CHAPTER 5

A Mathematical Trivium

V. I. Arnold

The standard of mathematical culture is falling; both undergraduate and graduate students educated at our colleges, including the Mechanics and Mathematics Department of Moscow State University, are becoming no less ignorant than the professors and teachers. What is the reason for this abnormal phenomenon? Following the general principle of propagation of knowledge, under normal conditions students usually know their subject better than their professors: new knowledge prevails not because old men learn it, but because new a generation enters the field already knowing it.

Among the many causes of this abnormal situation I would like to single out those for which we are responsible ourselves, so that we can try to correct what is within our reach. One such cause, I believe, is our system of examinations, which is specially designed for the systematic production of rejects, that is, pseudo-scientists who learn mathematics like Marxism: they cram themselves with formulae and rote-learning of answers to the most frequent examination questions.

How can the standard of training of a mathematician be measured? Neither a list of courses nor their syllabi determine the standard. The only way to determine what we have actually taught our students is to list the problems which they should be able to solve as a result of our instruction.

I am not talking about some difficult problems, but rather about those simple questions which form the absolutely essential minimum. There need not necessarily be many of these problems, but we must insist that the students are able to solve them. I. E. Tamm used to tell the story that having fallen into the hands of the Makhnovtsy[1] during the Russian Civil War, he said under interrogation that he was a student at the physics and mathematics department. He saved his life only because he could solve a problem in the theory of infinite series he was immediately given to check if he is telling the truth. Our students should be prepared for such ordeals!

Throughout the world a mathematics exam consists of presenting written solution of problems. The written character of the test is everywhere considered just as much a necessary attribute of a democratic society as the choice between several candidates in an election. In fact, in an oral examination the student is completely defenceless. While conducting examinations in the Division of Differential Equations at the Mechanics and Mathematics Department of Moscow State University,

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[1] *Editors' remark*: Nestor Makhno was the commander of an independent anarchist army in Ukraine that led a guerrilla campaign during the Russian Civil War.

I overheard examiners at a nearby table failing students who gave immaculate answers (which perhaps exceeded the level of comprehension of the examiner). Also, there have been cases where they examiners have failed a student on purpose (sometimes I was able to save the situation by entering the examination room).

Written work is a document and the examiner has to be more objective in grading it (particularly if the work to be graded is anonymous, as it should be).

There is another important advantage of written examinations: the problems are preserved and can be published or passed on to the students of the next year to help them to prepare for their exams. In addition, these problems determine both the level of the course and the quality of the teacher who compiled them. The teacher's strong and weak points can be seen at once, and specialists can immediately assess the teacher both in what he wants to teach the students and how he succeeded in doing this.

Incidentally, in France the problems in the *Concours général*, common to the entire country and roughly equivalent to our Olympiad, are compiled by teachers sending their problems to Paris where the best are chosen. The Ministry obtains objective data about the standard of its teachers by comparing first the problem sets, and second the results of their students. However, in our situation teachers are assessed, as we all know, according to such indicators as their appearance, quickness of speech, and ideological "correctness".

It is not surprising that other countries are unwilling to recognize our diplomas (in future I think this will even extend to diplomas in mathematics). Assessments obtained from oral exams that leave no records cannot be objectively compared with anything else and have an extremely vague and relative value, wholly depending on standards of teaching and the requirements established in a particular college. With the same syllabi and grades the knowledge and ability of the graduates may vary (in an appropriate sense) by a factor of ten. In addition, an oral exam can be far more easily falsified (this has even happened with us at the Mechanics and Mathematics Department of Moscow State University, where, as a blind teacher once said, a good mark must be given to a student whose report is "very close to the textbook", even if he cannot answer a single question).

The essence and the shortcomings of our system of mathematical education have been brilliantly described by Richard Feynman in his memoirs (*Surely you're joking, Mr. Feynman* (Norton, New York 1984), in the chapter on physics education in Brazil, a Russian translation of which was published in *Uspekhi Fizicheskikh Nauk* **148**:3 (1986)).

In Feynman's words, these students understand nothing, but never ask questions, so that they appear to understand everything. If anybody begins to ask questions, he is quickly put in his place, as he is wasting the time of the lecturer dictating the lecture and of students copying it down. The result is that no one can apply anything they have been taught to even a single example. The examinations (dogmatic like ours: state the definition, state the theorem) are always successfully passed. The students reach a state of "self-propagating pseudo-education" and can teach future generations in the same way. But all this activity is completely meaningless, and in fact our output of specialists is to a significant extent craft, illusion, and sham: these so-called specialists cannot solve simplest problems, and do not possess even rudiments of their profession.

Thus, *to put an end to this spurious enhancement of the results, we must specify not a list of theorems, but a collection of problems which students should be able to solve. These lists of problems must be published annually (I think there should be ten problems for each one-semester course).* Then we shall see what we really teach our students and how successful we are. And in order for students to learn how to use their knowledge, *all examinations must be written examinations.*

Naturally the list of problems will vary from college to college and from year to year. Then one can compare levels of different teachers and of students graduating at different years. A student who takes much more than five minutes to calculate the mean value of $\sin^{100} x$ with 10% accuracy did not maste mathematics even if he has studied non-standard analysis, universal algebras, supermanifolds, or embedding theorems.

The compilation of model problems is a laborious job, but I think it must be done. As an attempt I give below a list of one hundred problems forming a mathematical minimum for a physics student. Model problems (unlike syllabi) are not uniquely defined, and many will probably not agree with me. Nonetheless I believe that it is necessary to start defining mathematical standards using written examinations and model problems. One wants to hope that in the future students will receive model problems for each course at the beginning of each semester, while oral examinations with cramming of rote learning will become a thing of the past.

