrm{Diff}^s(M)$.

Proof. It is easy to check that $\int_M \varphi(t,x)\, d\mu$ is constant in time, which allows one to solve the equation $\Delta f(t,x) = -\partial\varphi/\partial t(t,x)$ for f, for any fixed time t. Using the explicit formula (4.9), we easily see that f is smooth in time and spatially in H^{s+1} if u_0 is in H^s.

For t in $[0, T_{\max})$, we define a time-dependent vector field by the formula $X(t,x) = \nabla f(t,x)/\varphi(t,x)$. Let $t \mapsto \xi(t)$ denote the flow of X starting at the identity (which exists for $t \in [0, T_{\max})$ and $x \in M$ by compactness of M). Using the definition of f and $\mathcal{L}_X(\varphi\mu) = \operatorname{div}(\varphi X)\mu$, we compute

$$\frac{d}{dt}\xi^*(\varphi\mu) = \xi^*\left(\frac{\partial\varphi}{\partial t}\mu + \mathcal{L}_X(\varphi\mu)\right) = 0.$$

Since $\varphi(0) = 1$ and $\xi(0) = e$ we have $\xi^*(\varphi\mu) = \mu$ for any $0 \le t < T_{\max}$. Denoting by $\eta(t)$ the inverse of the diffeomorphism $\xi(t)$, we find that $\eta^*\mu = \varphi\mu$, from which it follows that $\mathrm{Jac}_\mu(\eta(t,x)) = \varphi(t,x)$ as desired. □

The method of Proposition 4.6 gives a particular choice of a diffeomorphism flow η, and hence a velocity field appearing in (4.2) and satisfying $\operatorname{div} u = \varphi$. The flow must break down at the critical time $T_{\max}$, since the vector field X becomes singular (when φ reaches zero). The difficulty here is that one constructs η indirectly, by first constructing $\xi = \eta^{-1}$, and it is this inversion procedure that breaks down at the blowup time $T_{\max}$.

For the Hunter–Saxton equation on $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ the related construction of weak solutions was explained in [Lene07]. In this case the flow is determined (up to rotations of the base point) by its Jacobian. If the initial velocity is not constant in any interval, then the singularities of the flow are isolated so that it is a homeomorphism (but not a diffeomorphism past the blowup time). In terms of the spherical picture, the square root map Φ from Theorem 3.1 maps only onto a small portion of the space of functions with fixed L^2 norm, but its inverse can be defined on the entire sphere. In higher dimensions if the Jacobian is not everywhere positive the situation is much more complicated. Nevertheless, in this case it may be possible to apply the techniques of Gromov and Eliashberg [GE73] in order to construct a map with a prescribed Jacobian. It would be interesting to extend Moser's argument to construct a global flow of homeomorphisms out of this flow of maps (past the blowup time).

4.3 Complete integrability. For a $2n$-dimensional Hamiltonian system, complete integrability means the existence of n functionally independent integrals $H_1, \dots, H_n$ in involution (one of which is the Hamiltonian of the system); in such a case the motion can be determined by quadrature. In infinite dimensions the situation is more subtle: the existence of infinitely many constants of motion may not suffice to determine the motion. Infinite-dimensional systems have been studied intensively since the discovery of the complete integrability of the Korteweg-de Vries equation. Other examples include one-dimensional equations like the Camassa–Holm and Hunter–Saxton equations, and two-dimensional examples like the Kadomtsev–Petviashvili, Ishimori, and Davey–Stewartson equations.

In addition to having an explicit formula for solutions (see Theorem 4.3), one can also construct infinitely many independent constants of motion, using the fact that geodesic motion on a sphere of any dimension is completely integrable. First consider the unit sphere $S^{n-1} \subset \mathbb{R}^n$, given by the equation $\sum_{j=1}^n q_j^2 = 1$ with $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ and equipped with its standard round metric. The geodesic flow in this metric is defined by the Hamiltonian $H = \sum_{j=1}^n p_j^2$ on the cotangent bundle T^*S^{n-1}. It is a classical example of a completely integrable system, which has the property that all of its orbits are closed.

Proposition 4.7. (*see e.g.,* [Bol10])

(i) *The functions* $h_{ij} = p_i q_j - p_j q_i$, $1 \le i < j \le n$ *on* $T^*\mathbb{R}^n$ *(as well as their reductions to* T^*S^{n-1}*) commute with the Hamiltonian* $H = \sum_{j=1}^n p_j^2$ *and generate the Lie algebra* $\mathfrak{so}(n)$.

(ii) *The functions*

$$H_k := \sum_{1\le i<j\le k} h_{ij}^2 = \sum_{j=1}^k p_j^2 \sum_{j=1}^k q_j^2 - \left(\sum_{j=1}^k q_j p_j\right)^2$$

for $k = 2, \dots, n$ *form a complete set of independent integrals in involution for the geodesic flow on the round sphere* $S^{n-1} \subset \mathbb{R}^n$*, that is* $\{H_i, H_j\} = 0$*, for any* $2 \le i, j \le n$.

Proof. The Hamiltonian functions h_{ij} in $T^*\mathbb{R}^n$ generate rotations in the (q_i, q_j)-plane in $\mathbb{R}^n$, which are isometries of S^{n-1}. These rotations commute with the geodesic flow on the sphere and hence $\{h_{ij}, H\} = 0$. A direct computation gives $\{h_{ij}, h_{jk}\} = h_{ik}$, which are the commutation relations of $\mathfrak{so}(n)$.

The involutivity of H_k is a routine calculation. □

Alternatively, one can consider the chain of subalgebras $\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(n)$. Then H_k is one of the Casimir functions for $\mathfrak{so}(k)$ and it therefore commutes with any function on $\mathfrak{so}(k)^*$. In particular, it commutes with all the preceding functions H_m for $m < k$. They are functionally independent because at each step H_k involves new functions h_{jk}. Note that on the cotangent bundle T^*S^{n-1} the function H_n coincides with the Hamiltonian H since $\sum_{j=1}^n q_j^2 = 1$ and $\sum_{j=1}^n p_i q_i = 0$ ("the tangent plane equation").

The same procedure allows one to construct integrals in infinite dimensions, for $S_r^\infty \subset L^2(M, d\mu)$. Similarly, on the cotangent space $T^*S_r^\infty$ with position coordinates q_i and momentum coordinates p_i, Hamiltonians $h_{ij} = p_i q_j - p_j q_i$ generate rotations of the sphere in the (q_i, q_j)-plane. They now form the Lie algebra $\mathfrak{so}(\infty)$ of the group of unitary operators on L^2 and generate an infinite sequence of functionally independent first integrals $\{H_k\}_{k=2}^\infty$ in involution. This sequence corresponds to the infinite chain of embeddings $\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(\infty)$ and provides infinitely many conserved quantities for the geodesic flow on the unit sphere $S_r^\infty \subset L^2(M, d\mu)$. We summarize the above consideration in the following

Theorem 4.8. *The Euler–Arnold equation (4.2) of the right-invariant $\dot{H}^1$-metric on the space of densities* $\mathrm{Dens}(M)$ *is an infinite-dimensional completely integrable dynamical system.*

Remark 4.9. In 1981 Arnold posed a problem of finding equations of mathematical physics which realize geodesic flows on infinite-dimensional ellipsoids (see Problem 1981-29 in *Arnold's problems*). The $\dot{H}^1$-geodesic equation on $\mathrm{Dens}(M)$ can be viewed as an example of such, being the geodesic flow on an infinite-dimensional sphere and manifesting a high degree of integrability, since all of its orbits are closed.

Furthermore, the geodesic flow on an n-dimensional ellipsoid (and sphere as the limiting case) is known to be a bi-hamiltonian dynamical system and its first integrals can be obtained by a procedure similar to the Lenard-Magri scheme. On the other hand, the one-dimensional Hunter–Saxton equation has a bi-Hamiltonian structure. It would be interesting to find explicitly a bi-Hamiltonian structure for the higher-dimensional equation (1.3) and relate the H_k functionals to the Lenard-Magri type invariants.

5 The Space of Metrics and the Diffeomorphism Group

Apart from the fact that the Euler–Arnold equations of H^1 metrics yield a number of interesting evolution equations of mathematical physics discussed above there is also a purely geometric reason to study them. Below we show that right-invariant Sobolev metrics of the type studied in this paper arise naturally on orbits of the diffeomorphism group acting on the space of all Riemannian metrics and volume forms on M. Our main references for the constructions recalled are [Ebi70, FG89].

Given a compact manifold M consider the set $\mathrm{Met}(M)$ of all (smooth) Riemannian metrics on M. This set acquires in a natural way the structure of a smooth Hilbert manifold.[8] The group $\mathrm{Diff}(M)$ acts on $\mathrm{Met}(M)$ by pull-back $g \mapsto \mathcal{P}_g(\eta) = \eta^* g$ and there is a natural geometry on $\mathrm{Met}(M)$ which is invariant under this action. If g is a Riemannian metric and A, B are smooth sections of the tensor bundle S^2T^*M, then the expression

$$\langle\langle A, B \rangle\rangle_g = \int_M \mathrm{Tr}\left(g^{-1} A\, g^{-1} B\right)\, d\mu_g \tag{5.1}$$

defines a (weak Riemannian) L^2-metric on $\mathrm{Met}(M)$. Here μ_g is the volume form of g. This metric is invariant under the action of $\mathrm{Diff}(M)$, see [Ebi70].

The space $\mathrm{Vol}(M)$ of all (smooth) volume forms on M also carries a natural (weak Riemannian) L^2-metric

$$\langle\langle \alpha, \beta \rangle\rangle_\nu = \frac{4}{n} \int_M \frac{d\alpha}{d\nu} \frac{d\beta}{d\nu}\, d\nu, \tag{5.2}$$

where $\nu \in \mathrm{Vol}(M)$ and α, β are smooth n-forms and which appeared already in the paper [FG89].[9] It is also invariant under the action of $\mathrm{Diff}(M)$ by pull-back $\mu \to \mathcal{P}_\mu(\eta) = \eta^*\mu$.

There is a map $\Xi\colon \mathrm{Met}(M) \to \mathrm{Vol}(M)$ which assigns to a Riemannian metric g the volume form μ_g. One checks that Ξ is a Riemannian submersion in the normalization of (5.2). Furthermore, for any g in $\mathrm{Met}(M)$ there is a map $\iota_g\colon \mathrm{Vol}(M) \times \{g\} \to \mathrm{Met}(M)$ given by

$$\iota_g(\nu) = \left(\frac{d\nu}{d\mu_g}\right)^{2/n} g,$$

which is an isometric embedding.