(1) Given the graph of a function, sketch the graph of the derivative and the graph of the anti-derivative of this function.

(2) Find the limit

$$\lim_{x\to 0} \frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}.$$

(3) Find the critical values and critical points of the mapping $z \mapsto z^2 + 2\bar{z}$. (Sketch the answer.)

(4) Calculate the 100th derivative of the function

$$\frac{x^2+1}{x^3-x}.$$

(5) Calculate the 100th derivative of the function

$$\frac{1}{x^2+3x+2}$$

at $x = 0$ with 10% relative error.

(6) In the (x, y)-plane sketch the curve given parametrically by

$$x = 2t - 4t^3, \quad y = t^2 - 3t^4.$$

(7) How many normals to an ellipse can be drawn from a given point of the plane? Find the region in which the number of normals is maximal.

(8) How many maxima, minima, and saddle points does the function

$$x^4 + y^4 + z^4 + u^4 + v^4$$

have on the surface

$$x + \cdots + v = 0, \quad x^2 + \cdots + v^2 = 1, \quad x^3 + ... + v^3 = C?$$

(9) Does every positive polynomial in two real variables attain its lower bound in the plane?

(10) Investigate the asymptotic behaviour of the solutions y of the equation $x^5 + x^2y^2 = y^6$ that tend to zero as $x \to 0$.

(11) Investigate the convergence of the integral

$$\iint_{-\infty}^{+\infty} \frac{dx\, dy}{1 + x^4y^4}.$$

(12) Find the flux of the vector field $\vec{r}/r^3$ through the surface

$$(x-1)^2 + y^2 + z^2 = 2.$$

(13) Calculate

$$\int_1^{10} x^x\, dx$$

with 5% relative error.

(14) Calculate

$$\int_{-\infty}^{\infty} (x^4 + 4x + 4)^{-100}\, dx$$

with at most 10% relative error.

(15) Calculate

$$\int_{-\infty}^{\infty} \cos(100(x^4 - x))\, dx$$

with 10% relative error.

(16) What fraction of the volume of a 5-dimensional cube is the volume of the inscribed ball? What fraction is it of a 10-dimensional cube?

(17) Find the distance between the center of gravity of a uniform 100-dimensional solid half-ball of radius 1 and the center of the sphere with 10% relative error.

(18) Calculate

$$\int \cdots \int e^{-\sum_{1 \le i \le j \le n} x_i x_j}\, dx_1 \cdots dx_n.$$

(19) Investigate the path of a light ray in a plane medium with refractive index $n(y) = y^4 - y^2 + 1$, using Snell's law $n(y)\sin\alpha = \text{const}$, where α is the angle made by the ray with the y-axis.

(20) Find the derivative of the solution of the equation $\ddot{x} = x + A\dot{x}^2$, with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$, with respect to the parameter A for $A = 0$.

(21) Find the derivative of the solution of the equation $\ddot{x} = \dot{x}^2 + x^3$ with initial conditions $x(0) = 0$, $\dot{x}(0) = A$ with respect to A for $A = 0$.

(22) Investigate the boundary of the domain of stability ($\max \operatorname{Re} \lambda_j < 0$) in the space of coefficients of the equation $\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$.

(23) Solve the quasi-homogeneous equation

$$\frac{dy}{dx} = x + \frac{x^3}{y}.$$

(24) Solve the quasi-homogeneous equation

$$\ddot{x} = x^5 + x^2\dot{x}.$$

(25) Can an asymptotically stable equilibrium position become unstable in the Lyapunov sense under linearization?

(26) Investigate the behaviour as $t \to +\infty$ of solutions of the systems

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 2\sin y - y - x, \end{cases} \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = 2x - x^3 - x^2 - \varepsilon y, \end{cases}$$

where $\varepsilon \ll 1$.

(27) Sketch the images of the solutions of the equation

$$\ddot{x} = F(x) - k\dot{x}, \quad F = -dU/dx,$$

in the (x, E)-plane, where $E = \dot{x}^2 + U(x)$, near non-degenerate critical points of the potential U.

(28) Sketch the phase portrait and investigate how it depends on variation of the small complex parameter ε:

$$\dot{z} = \varepsilon z - (1 + i)z|z|^2 + \bar{z}^4.$$

(29) A charge moves with velocity 1 in a plane under the action of a strong magnetic field $B(x, y)$ perpendicular to the plane. To which side will the center of the Larmor circle drift? Calculate the velocity of this drift (to a first approximation). [Mathematically, this concerns the curves of curvature NB as $N \to \infty$.]

(30) Find the sum of the indexes of the singular points other than zero of the vector field $z\bar{z}^2 + z^4 + 2\bar{z}^4$.

(31) Find the index of the singular point 0 of the vector field with components $(x^4 + y^4 + z^4, x^3y - xy^3, xyz^2)$.

(32) Find the index of the singular point 0 of the vector field

$$\operatorname{grad}(xy + yz + zx).$$

(33) Find the linking coefficient of the phase trajectories of the equation of small oscillations $\ddot{x} = -4x, \ddot{y} = -9y$ on a level surface of the total energy.

(34) Investigate the singular points on the curve $y = x^3$ in the projective plane.

(35) Sketch the geodesics on the surface

$$(x^2 + y^2 - 2)^2 + z^2 = 1.$$

(36) Sketch the evolvent of the cubic parabola $y = x^3$. (The evolvent is the locus of the points $\vec{r}(s) + (c - s)\dot{\vec{r}}(s)$, where s is the arc-length of the curve $\vec{r}(s)$ and c is a constant).

(37) Prove that in Euclidean space the surfaces

$$((A - \lambda E)^{-1}x, x) = 1$$

passing through the point x and corresponding to different values of λ are pairwise orthogonal. (A is a symmetric operator without multiple eigenvalues.)