For any $\mu \in \mathrm{Vol}(M)$ the inverse image $\mathrm{Met}_\mu(M) = \Xi^{-1}[\mu]$ can be given a structure of a submanifold in the space of Riemannian metrics whose volume form is μ. The metric (5.1) induces a metric on $\mathrm{Met}_\mu(M)$, which turns it into a globally symmetric space. The natural action on $\mathrm{Met}_\mu(M)$ is again given by pull-back by elements of the group $\mathrm{Diff}_\mu(M)$.

The sectional curvature of the metric (5.1) on $\mathrm{Met}(M)$ was computed in [FG89] and found to be nonpositive. The corresponding sectional curvature of $\mathrm{Met}_\mu(M)$ is also nonpositive. On the other hand, the space $\mathrm{Vol}(M)$ equipped with L^2-metric (5.2) turns out to be flat.

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[8] Indeed, the closure of C^∞ metrics in any Sobolev H^s norm with $s > n/2$ is an open subset of $H^s(S^2T^*M)$.

[9] The space $\mathrm{Vol}(M)$ of volume forms on M contains the codimension one submanifold $\mathrm{Dens}(M) \subset \mathrm{Vol}(M)$ of those forms whose total volume is normalized.

We now explain how these structures relate to our paper. Observe that the pull-back actions of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ and $\mathrm{Vol}(M)$ (and similarly, the action of $\mathrm{Diff}_\mu(M)$ on $\mathrm{Met}_\mu(M)$) leave the corresponding metrics (5.1) and (5.2) invariant. This allows one to construct geometrically natural right-invariant metrics on the orbits of a (suitably chosen) metric or volume form.

We first consider the action of the full diffeomorphism group $\mathrm{Diff}(M)$ on the space of Riemannian metrics $\mathrm{Met}(M)$.

Theorem 5.1. *If $g \in \mathrm{Met}(M)$ has no nontrivial isometries, then the map $\mathcal{P}_g \colon \mathrm{Diff}(M) \to \mathrm{Met}(M)$ is an immersion, and the metric (5.1) induces a right-invariant metric on $\mathrm{Diff}(M)$ given at the identity by*

$$\begin{aligned}\langle\!\langle u, v\rangle\!\rangle &= \langle\!\langle \mathcal{L}_u g, \mathcal{L}_v g\rangle\!\rangle_g \\ &= 2\int_M \langle du^\flat, dv^\flat\rangle\, d\mu + 4\int_M \langle \delta u^\flat, \delta v^\flat\rangle\, d\mu - 4\int_M \mathrm{Ric}(u,v)\, d\mu, \end{aligned} \tag{5.3}$$

for any vector fields $u, v \in T_e\mathrm{Diff}(M)$ and where Ric *stands for the Ricci curvature of M.*

Remark 5.2. If the metric g is Einstein then $\mathrm{Ric}(u,v) = \lambda\langle u, v\rangle$ for some constant λ and the induced metric in (5.3) becomes a special case of the Sobolev a-b-c metric (1.2) with $a = -4\lambda$, $b = 4$ and $c = 2$.

Proof. First, observe that the differential of the pull-back map $\mathcal{P}_g(\eta)$ with respect to η is given by the formula

$$(\mathcal{P}_g)_{*\eta}(v \circ \eta) = \eta^*(\mathcal{L}_v g),$$

for any $v \in T_e\mathrm{Diff}(M)$ and $\eta \in \mathrm{Diff}(M)$, where $\mathcal{L}_v$ stands for the Lie derivative. If g has no nontrivial isometries then it has no Killing fields and therefore the differential $\mathcal{P}_{g*}$ is a one-to-one map. The last identity in (5.3) involving the Ricci curvature is obtained by rewriting the inner product $\langle\!\langle u, v\rangle\!\rangle = \int_M \langle \mathcal{L}_u g, \mathcal{L}_v g\rangle\, d\mu$ explicitly in terms of d and δ. Right-invariance follows from invariance of the metric under the action of diffeomorphisms. □

Remark 5.3. More generally, if g has non-trivial isometries, then the above procedure yields a right-invariant metric on the homogeneous space $\mathrm{Diff}(M)/\mathrm{Iso}_g(M)$; see the diagram (5.5) below.

In exactly the same manner we obtain an immersion of the volumorphism group $\mathrm{Diff}_\mu(M)$ into $\mathrm{Met}_\mu(M)$.

Corollary 5.4. *If $g \in \mathrm{Met}_\mu(M)$ has no nontrivial isometries then the map $\mathcal{P}_g \colon \mathrm{Diff}_\mu(M) \to \mathrm{Met}_\mu(M)$ is an immersion and (5.1) restricts to a right-invariant metric on $\mathrm{Diff}_\mu(M)$.*

Finally, we perform an analogous construction for the action of $\mathrm{Diff}(M)$ on the space of volume forms $\mathrm{Vol}(M)$. In this case the isotropy subgroup is $\mathrm{Diff}_\mu(M)$ and we obtain a metric on the quotient space $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$.

PROPOSITION 5.5. *If μ is a volume form on M then the map $\mathcal{P}_\mu\colon \mathrm{Diff}(M) \to \mathrm{Vol}(M)$ defines an immersion of the homogeneous space $\mathrm{Dens}(M)$ into $\mathrm{Vol}(M)$ and the right-invariant metric induced by (5.2) has the form*

$$\langle\!\langle u, v\rangle\!\rangle = \langle\!\langle \mathcal{L}_u\mu, \mathcal{L}_v\mu\rangle\!\rangle_\mu = \frac{4}{n}\int_M \operatorname{div} u \cdot \operatorname{div} v \, d\mu. \tag{5.4}$$

Proof. The differential of the pullback map is

$$(\mathcal{P}_\mu)_{*\eta}(v \circ \eta) = \eta^*(\mathcal{L}_v\mu)$$

for any $v \in T_e\mathrm{Diff}(M)$ and $\eta \in \mathrm{Diff}(M)$. Right-invariance and the fact that $\mathcal{L}_v\mu = (\operatorname{div} v)\,\mu$ yields the desired formula. □

The three immersions described in Theorem 5.1, Corollary 5.4 and Proposition 5.5 can be summarized in the following diagram.

$$\begin{array}{ccccccc}
\mathrm{Iso}_g(M) & \xrightarrow{\text{emb}} & \mathrm{Diff}_\mu(M) & \xrightarrow{\text{proj}} & \mathrm{Diff}_\mu(M)/\mathrm{Iso}_g(M) & \xrightarrow{\mathcal{P}_g} & \mathrm{Met}_\mu(M) \\
\| & & \downarrow{\scriptstyle\text{emb}} & & \downarrow{\scriptstyle\text{emb}} & & \downarrow{\scriptstyle\text{emb}} \\
\mathrm{Iso}_g(M) & \xrightarrow{\text{emb}} & \mathrm{Diff}(M) & \xrightarrow{\text{proj}} & \mathrm{Diff}(M)/\mathrm{Iso}_g(M) & \xrightarrow{\mathcal{P}_g} & \mathrm{Met}(M) \\
\downarrow{\scriptstyle\text{emb}} & & \| & & \downarrow{\scriptstyle\text{proj}} & & \downarrow{\scriptstyle\Xi} \\
\mathrm{Diff}_\mu(M) & \xrightarrow{\text{emb}} & \mathrm{Diff}(M) & \xrightarrow{\text{proj}} & \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M) & \xrightarrow{\mathcal{P}_\mu} & \mathrm{Vol}(M)
\end{array} \tag{5.5}$$

The first three terms of each row in (5.5) form smooth fiber bundles in the obvious way. The third column is a smooth fiber bundle since $\mathrm{Iso}_g(M) \subset \mathrm{Diff}_\mu(M)$. The fourth column is a trivial fiber bundle which already appeared in [FG89].

REMARK 5.6. While curvatures of the spaces $\mathrm{Met}(M)$, $\mathrm{Met}_\mu(M)$ and $\mathrm{Vol}(M)$ have relatively simple expressions, the induced metrics above on the corresponding homogeneous spaces

$$\mathrm{Diff}(M)/\mathrm{Iso}_g(M), \quad \mathrm{Diff}_\mu(M)/\mathrm{Iso}_g(M) \quad \text{and} \quad \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$$

turn out to have complicated geometries (with the exception of $\mathrm{Dens}(M)$ discussed in the previous sections). For example, one can show that the sectional curvature of $\mathrm{Diff}(M)/\mathrm{Iso}_g(M)$ in the induced metric assumes both signs, see [KLMP11].

6 Applications and Discussion

Here we discuss connections of the above metrics on the space of densities to gradient flows, shape theory, and Fredholmness.

6.1 Gradient flows. The L^2-Wasserstein metric (2.10) on the space of densities was used to study certain dissipative PDE (such as the heat and porous medium equations) as gradient flow equations on $\mathrm{Dens}(M)$, see [Ott01, Vil09]. It turns out that the $\dot{H}^1$-metric yields the heat-like equation as a gradient equation on the infinite-dimensional L^2-sphere.

PROPOSITION 6.1. *The $\dot{H}^1$-gradients of the potentials*

$$H(f) = \int_M h(f)\, d\mu \quad \text{and} \quad F(f) = -\frac{1}{2}\int_M \langle \nabla f, \nabla f\rangle\, d\mu$$

where $f \in S_r^\infty \cap H^{s-1}(M)$ is the square root of the Radon–Nikodym derivative, $f^2 = d\lambda/d\mu$, on the space of densities and $s > n/2+2$, are given by the formulas

$$\operatorname{grad} H(f) = h'(f) - c_h f \quad \text{and} \quad \operatorname{grad} F(f) = \Delta f - cf$$

for any function $h \in C^\infty(\mathbb{R})$ with bounded derivatives and where ∇ and Δ denote the gradient and the Laplace–Beltrami operator on M. Here the constants c_h and c are given by

$$c_h = \mu(M)^{-1}\int_M h'(\mathrm{Jac}_\mu^{1/2}\eta)\,\mathrm{Jac}_\mu^{1/2}\eta\, d\mu,$$

$$c = -\mu(M)^{-1}\int_M |\nabla \mathrm{Jac}_\mu^{1/2}\eta|^2\, d\mu.$$

Sketch of proof. For a small real parameter ϵ and any mean-zero function β on M, write the expression

$$H(f+\epsilon\beta) = \int_M h(f+\epsilon\beta)\, d\mu = \int_M h(f)\, d\mu + \epsilon \int_M h'(f)\beta\, d\mu + \mathcal{O}(\epsilon^2).$$