(38) Calculate the integral of the Gaussian curvature of the surface

$$z^4 + (x^2 + y^2 - 1)(2x^2 + 3y^2 - 1) = 0.$$

(39) Calculate the Gauss integral

$$\iint \frac{(d\vec{A}, d\vec{B}, \vec{A} - \vec{B})}{|\vec{A} - \vec{B}|^3},$$

where $\vec{A}$ runs along the curve $x = \cos\alpha$, $y = \sin\alpha$, $z = 0$, and $\vec{B}$ along the curve $x = 2\cos^2\beta$, $y = \frac{1}{2}\sin\beta$, $z = \sin 2\beta$.

(40) Find the parallel displacement of a vector pointing north at Leningrad (latitude $60°$) from West to East along a closed parallel.

(41) Find the geodesic curvature of the line $y = 1$ in the upper half-plane with the Lobachevskii–Poincaré metric

$$ds^2 = (dx^2 + dy^2)/y^2.$$

(42) Do the medians of a triangle meet in a single point in the Lobachevskii plane? What about the altitudes?

(43) Find the Betti numbers of the surface $x_1^2 + \cdots + x_k^2 - y_1^2 - \cdots - y_l^2 = 1$ and the set $x_1^2 + \cdots + x_k^2 \leq 1 + y_1^2 + \cdots + y_l^2$ in the $(k + l)$-dimensional vector space.

(44) Find the Betti numbers of the surface $x^2 + y^2 = 1 + z^2$ in the three-dimensional projective space. The same for the surfaces $z = xy$, $z = x^2$, $z^2 = x^2 + y^2$.

(45) Find the self-intersection index of the surface $x^4 + y^4 = 1$ in the projective plane $\mathbb{CP}^2$.

(46) Map the interior of the unit disc conformally onto the first quadrant.

(47) Map the exterior of a disc conformally onto the exterior of a given ellipse.

(48) Map the half-plane without a segment perpendicular to its boundary conformally onto the halfplane.

(49) Calculate

$$\oint_{|z|=2} \frac{dz}{\sqrt{1 + z^{10}}}.$$

(50) Calculate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + x^2}\, dx.$$

(51) Calculate the integral

$$\int_{-\infty}^{\infty} e^{ikx}\, \frac{1 - e^x}{1 + e^x}\, dx.$$

(52) Calculate the first term of the asymptotic expansion as $k \to \infty$ of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx} dx}{\sqrt{1 + x^{2n}}}.$$

(53) Investigate the singular points of the differential form $dt = dx/y$ on the compact Riemann surface $y^2/2 + U(x) = E$, where U is a polynomial and E is not a critical value.

(54) Let $\ddot{x} = 3x - x^3 - 1$. In which of the potential wells is the period of oscillation greater (in the more shallow one or in the deeper one) for equal values of the total energy?

(55) Investigate topologically the Riemann surface of the function

$$w = \arctan z.$$

(56) How many handles does the Riemann surface of the function

$$w = \sqrt{1 + z^n}$$

Have?

(57) Find the dimension of the solution space of the problem $\partial u/\partial\bar{z} = \delta(z-i)$ for $\operatorname{Im} z \geq 0$, $\operatorname{Im} u(z) = 0$ for $\operatorname{Im} z = 0$, $u \to 0$ as $z \to \infty$.

(58) Find the dimension of the solution space of the problem $\partial u/\partial\bar{z} = a\delta(z-i) + b\delta(z+i)$ for $|z| \leq 2$, $\operatorname{Im} z = 0$ for $|z| = 2$.

(59) Investigate the existence and uniqueness of the solution of the problem $yu_x = xu_y$, $u|_{x=1} = \cos y$ in a neighbourhood of the point $(1, y_0)$.

(60) Is there a solution of the Cauchy problem

$$x(x^2+y^2)\frac{\partial u}{\partial x} + y^3\frac{\partial u}{\partial y} = 0, \quad u|_{y=0} = 1,$$

in a neighbourhood of the point $(x_0, 0)$ of the x-axis? Is it unique?

(61) What is the largest value of t for which the solution of the problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \sin x, \quad u|_{t=0} = 0,$$

can be extended to the interval $[0, t)$?

(62) Find all solutions of the equation $y\,\partial u/\partial x - \sin x\,\partial u/\partial y = u^2$ in a neighbourhood of the point $(0, 0)$.

(63) Is there a solution of the Cauchy problem $y\,\partial u/\partial x + \sin x \partial u/\partial y = y$, $u|_{x=0} = y^4$ on the whole (x, y) plane? Is it unique?

(64) Does the Cauchy problem $u|_{y=x^2} = 1$, $(\nabla u)^2 = 1$ have a smooth solution in the domain $y \geq x^2$? In the domain $y \leq x^2$?

(65) Find the mean value of the function $\ln r$ on the circle $(x-a)^2+(y-b)^2 = R^2$ (of the function $1/r$ on the sphere).

(66) Solve the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \text{ for } && x^2+y^2 < 1; \\ u &= 1 \quad \text{for } && x^2+y^2 = 1,\ y > 0; \\ u &= -1 \text{ for } && x^2+y^2 = 1,\ y < 0. \end{aligned}$$

(67) What is the dimension of the space of solutions of the problem

$$\Delta u = 0 \text{ for } x^2+y^2 > 1, \quad \partial u/\partial n = 0 \text{ for } x^2+y^2 = 1$$

that are continuous on $x^2+y^2 \geq 1$?

(68) Find

$$\inf \iint_{x^2+y^2 \leq 1} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy$$

for C^∞-functions u that vanish at 0 and are equal to 1 on $x^2+y^2 = 1$.

(69) Prove that the solid angle based on a given closed contour is a function of the vertex of the angle that is harmonic outside of the contour.