Using the L^2 metric on $S_r^\infty \subset L^2(M, d\mu)$ to identify the variational derivative $\delta H/\delta f$ of H with its gradient $\operatorname{grad} H$, compute

$$\Big\langle \frac{\delta H}{\delta f}, \beta \Big\rangle = \frac{d}{d\epsilon} H(f+\epsilon\beta)\big|_{\epsilon=0} = \int_M h'(f)\beta\, d\mu,$$

which gives the gradient $\delta H/\delta f = h'(f)$ of H in the ambient L^2-space. To find the gradient of H on the space of densities, we need to project $\delta H/\delta f$ to the tangent space $T_f S_r^\infty$. This is equivalent to subtracting $c_h f$ with an appropriate coefficient c_h to make the result L^2-orthogonal to f itself. Under our assumptions, the difference $h'(f) - c_h f$ still belongs to H^{s-1}, and the whole argument can be carried out in the Sobolev framework. The computation of the gradient of F is similar. □

It follows from the above proposition that the associated gradient flow equation on the space $S_r^\infty \cap H^{s-1}(M)$ can be interpreted as the heat-like equation

$$\partial_t f = \operatorname{grad} F(f) = \Delta f - cf.$$

Observe that the heat equation can be obtained from the Boltzmann (relative) entropy functional $E(\lambda) = \int_M \lambda \log \lambda \, d\mu$ in the L^2-Wasserstein metric on the density space $\mathrm{Dens}(M)$; see e.g., [Ott01].

6.2 Shape theory. It is tempting to apply the distance $\mathrm{dist}_{\dot{H}^1}$ to problems of computer vision and shape recognition. Given a bounded domain E in the plane (a 2D "shape") one can mollify the corresponding characteristic function χ_E and associate with it (up to a choice of the mollifier) a smooth measure ν_E normalized to have total volume equal to 1. One can now use the above formula (3.2) to introduce a notion of "distance" between two 2D "shapes" E and F by integrating the product of the corresponding Radon–Nikodym derivatives with respect to the 2D Lebesgue measure.

In this context it is interesting to compare the spherical metric to other right-invariant Sobolev metrics that have been introduced in shape theory. For example, in [SM06] the authors proposed to study 2D "shapes" using a certain Kähler metric on the Virasoro orbits of type $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$.

This metric is particularly important because it is related to the unique complex structure on the Virasoro orbits. Furthermore, it has negative sectional curvature, which provides uniqueness of the corresponding geodesics.

The paper of Younes et al. [YMSM08] discusses a metric on the space of immersed curves which is also isometric to an infinite-dimensional round sphere and hence has explicit geodesics. Its relation with the above metric is similar to the relation of distances between the characteristic functions of shapes and between their boundary curves. In [You10] a one-dimensional version of (3.2) is used to define distances between densities on an interval.

6.3 Affine connections and duality. One of the problems in geometric statistics is to construct an infinite-dimensional theory of so-called dual connections (see [AN00], Section 8.4). In this section we describe a family of such connections $\nabla^{(\alpha)}$, as well as their geodesic equations, on the density space $\mathrm{Dens}(M)$ in the case when $M = S^1$, which generalize the α-connections of Chentsov [Che82].

Identify the space of densities with the set of circle diffeomorphisms which fix a prescribed point: $\mathrm{Dens}(S^1) \simeq \{\eta \in \mathrm{Diff}(S^1) : \eta(0) = 0\}$. Set $A = -\partial_x^2$ and given a smooth mean-zero periodic function u define the operator A^{-1} by

$$A^{-1}u(x) = -\int_0^x \int_0^y u(z)\, dz dy + x \int_0^1 \int_0^y u(z)\, dz dy.$$

Let v and w be smooth mean-zero functions on the circle and denote by $V = v \circ \eta$ and $W = w \circ \eta$ the corresponding vector fields on $\mathrm{Dens}(S^1)$. For any $\alpha \in \mathbb{R}$ define

$$\eta \to (\nabla_V^{(\alpha)} W)(\eta) = \Big(w_x v + \Gamma_e^{(\alpha)}(v, w) \Big) \circ \eta,$$

where

$$\Gamma_e^{(\alpha)}(v,w) = \frac{1+\alpha}{2} A^{-1}\partial_x(v_x w_x). \tag{6.6}$$

Following [AN00] we say that two connections ∇ and ∇^* on $\mathrm{Dens}(S^1)$ are dual with respect to $\langle\langle\cdot,\cdot\rangle\rangle$ if $U\langle\langle V,W\rangle\rangle = \langle\langle\nabla_U V, W\rangle\rangle + \langle\langle V, \nabla_U^* W\rangle\rangle$ for any smooth vector fields U, V and W. One can prove the following result.

Theorem 6.2.

(i) *For each $\alpha \in \mathbb{R}$ the map $\nabla^{(\alpha)}$ is a right-invariant torsion-free affine connection on* $\mathrm{Dens}(S^1)$ *with Christoffel symbols* $\Gamma^{(\alpha)}$.