(70) Calculate the mean value of the solid angle by which the disc $x^2+y^2 \leq 1$ lying in the plane $z = 0$ is seen from points of the sphere $x^2+y^2+(z-2)^2 = 1$.

(71) Calculate the charge density on the conducting boundary $x^2+y^2+z^2 = 1$ of a cavity in which a charge $q = 1$ is placed at distance r from the center.

(72) Calculate to the first order in ε the effect that the flattening of the earth ($\varepsilon \approx 1/300$) has on the gravitational field of the earth at the distance of the moon (assuming the earth to be homogeneous).

(73) Find (to the first order in ε) the influence of the imperfection of an almost spherical capacitor $R = 1 + \varepsilon f(\phi, \theta)$ on its capacity.

(74) Sketch the graph of $u(x,1)$, if $0 \le x \le 1$,

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \qquad u|_{t=0} = x^2, \quad u|_{x^2=x} = x^2.$$

(75) On account of the annual fluctuation of temperature the ground at a given town freezes to a depth of 2 metres. To what depth would it freeze on account of the daily fluctuation of the same amplitude?

(76) Investigate the behaviour at $t \to \infty$ of the solution of the problem

$$u_t + (u \sin x)_x = \varepsilon u_{xx}, \qquad u|_{t=0} \equiv 1, \quad \varepsilon \ll 1.$$

(77) Find the eigenvalues and their multiplicities of the Laplace operator $\Delta = \operatorname{div}\operatorname{grad}$ on a sphere of radius R in Euclidean space of dimension n.

(78) Solve the Cauchy problem

$$\frac{\partial^2 A}{\partial t^2} = 9\frac{\partial^2 A}{\partial x^2} - 2B, \quad \frac{\partial^2 B}{\partial t^2} = 6\frac{\partial^2 B}{\partial x^2} - 2A,$$

$$A|_{t=0} = \cos x, \quad B|_{t=0} = 0, \quad \left.\frac{\partial A}{\partial t}\right|_{t=0} = \left.\frac{\partial B}{\partial t}\right|_{t=0} = 0.$$

(79) How many solutions does the boundary-value problem

$$u_{xx} + \lambda u = \sin x, \quad u(0) = u(\pi) = 0$$

have?

(80) Solve the equation

$$\int_0^1 (x+y)^2 u(x)\, dx = \lambda u(y) + 1.$$

(81) Find the Green's function of the operator $d^2/dx^2 - 1$ and solve the equation

$$\int_{-\infty}^{\infty} e^{-|x-y|} u(y)\, dy = e^{-x^2}.$$

(82) For what values of the velocity c does the equation $u_t = u - u^2 + u_{xx}$ have a solution in the form of a travelling wave $u = \phi(x - ct)$, $\phi(-\infty) = 1$, $\phi(\infty) = 0$, $0 \le u \le 1$.

(83) Find solutions of the equation $u_t = u_{xxx} + uu_x$ in the form of a travelling wave $u = \phi(x - ct)$, $\phi(\pm\infty) = 0$.

(84) Find the number of positive and negative squares in the canonical form of the quadratic form $\sum_{i<j}(x_i - x_j)^2$ in n variables. The same for the form $\sum_{i<j} x_i x_j$.

(85) Find the lengths of the principal axes of the ellipsoid

$$\sum_{i \le j} x_i x_j = 1$$

(86) Through the center of a cube (tetrahedron, icosahedron) draw a line in such a way that the sum of the squares of its distances from the vertices is a) minimal, b) maximal.

(87) Find the derivatives of the lengths of the semiaxes of the ellipsoid $x^2 + y^2 + z^2 + xy + yz + zx = 1 + \varepsilon xy$ with respect to ε at $\varepsilon = 0$.

(88) List all the figures that can be obtained by intersecting the infinite-dimensional cube $|x_k| \le 1$, $k = 1, 2, \dots$ with a two-dimensional plane?

(89) Calculate the sum of vector products $[[x,y],z] + [[y,z],x] + [[z,x],y]$,

(90) Calculate the sum of matrix commutators $[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$, where $[A,B] = AB - BA$.

(91) Find the Jordan normal form of the operator $e^{d/dt}$ in the space of quasi-polynomials $\{e^{\lambda t}p(t)\}$ where the degree of the polynomial p is less than 5, and of the operator $\mathrm{ad}_A : B \mapsto [A,B]$, in the space of $n \times n$ matrices B, where A is a diagonal matrix.

(92) Find the orders of the subgroups of the rotation group of the cube, and find its normal subgroups.

(93) Decompose the space of functions defined on the vertices of a cube into invariant subspaces irreducible with respect to the group of a) its symmetries, b) its rotations.

(94) Decompose a 5-dimensional real vector space into the irreducible invariant subspaces of the group generated by cyclic permutations of the basis vectors.

(95) Decompose the space of homogeneous polynomials of degree 5 in (x, y, z) into irreducible subspaces invariant with respect to the rotation group $SO(3)$.

(96) Each of 3600 subscribers of a telephone exchange calls it once an hour on average. What is the probability that in a given second 5 or more calls are received? Estimate the mean interval of time between two such seconds $(i, i+1)$.

(97) A particle performing a random walk on the integer points of the semi-axis $x \geq 0$ moves a distance 1 to the right with probability a, and to the left with probability b, and stands still in the remaining cases (if $x = 0$, it stands still instead of moving to the left). Determine the steady-state probability distribution, and also the expected value of x and x^2 over a long time, if the particle starts at the point 0.

(98) In the game of "Fingers", N players stand in a circle and simultaneously thrust out their right hands, each with a certain number of fingers showing. The total number of fingers shown is counted out round the circle from the leader, and the player on whom the count stops is the winner. How large must N be for a suitably chosen group of $N/10$ players to contain a winner with probability at least 0.9? How does the probability that the leader wins behave as $N \to \infty$?