(ii) $\nabla^{(0)}$ *is the Levi-Civita connection of the $\dot{H}^1$-metric* (3.1) *and $\nabla^{(-1)}$ is flat.*

(iii) *The connections $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ are dual with respect to the $\dot{H}^1$-metric for any $\alpha \in \mathbb{R}$.*

(iv) *The equation of geodesics of the affine α-connection $\nabla^{(\alpha)}$ coincides with the generalized Proudman–Johnson equation*

$$u_{txx} + (2-\alpha)u_x u_{xx} + u u_{xxx} = 0.$$

The cases $\alpha = 0$ and $\alpha = -1$ correspond to one-dimensional completely integrable systems: the HS equation (2.8) *and the μ-Burgers equation, respectively.*

For the latter statement we note that the equation for geodesics of $\nabla^{(\alpha)}$ on $\mathrm{Dens}(S^1)$ reads

$$\ddot{\eta} + \Gamma_\eta^{(\alpha)}(\dot{\eta},\dot{\eta}) = 0$$

where $\Gamma_\eta^{(\alpha)}$ is the right-translation of $\Gamma_e^{(\alpha)}$. Substituting $\dot{\eta} = u \circ \eta$ gives

$$u_t + u u_x + \Gamma_e^{(\alpha)}(u,u) = 0$$

and using (6.6) and differentiating both sides of the equation twice in the x variable completes the proof. The generalized Proudman–Johnson equation can be found e.g. in [Oka09].

Remark 6.3. From the formula (6.6) we see that the Christoffel symbols $\Gamma^{(\alpha)}$ do not lose derivatives. In fact, with a little extra work it can be shown that this implies that $\nabla^{(\alpha)}$ is a smooth connection on the H^s Sobolev completion of $\mathrm{Dens}(S^1)$ for $s > 3/2$. Consequently, one establishes the existence and uniqueness in H^s of local (in time) geodesics of $\nabla^{(\alpha)}$ using the methods of [MP10].

Dual connections of Amari have not yet been fully explored in infinite dimensions. We add here that as in finite dimensions [AN00] there is a simple relation between the curvature tensors of $\nabla^{(\alpha)}$ i.e. $R^{(\alpha)} = (1-\alpha^2)R^{(0)}$ where $R^{(0)}$ is the curvature of the round metric on $\mathrm{Dens}(S^1)$. It follows that the dual connections $\nabla^{(-1)}$ and $\nabla^{(1)}$ are flat and in particular there is a chart on $\mathrm{Dens}(S^1)$ in which the geodesics of the latter are straight lines.

6.4 The exponential map on $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$. Finally we describe the structure of singularities of the exponential map of our right-invariant $\dot{H}^1$-metric on the space of densities. Recall from Proposition 3.5 that the diameter of $\mathrm{Dens}(M)$ with respect to the metric (3.1) is equal to $\pi\sqrt{\mu(M)}/2$. This immediately implies the following.

PROPOSITION 6.4. *Any geodesic in* $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ *through the reference density is free of conjugate points.*

Using the techniques of [MP10] one can show that the Riemannian exponential map of (3.1) on $\mathrm{Dens}(M)$ is a nonlinear Fredholm map. In other words, its differential is a bounded Fredholm operator (on suitable Sobolev completions of tangent spaces) of index zero for as long as the solution is defined. The fact that this is true for the general right-invariant *a-b-c* metric given at the identity by (1.2) on $\mathrm{Diff}(M)$ or $\mathrm{Diff}_\mu(M)$ also follows from the results of [MP10]. More precisely, we have the following

Theorem 6.5. *For any Sobolev index $s > n/2 + 1$, the Riemannian exponential map of* (3.1) *on the quotient* $\mathrm{Diff}^s(M)/\mathrm{Diff}^s_\mu(M)$ *of the H^s completions is Fredholm up to the blowup time $t = T_{\max}$ given in* (4.8).

The proof of Fredholmness given in [MP10] is based on perturbation techniques. The basic idea is that the derivative of the exponential map along any geodesic $t \mapsto \eta(t) = \exp_e(tu_0)$ can be expressed as $(\exp_e)_{*tu_0} = t^{-1} dL_{\eta(t)} \Psi(t)$, where $\Psi(t)$ is a time dependent operator satisfying the equation

$$\Psi(t) = \int_0^t \Lambda(\tau)^{-1}\, d\tau + \int_0^t \Lambda(\tau)^{-1} B\big(u_0, \Psi(\tau)\big)\, d\tau, \tag{6.7}$$

and where $\Lambda = \mathrm{Ad}^*_\eta \mathrm{Ad}_\eta$ (as long as $t < T_{\max}$). If the linear operator $w \mapsto B(u_0, w)$ is compact for any sufficiently smooth u_0 then $\Psi(t)$ is Fredholm being a compact perturbation of the invertible operator defined by the integral $\int_0^t \Lambda(\tau)^{-1}\, d\tau$. In the same way one can check that this is indeed the case for the homogeneous space of densities with the right-invariant metric (3.1). We will not repeat the argument here and refer to [MP10] for details.