(99) One player conceals a 10 or 20 copeck coin, and the other guesses its value. If he is right he gets the coin, if wrong he pays 15 copecks. Is this a fair game? What are the optimal mixed strategies for both players?

(100) Find the mathematical expectation of the area of the projection of a cube with edge of length 1 onto a plane with an isotropically distributed random direction of projection.

CHAPTER 6

Comments on "A Mathematical Trivium"

Boris Khesin and Serge Tabachnikov

What follows are solutions, hints, and comments to some of the problems from Arnold's *Trivium.* This compendium is not uniform: some problems are standard exercises that every mathematician should do once in a lifetime (such as problem 90), some would be easy to those who have mastered a particular subject (for example, problem 99 is a test on the basics of game theory), and some are ingeniously constructed, akin to sophisticated chess problems (e.g., problem 2). The reader interested in a more-or-less complete set of solutions is referred to the blog [**21**] (in French).[1]

The problems in the Trivium cover a large part of mathematics but not all of it: one will not find number theoretical or combinatorial problems here. Of course, the selection of problems reflected mathematical interests and tastes of the author (at the time of writing). Overall, this collection represents well what Arnold expected from his students.

One of us (S.T.), when a graduate student at the Moscow State University, had to take a special topic examination from V. Arnold (every graduate student had to take three such exams). Arnold listened to the request and asked: "Can you draw a swallow tail?" (the discriminant surface of quartic polynomials $x^4 + ax^2 + bx + c$, see Figure 1). Only after this 'placement test' was more-or-less successfully passed, he agreed to discuss the matter.

When the other author of these comments (B.K.) asked Arnold to become his advisor (at the Moscow State this choice was - and still is - to be made during one's sophomore year), Arnold suggested first to solve "Test 1" given at the end of his book [**2**] as a take-home exam. As a matter of fact, almost all of Arnold's students (at least in the 80's) underwent a similar 'placement test'. We believe that that test served as a prototype for the whole "Mathematical Trivium" by Arnold. We present "Test 1" from [**2**] at the end of the comments.

One cannot help wondering what other mathematicians' (say Hilbert's or Poincaré's) Trivia would look like. By a wild flight of imagination, one could even think of those of Gauss, Euler, or Newton (the reader has noticed that Arnold's collection is somewhat tilted toward Newton and his time, as witnessed by the book [**3**]).

A brief comment on the name. The term *trivium* refers to the three subjects that were taught first at medieval universities: grammar, logic, and rhetoric. This was followed by the *quadrivium*, consisting of geometry, arithmetic, astronomy, and

[1] Warning: Some of the problems were translated to French with distortion, and this affected some solutions.

music. The trivium and quadrivium resulted in the seven liberal arts of classical study.

Note that Arnold also wrote a sequel, the paper "Mathematical Trivium II", which, unlike the list of 100 problems under discussion, consists of about a dozen typical final exams in courses (in analysis, differential equations, group theory, etc.) given at mathematics departments around the world, see [**5**].

A final remark: Mathematical Trivium overlaps with another collection of problems written by V. Arnold, this one for children from 5 to 15 years old [**8**]. Namely, problems 6, 13, 86 from the Trivium (all commented upon below) are included into [**8**], and problem 65 appears in both collections. Curiously, in the latter problem the students (i.e., the readers of Trivium) have to consider the 2-dimensional case, whereas the 'kids' (solving [**8**]) are to deal with the 3-dimensional case!

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2. The answer is 1, see [**3**]. To quote from that book:

> Here is an example of a problem that people like Barrow, Newton and Huygens would have solved in a few minutes and which present-day mathematicians are not, in my opinion, capable of solving quickly...

6. This is a curve with two cusps and one self-intersection. It appears (upside down) as a section of the swallowtail in $\mathbb{R}^3$, see Figure 1.

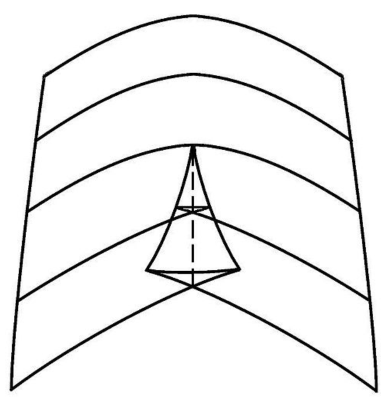

FIGURE 1. Swallowtail

7. The generic answer is 2 or 4, depending on whether the point is inside or outside of the evolute. At a smooth point of the evolute, the answer is 3, and at its cusp, the answer is 2. See Figure 2.

A version of this problem, to describe the set of points outside of an ellipse from which one can drop the greatest number of perpendiculars to the ellipse, was offered at the First All-Union Mathematical Olympiad for college students in 1974, see [**4**]. Out of 89 carefully selected participants, only one completely solved the problem, and 39 did not even attempt to solve it.

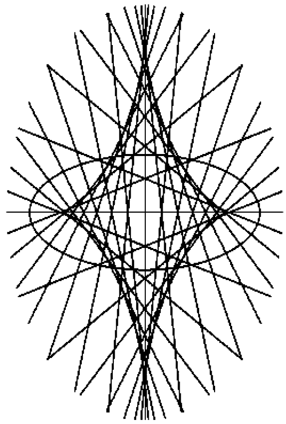

FIGURE 2. The evolute (the envelope of the normals) of an ellipse

9. No; example: $x^2+(xy-1)^2$. This problem was offered at the ‘Mathematical Wrangle’ (Matematicheskiy Boy) at the Soviet All-Union Mathematics Olympiad in 1973.

10. For a curve defined by a given polynomial in x and y one can find each branch of the curve with undetermined coefficients in the form of the Puiseux series $y(x) = \sum_{k=k_0}^{\infty} a_k x^{k/n}$ in fractional powers of x, where n and k_0 are defined by the leading term and slope of the segments facing the origin of the corresponding Newton polygon.

In this problem, after substituting $u = y^2$, in order to find expansion of the curve given by the equation $f(x,u) = x^5 + x^2u - u^3 = 0$ we first construct its Newton polygon; i.e., mark on the two-dimensional plane $\mathbb{Z}^2$ the integer points $A = (5,0), B = (2,1)$, and $C = (0,3)$, corresponding to the powers of monomials x^5, x^2u, and u^3, respectively, see Figure 3.

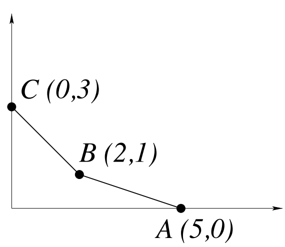

FIGURE 3. The Newton polygon of the polynomial $x^5 + x^2u - u^3$

Then the branches of the curve at the origin in the (x,u)-plane are described by the parts of the original equation corresponding to those sides of the Newton polygon that face the origin: $x^5 + x^2u = x^2(x^3+u)$ and $x^2u - u^3 = u(x^2 - u^2)$ corresponding to the segments $[A,B]$ and $[B,C]$, respectively. Then the first approximations of the branches of the curve defined by $x^5 + x^2u - u^3 = 0$ are the components of these curves $x^3 + u = 0$ and $x^2 - u^2 = 0$. Namely, the curve $x^3 + u = 0$ is a cubic parabola $u = -x^3$, so in terms of the initial variable y we obtain $y^2 = -x^3$; therefore, $y = ix^{3/2}$. This defines the asymptotics of the branch corresponding to $[A, B]$. Similarly, the equation $x^2 - u^2 = (x-u)(x+u) = 0$ defines two lines, $u = x$ and $u = -x$; i.e., two different branches, $y = x^{1/2}$ and $y = ix^{1/2}$ corresponding to the side $[B, C]$.

To find a more detailed asymptotic given by a full-fledged expansion one proceeds as follows. For the branch corresponding to the side $[A, B]$ one plugs the expression $u = -x^3(1 + \sum_{k=1}^{\infty} a_k x^k)$ with undetermined coefficients into the original equation. The other two branches, which correspond to $[B, C]$, are defined by plugging the expansions $u = \pm x(1 + \sum_{k=1}^{\infty} b_k^{\pm} x^k)$. To obtain the expansion in terms of y one can either take the square root of the above expansions for u or, alternatively, find the corresponding expansions for the branches directly from the original equation by using Puiseux series $y = ix^{3/2} + \dots$, $y = x^{1/2} + \dots$, and $y = ix^{1/2} + \dots$.[2]

12. Note that this vector field is $\nabla(-1/r)$, the gradient of the potential function $-1/r$. By the divergence theorem the given integral over the sphere equals the integral of the divergence of this field, $\operatorname{div}(\nabla(-1/r)) = \Delta(-1/r)$, over the ball bounded by the sphere. In turn, the function $1/r$ is a multiple of the Green function in $\mathbb{R}^3$, so its integral over a domain depends on whether the source point is inside the sphere or not.

13. The answer is surprisingly neat: $3 \cdot 10^9$.

Here is a sketch of a solution, adapted from [**21**]. Consider the function

$$f(x) = \frac{x^x}{\ln x + 1}.$$

Then,

$$x^x = f'(x) + \frac{x^{x-1}}{(\ln x + 1)^2};$$

hence,

$$\int_1^{10} x^x \, dx = f(10) - f(1) + \int_1^{10} \frac{x^{x-1}}{(\ln x + 1)^2} \, dx \approx f(10),$$

since x^x is sufficiently larger than $x^{x-1}/(\ln x + 1)^2$ and than $f(1) = 1$.

This problem was also offered at the First All-Union Mathematical Olympiad for college students in 1974. Only two students solved the problem, and 50 did not even attempt to solve it. See [**4**].

A version of this problem, to evaluate $\int_1^{100} x^x \, dx$, with 5% accuracy, is found in [**17**] (problem 174). The problem is attributed to M. Klamkin. The solution makes use of the inequalities

$$\frac{b^b - a^a}{1 + \ln b} \le \int_a^b x^x \, dx \le \frac{b^b - a^a}{1 + \ln a}.$$

As we mentioned above, this very problem appears in the list for kids from 5 to 15 [**8**]!

14–15. In both of these problems after some tricks, (see the corresponding solutions in [**21**]), one is taking the first two terms of an appropriate series expansion. One cannot help but quote from [**3**]:

[2] We are grateful to O.Viro for an improvement of this comment.

> As for the convergence, these series converge so rapidly that Newton, although he did not strictly prove convergence, had no doubt about it. He had the definition of convergence and explicitly calculated series for specific examples with an enormous number of digits (in the letter to Leibniz Newton wrote that he was ashamed to admit to how many digits he took these calculations).

16–17. The phenomenon of accumulation of volume near the boundary of the ball as the dimension goes to infinity is discussed; e.g., in [**20**].

18. The higher-dimensional Gauss integral

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} A_{ij}x_i x_j\right) d^n x = \sqrt{\frac{(2\pi)^n}{\det A}}$$

reduces the calculation to finding the determinant of the $n\times n$ matrix A. In our case, $A/2$ has 1's on the diagonal and 1/2's otherwise. Its determinant can be thought of as the Gram determinant for n unit vectors forming a regular tetrahedron in $\mathbb{R}^n$, that is, the vectors connecting one vertex of a unit regular tetrahedron in $\mathbb{R}^n$ with n other vertices. The Gram determinant is the square of the volume of the parallelotope formed by these vectors, which in turn, equals $(n+1)/2^n$.