REMARK 6.6. We emphasize that the perturbation argument described above works only for sufficiently short geodesic segments in the space of densities. Recall that for the round sphere in a Hilbert space the Riemannian exponential map cannot be Fredholm for a sufficiently long geodesic because any geodesic starting at one point has a conjugate point of infinite order at the antipodal point. In the case of the metric (3.1) on the space of densities one checks that $\|\Lambda(t)^{-1}\| \nearrow \infty$ as $t \nearrow T_{\max}$ since it depends on the C^1 norm of η via the adjoint representation. Therefore the argument of [MP10] breaks down here past the blowup time as equality (6.7) becomes invalid.[10]

Appendix A: The Euler–Arnold Equation of the *a*-*b*-*c* Metric

In this Appendix we compute the general Euler–Arnold equation for the a-b-c metric (1.2) on the full diffeomorphism group $\mathrm{Diff}(M)$, and consider the degenerations of the metric in case one or more of the parameters vanish. It is convenient to proceed with the derivation of the Euler–Arnold equation in the language of differential forms. As usual, the symbols $\flat$ and $\sharp = \flat^{-1}$ denote the isomorphisms between vector fields and one-forms induced by the Riemannian metric on M. While we use d and δ notations throughout, we will continue to employ the more familiar formulas when available. For example, in any dimension we have $\delta u^\flat = -\operatorname{div} u$ for any vector field u, while if $n=1$ then $du^\flat = 0$. For $n=1$ the metric (1.2) simplifies to

$$\langle\!\langle u, v\rangle\!\rangle = a\int_{S^1} uv\,dx + b\int_{S^1} u_x v_x\,dx.$$

Recall also that the (regular) dual $T_e^*\mathrm{Diff}(M)$ of the Lie algebra $T_e\mathrm{Diff}(M)$ admits the orthogonal Hodge decomposition[11]

$$T_e^*\mathrm{Diff}(M) = d\Omega^0(M) \oplus \delta\Omega^2(M) \oplus \mathcal{H}^1, \tag{A.1}$$

where $\Omega^k(M)$ and $\mathcal{H}^k$ denote the spaces of smooth k-forms and harmonic k-forms on M, respectively.

We now proceed to derive the Euler–Arnold equation of the a-b-c metric (1.2). Let $A : T_e\mathrm{Diff}(M) \to T_e^*\mathrm{Diff}(M)$ be the self-adjoint elliptic operator

$$Av = av^\flat + bd\delta v^\flat + c\delta dv^b \tag{A.2}$$

(the inertia operator) so that

$$\langle\!\langle u, v\rangle\!\rangle = \int_M \langle Au, v\rangle\,d\mu, \tag{A.3}$$

for any pair of vector fields u and v on M.

Theorem A.1. *The Euler–Arnold equation of the general Sobolev H^1 metric (1.2) on* $\mathrm{Diff}(M)$ *has the form*

$$\begin{aligned} Au_t = &-a\Big((\operatorname{div} u)\,u^\flat + \iota_u du^\flat + d\langle u,u\rangle\Big) - b\Big((\operatorname{div} u)\,d\delta u^\flat + d\iota_u d\delta u^\flat\Big)\\ &-c\Big((\operatorname{div} u)\,\delta du^\flat + \iota_u d\delta du^\flat + d\iota_u\delta du^\flat\Big) \end{aligned} \tag{A.4}$$

where A is given by (A.2) and u is assumed to be a time-dependent vector field of Sobolev class H^s with $s > \frac{n}{2}+1$ on the manifold M.

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[10] It is tempting to interpret this phenomenon as the infinite multiplicity of conjugate points on the Hilbert sphere *forcing* the classical solutions of (4.2) to break down before the conjugate point is reached.

[11] Orthogonality of the components in (A.1) is established for suitable Sobolev completions with respect to the induced metric on differential forms $\langle\!\langle \alpha^\sharp, \beta^\sharp\rangle\!\rangle$.

Proof. By definition (2.2) of the bilinear operator B, for any vectors u, v and w in $T_e\mathrm{Diff}(M)$ we have

$$\langle\!\langle B(u,v), w\rangle\!\rangle = \langle\!\langle u, \mathrm{ad}_v w\rangle\!\rangle = -\int_M \langle Au, [v,w]\rangle\, d\mu. \tag{A.5}$$

Integrating over M the following identity

$$\langle Au, [v,w]\rangle = \big\langle d\langle Au, w\rangle, v\big\rangle - \big\langle d\langle Au, v\rangle, w\big\rangle - dAu(v,w)$$

and using

$$\int_M \big\langle d\langle Au, w\rangle, v\big\rangle\, d\mu = -\int_M \langle Au, w\rangle \operatorname{div} v\, d\mu$$

we get

$$\langle\!\langle u, \mathrm{ad}_v w\rangle\!\rangle = \int_M \big\langle (\operatorname{div} v)Au + d\langle Au, v\rangle + \iota_v dAu, w\big\rangle\, d\mu.$$

On the other hand, we have

$$\langle\!\langle B(u,v), w\rangle\!\rangle = \int_M \langle AB(u,v), w\rangle\, d\mu$$

and, since w is an arbitrary vector field on M, comparing the two expressions above, we obtain

$$B(u,v) = A^{-1}\big((\operatorname{div} v)Au + d\langle Au, v\rangle + \iota_v dAu\big). \tag{A.6}$$

Setting $v = u$, isolating the coefficients a, b, and c, and using (2.1) yields the equation (A.4). The simplification in the b term comes from $d^2 = 0$.