19. The graph $y = 1/\sin x$ is a trajectory.

29. By definition, the Larmor circle is the osculating circle of the trajectory, and the locus of its centers is the evolute of the trajectory, the envelope of its normals. Therefore the center of the Larmor circle drifts in the direction orthogonal to the trajectory.

33. These phase trajectories give the Hopf fibration of the 3-dimensional sphere, the level surface of the total energy. Different Hooke coefficients imply that the periods of oscillations in the $(x,\dot{x})$- and the $(y,\dot{y})$-planes differ. For the common period, one trajectory traverses the circle twice, and the other trice.

34. The curve $y = x^3$ has a cusp at infinity: in the respective affine chart it is a semi-cubic parabola. The curve is projectively self-dual: the cusp at infinity corresponds to the inflection point at the origin. V. Arnold was interested in projectively self-dual curves and posed the problem of their classification in [**7**] (problem 1994-17).

35. Use Clairaut's theorem for geodesics on a surface of revolution: along such a geodesic, the quantity $\rho \sin\alpha$ is constant, where ρ is the distance to the axis of rotation, and α is the angle between the geodesic and meridians of the surface.

36. See Figure 4. Note that there is a 1-parameter family of involutes (also called evolvents): they form an equidistant family of curves.

Given a curve, move each point along the normal line the same distance t; the locus of these points is an equidistant curve. If the original curve is a source of light, the the equidistant curves consist of the points reached by light at a given time. If the initial curve is smooth then the equidistant curves are also smooth for small values of t but, typically, they eventually develop singularities.

The involutes of a given curve constitute an equidistant family. Equivalently, the evolutes of the equidistant curves coincide.

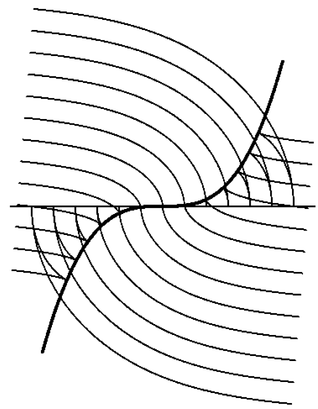

FIGURE 4. Involutes of a cubic parabola

37. This problem concerns the classical theory of elliptic coordinates, due to Jacobi. The dual problem reduces to the orthogonality of the eigen directions of a symmetric matrix. For proofs and discussions, see, e.g., [**11**].

38. This surface is a torus, its Euler characteristic is zero. By the Gauss-Bonnet theorem, the total curvature is also zero.

39. The Gauss integral is a multiple of the linking number of the corresponding curves.

40. This problem is solved using the Gauss-Bonnet theorem. See [**1**] where this question is discussed in the context of the Foucault pendulum (the one at the St. Isaac's Cathedral at St. Petersburg).

41. This curve is a horocycle; its curvature is 1.

42. Arnold was thinking about variations of this problem over the years. He gave a proof for the altitudes, deducing this fact from the Jacobi identity in the Lie algebra $sl(2,R)$ of isometries of the hyperbolic plane. A similar, but simpler, proof can be given in the spherical geometry (the Lie algebra $so(3)$). See [**9**], and [**14**] for an exposition. Incidentally, the notion of center of mass is also well defined in the spherical and hyperbolic geometry; see [**12**].

53. One can show that the 1-form dx/y has no poles or zeros for finite x and y. The problem reduces to the study of the point(s) at infinity, compactifying the Riemann surface.

54. The potential energy is a polynomial of degree 4, and hence the corresponding (compactified) energy surface is an elliptic curve (cf. problem 53). The form $dt = dx/y$, where $y = \dot{x}$, is a holomorphic form on the curve, while oscillations in the two wells correspond to homologically equivalent cycles.

61. The equation describes velocity of a one-dimensional gas in a force field. Its equation of characteristics is the system $\dot{x} = u, \dot{u} = \sin x$, which is the physical pendulum equation: $\ddot{x} = \sin x$. The pendulum trajectories near stable equilibria $x = \pi + 2\pi k$ behave like those of the mathematical pendulum equation $\ddot{x} = -x$ after the shift of variable. The projections from the (x,u)-plane onto the x-axis of the trajectories of the latter equation, that start at $u|_{t=0} = 0$, coincide after a quarter-period, that is, after $t = \pi/2$. Thus the solution u as a function of x should have different values at the same value of x starting at this time $t = \pi/2$: this corresponds to merging a shock wave for the gas motion. The same is true for the equation of physical pendulum, as x tends to any stable equilibrium. Hence the continuation of u as a function of x beyond $\pi/2$ is impossible.

62–63. Rewrite the equation as a Lie derivative of u along a vector field. For example, the equation of problem 62 is $L_v u = u^2$ where $v = y\partial/\partial x - \sin x\partial/\partial y$. Then make use of the topology of the field's orbits.

64. The problem reduces to investigating the smoothness of the fronts generated by the parabola (that is, its equidistant curves), inside and outside. Inside of the parabola, one encounters a focal point.

70. Use problem 69: such a solid angle is a harmonic function in $\mathbb{R}^3$ outside the circle. Hence its mean value on the sphere is equal to its value at the sphere center.

75. Heat propagation is described by the heat equation

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

where $T(x,t)$ is the temperature at depth x at time t. This equation is invariant under the one-parameter group $(x,t) \mapsto (cx, c^2 t)$. Therefore if the time is changed by a factor of 365 then the depth is changed by a factor of $\sqrt{365} \approx 19$. Hence the ground would freeze at depth of about 10 cm.