The requirements on the smoothness of vector fields u follow from the Hilbert manifold structure on diffeomorphism groups, see Remark 3.3. □

Remark A.2 (Wellposedness of the Cauchy problem). In order to study wellposedness of the Cauchy problem for Euler–Arnold equation (A.4), it is convenient to switch to Lagrangian coordinates and consider the corresponding geodesic equation in the H^s Sobolev framework on $\mathrm{Diff}^s(M)$, with a suitably large Sobolev index s ($s > \frac{n}{2} + 1$). The right-invariant metric defined by (A.3) admits a smooth Levi-Civita connection on $\mathrm{Diff}^s(M)$, and therefore its geodesics can be constructed by Picard iterations as solutions to an ordinary differential equation on a smooth Hilbert manifold (cf. Remark 3.3). This approach has been employed in several particular cases listed in the remark below.

We point out however that the two Cauchy problems in the Lagrangian and Eulerian formulations are not equivalent. For example, for the Lagrangian framework, as a consequence of the fundamental theorem of ODE the geodesics η in $\mathrm{Diff}^s(M)$ will depend smoothly (with respect to H^s norms) on the initial data u_0. On the other hand, in the Eulerian setting the solution map $u_0 \to u(t)$ for the corresponding PDE (A.4) viewed as a map from H^s into $C([0,T], H^s)$, while retaining continuity in general, may not be even Lipschitz. This is essentially due to derivative loss which occurs upon changing back from Lagrangian to Eulerian coordinates, as it involves the inversion map $u(t) = \dot\eta(t) \circ \eta^{-1}(t)$.

Remark A.3. Special cases of the Euler–Arnold equation (A.4) include several well-known evolution PDE.

- For $n = 1$ and $a = 0$, we obtain the Hunter–Saxton equation (2.8).
- For $n = 1$ and $b = 0$, we get the (inviscid) Burgers equation $u_t + 3uu_x = 0$.
- For $n = 1$ and $a = b = 1$, we obtain the Camassa–Holm equation $u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$.
- For any n when $a = 1$ and $b = c = 0$ we get the multi-dimensional (right-invariant) Burgers equation $u_t + \nabla_u u + u(\operatorname{div} u) + \frac{1}{2}\nabla|u|^2 = 0$, referred to as the template matching equation.
- For any n and $a = b = c = 1$ we get the EPDiff equation $m_t + \mathcal{L}_u m + m \operatorname{div} u = 0$, where $m = u^\flat - \Delta u^\flat$; see e.g., [HMR98].

Now observe that if $a = 0$ then the a-b-c metric becomes degenerate and can only be viewed as a (weak) Riemannian metric when restricted to a subspace. There are three cases to consider.

(1) $a = 0, b \neq 0, c = 0$: the metric is nondegenerate on the homogeneous space $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ which can be identified with the space of volume forms or densities on M. This is our principal example of the paper, studied in Sections 3 and 4.

(2) $a = 0, b = 0, c \neq 0$: the metric is nondegenerate on the group of (exact) volumorphisms and the Euler–Arnold equation is (A.8), see Corollary A.5 below.

(3) $a = 0, b \neq 0, c \neq 0$: the metric is nondegenerate on the orthogonal complement of the harmonic fields. This is neither a subalgebra nor the complement of a subalgebra in general and thus the approach of taking the quotient modulo a subgroup developed in the other cases cannot be applied here. However, in the special case when M is the flat torus $\mathbb{T}^n$ the harmonic fields are the Killing fields which do form a subalgebra (whose subgroup $\mathrm{Isom}(\mathbb{T}^n)$ consists of the isometries). In this case we get a genuine Riemannian metric on the homogeneous space $\mathrm{Diff}(\mathbb{T}^n)/\mathrm{Isom}(\mathbb{T}^n)$.

In cases (1) and (3) above one needs to make sure that the degenerate (weak Riemannian) metric descends to a non-degenerate metric on the quotient. This can be verified using the general condition (2.7) in Proposition 2.3. We have already done this for case (1) in Theorem 4.1; the proof for case (3) is similar.

We now return to the nondegenerate a-b-c metric ($a \neq 0$) and restrict it to the subgroup of volumorphisms (or exact volumorphisms). Observe that one obtains the corresponding Euler–Arnold equations with $b = 0$ directly from (A.4) using appropriate Hodge projections.

COROLLARY A.4. *The Euler–Arnold equation of the a-b-c metric (1.2) restricted to the subgroup $\mathrm{Diff}_\mu(M)$ has the form*

$$au_t^\flat + c\delta du_t^\flat + a\iota_u du^\flat + c\iota_u d\delta du^\flat = d\Delta^{-1}\delta\left(a\iota_u du^\flat + c\iota_u d\delta du^\flat\right). \tag{A.7}$$

The Euler–Arnold equation (A.7) is closely related to the H^1 Euler-α equation which was proposed as a model for large-scale motions by Holm, Marsden and Ratiu [HMR98]; in fact if the first cohomology is trivial they are identical (with $\alpha^2 = c/a$).

There is also a "degenerate analogue" of the latter equation which corresponds to the case where $a = b = 0$:

COROLLARY A.5. *The Euler–Arnold equation of the right-invariant metric (1.2) with $a = b = 0$ on the subgroup of exact volumorphisms is*

$$\delta du_t^\flat + P\mathcal{L}_u(\delta du^\flat) = 0, \tag{A.8}$$

where P is the orthogonal Hodge projection onto $\delta\Omega^2(M)$.

This represents a limiting case of the Euler-α equation as $\alpha \to \infty$.

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