77. The eigenfunctions of the Laplacian on the unit sphere in $\mathbb{R}^n$ are the spherical functions, that is, the restrictions on the sphere of the harmonic polynomials in $\mathbb{R}^n$. The harmonic polynomials of the same degree correspond to the same eigenvalue of the Laplacian. The dimension of the space of harmonic polynomials of degree k equals $\binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}$. See, e.g., [**6**] for the theory of spherical functions.

83. This is the famous Korteweg–de Vries equation; the problem concerns its soliton solutions, see; e.g., [**15**].

84–85. The computations are simplified if one notices that the quadratic forms are symmetric and hence have $n-1$ equal eigenvalues.

86. The sum in question is the moment of inertia of the vertices about the line. Due to the symmetries of a regular polyhedron, its ellipsoid of inertia is a round sphere, and the sum of squares is the same for all lines.

This problem is discussed in [**10**] in the section "Symmetries (and the Curie Principle)". The problem is attributed to Landay and Lifshitz [**18**]. The Curie Dissymmetry Principle reads: a physical effect cannot have a dissymmetry absent from its efficient cause.

94. The irreducible components include two 2-dimensional and one 1-dimensional spaces, corresponding to two pairs of complex conjugated 5th roots of unity and 1. This decomposition is extensively discussed in [**3**] in the context of quasicrystals and the Penrose tilings.

99. (adapted from [**21**]) Let Player 1 conceal the 10-copeck coin with probability p (and respectively, he would conceal the 20-copeck coin with probability $1-p$), while we let Player 2 call the 10-copeck coin with probability q. Then the expectation value of the gain for Player 2 is

$$E = 10pq + 20(1-p)(1-q) - 15p(1-q) - 15(1-p)q$$

copecks. Rewrite this as follows: $E = 60(p - \frac{35}{60})(q - \frac{35}{60}) - \frac{35\cdot 35}{60} + 20$. Hence if $p > 35/60 = 7/12$, the second player has to choose q equal to 1, and if $p < 7/12$, the second player chooses q equal to 0 to maximize his gain. Similarly, if $q \neq 7/12$ the first player can choose an appropriate p to his advantage and bigger loss of the second player. The optimal strategy for both players is choosing $p = q = 7/12$. The expected gain for the second player in this case is $E = 20 - (35 \cdot 35)/60 = -25/60 = -5/12$ copecks. Thus the game is not fair: with the best strategy Player 2 loses at average 5/12 copecks in each round.

100. According to the Cauchy-Crofton formula, the area of a closed convex surface S in space is

$$\frac{1}{\pi}\int A(P_l(S))\,dl$$

where $P_l(S)$ is the projection onto the plane orthogonal to direction l and $A(P_l(S))$ is the area of this projection. The integral is over the sphere of all directions with respect to the uniform measure on the sphere. Applied to the unit cube, the average area of the projection is 3/2. See, e.g., [**13, 16**] for details.

More generally, given a convex body $K \subset \mathbb{R}^n$, one may consider the mean volume of its projection on a k-dimensional subspace. Up to universal constants, these quantities are called intrinsic volumes. They appear as coefficients in Steiner's formula for the volume of the ε-neighborhood of K: this volume is a polynomial of degree n in ε. If the boundary of K is a smooth hypersurface then these quantities are equal to the normalized integrals of the elementary symmetric polynomials of its principal curvatures. Arnold also discusses this matter in [**10**].

By the way, the same Steiner's formula (for large ε!) solves the following Box in a Box problem (see [**19**]):

> Let the cost of a rectangular box be given by the sum of its length, width, and height. Prove or disprove: It is impossible to fit a box into a cheaper box.

As we discussed in our introduction to these comments, we finish with another examination, taken from [**2**]. The reader will see many a familiar theme; we hope that (s)he will find this test interesting and instructive.

> In the four-hour written examination, 15 interrelated problems are given. Within square brackets, we indicate the point value of each problem. These values are revealed to the students beforehand.

Test 1

$$\ddot{x} = -\sin x + \varepsilon \cos t. \tag{1}$$

I. Let $\varepsilon = 0$.

(1) Linearize at the point $x = \pi,\ \dot{x} = 0$. [1]

(2) Is this equilibrium position stable? [1]

(3) Find the Jacobian matrix of the mapping of the phase flow at the point $x = \pi,\ \dot{x} = 0$ at time $t = 2\pi$. [3]

(4) Find the derivative of the solution with initial condition $x = \pi,\ \dot{x} = 0$ with respect to the parameter ε at $\varepsilon = 0$. [5]

(5) Draw the graph of the solution and its derivative with respect to t under the initial condition $x = 0,\ \dot{x} = 2$. [3]

(6) Find this solution. [3]

II. Let Eq. (2) be the linearized equation along the solution indicated in problem **5**.

(7) Does Eq. (2) have unbounded solutions? [8]

(8) Does Eq. (2) have nonzero bounded solutions? [8]

(9) Find the Wronskian of a fundamental system of solutions of Eq. (2) given that $W(0) = 1$. [5]

(10) Write out Eq. (2) explicitly and solve it. [10]

(11) Find the eigenvalues and eigenvectors of the monodromy operator for the linearized equation along the solution with initial condition $x = \pi/2,\ \dot{x} = 0$. [16]

(12) Prove that Eq. (1) has a 2π-periodic solution depending smoothly on ε and vanishing at $x = \pi$ for $\varepsilon = 0$. [6]

(13) Find the derivative of this solution with respect to ε at $\varepsilon = 0$. [6]

III. Consider the equation $u_t + uu_x = -\sin x$.

(14) Write out the equation of characteristics. [2]

(15) Find the largest value of t for which the solution of the Cauchy problem with $u|_{t=0} = 0$ can be extended to $[0, t)$. [8]

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