THE EULER AND NAVIER-STOKES EQUATIONS ON THE HYPERBOLIC PLANE

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ABSTRACT. We show that non-uniqueness of the Leray-Hopf solutions of the Navier–Stokes equation on the hyperbolic plane $\mathbb{H}^2$ observed in [CC] is a consequence of the Hodge decomposition. We show that this phenomenon does not occur on $\mathbb{H}^n$ whenever $n \geq 3$. We also describe the corresponding general Hamiltonian setting of hydrodynamics on complete Riemannian manifolds, which includes the hyperbolic setting.

INTRODUCTION

Consider the initial value problem for the Navier-Stokes equations on a complete $n$-dimensional Riemannian manifold $M$

\begin{align}
\frac{\partial v}{\partial t} + \nabla v v - L v &= -\text{grad} p, \quad \text{div } v = 0, \\
v(0,x) &= v_0(x).
\end{align}

The symbol $\nabla$ denotes the covariant derivative and $L = \Delta - 2r$ where $\Delta$ is the Laplacian on vector fields and $r$ is the Ricci curvature of $M$. Dropping the linear term $Lv$ from the first equation in (1) leads to the Euler equations of hydrodynamics

\begin{align}
\frac{\partial v}{\partial t} + \nabla v v &= -\text{grad} p, \quad \text{div } v = 0.
\end{align}

Most of the work on well-posedness of the Navier-Stokes equations has focused on the cases where $M$ is either a domain in $\mathbb{R}^n$ or the flat $n$-torus $\mathbb{T}^n$. In fundamental contributions J. Leray and E. Hopf established existence of an important class of weak solutions described as those divergence-free vector fields $v$ in $L^\infty([0, \infty), L^2) \cap L^2([0, \infty), H^1)$ which solve the Navier-Stokes equations in the sense of distributions and satisfy

\begin{align}
\|v(t)\|_{L^2}^2 + 4 \int_0^t \|\text{Def } v(s)\|_{L^2}^2 ds \leq \|v_0\|_{L^2}^2 \quad \text{and} \quad \lim_{t \searrow 0} \|v(t) - v_0\|_{L^2} = 0
\end{align}

for any $0 \leq t < \infty$ and where $\text{Def } v = \frac{1}{2}(\nabla v + \nabla v^T)$ is the so-called deformation tensor. When $n = 2$ using interpolation inequalities and energy estimates it is possible to show that the Leray-Hopf solutions are unique and regular but the problem is in general open for $n = 3$, see e.g. [CF] or [MB].

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There have also been studies on curved spaces, which with few exceptions have been confined to compact manifolds (possibly with boundary), see e.g. [Ta] and the references therein. In a recent paper Chan and Czubak [CC] studied the Navier-Stokes equation on the hyperbolic plane $\mathbb{H}^2$ and more general non-compact manifolds of negative curvature. In particular, using the results of Anderson [An] and Sullivan [Su] on the Dirichlet problem at infinity, they showed that in the former case the Cauchy problem (1)-(2) admits non-unique Leray-Hopf solutions.

Our goal in this note is to provide a direct formulation of the non-uniqueness of the Leray-Hopf solutions on $\mathbb{H}^2$ which turns out to rely on the specific form of the Hodge decomposition for 1-forms (or vector fields) in this case. We also show that no such phenomenon can occur in the hyperbolic space $\mathbb{H}^n$ with $n \geq 3$. As a by-product, we describe the corresponding Hamiltonian setting of the Euler equations on complete Riemannian manifolds (in particular, hyperbolic spaces).

We point out that this type of non-uniqueness cannot be found in the Euler equations. Furthermore, it is of a different nature than the examples constructed e.g., by Shnirelman [Sh] or De Lellis and Székelyhidi [DS]. On the other hand, it is similar to non-uniqueness of solutions of the Navier-Stokes equations defined in unbounded domains of the higher-dimensional Euclidean space, cf. Heywood [He].

1. Stationary harmonic solutions of the Euler equations

Our main result is summarized in the following theorem.

**Theorem 1.1.**

(i) There exists an infinite-dimensional space of stationary $L^2$ harmonic solutions of the Euler equations on $\mathbb{H}^2$.

(ii) There are no stationary $L^2$ harmonic solutions of the Euler equations on $\mathbb{H}^n$ for any $n > 2$.

**Proof.** Recall the Hamiltonian formulation of the Euler equations (3) on a complete Riemannian manifold $M$, see e.g. [AK]. Consider the Lie algebra $\mathfrak{g}_{\text{reg}} = \text{Vect}_\mu(M)$ of (sufficiently smooth) divergence-free vector fields on $M$ with finite $L^2$ norm. Its dual space $\mathfrak{g}_{\text{reg}}^*$ has a natural description as the quotient space $\Omega^1_{L^2}/d\Omega^0_{L^2}$ of the $L^2$ 1-forms modulo (the $L^2$ closure of) the exact 1-forms on $M$. Namely, the pairing between cosets $[\beta] \in \Omega^1_{L^2}/d\Omega^0_{L^2}$ of 1-forms $\beta \in \Omega^1_{L^2}$ and vector fields $w \in \text{Vect}_\mu(M)$ is given by

$$\langle [\beta], w \rangle := \int_M (\iota_w \beta) \, d\mu,$$

where $\iota_w$ is the contraction of a differential form with a vector field $w$, and $\mu$ is the Riemannian volume form on $M$.

Let $A : \mathfrak{g}_{\text{reg}} \to \mathfrak{g}_{\text{reg}}^*$ denote the inertia operator defined by the Riemannian metric. The operator $A$ assigns to a vector field $v \in \text{Vect}_\mu(M)$ the coset $[v^\flat]$ of the corresponding 1-form $v^\flat$ via the pairing given by the metric. The coset is defined as the 1-form up
to adding differentials of the $L^2$ functions on $M$. Thus, in the Hamiltonian framework the Euler equation reads
\[
\frac{d}{dt}[v^\flat] = -Lv[v^\flat],
\]
where $[v^\flat] \in \Omega^1_{L^2}/\delta\Omega^2_{L^2}$ and $Lv$ is the Lie derivative in the direction of the vector field $v$.

The space $\Omega^1_{L^2}$ of the $L^2$ 1-forms on a complete manifold $M$ admits the Hodge-Kodaira decomposition
\[
\Omega^1_{L^2} = \delta\Omega^2_{L^2} \oplus H^1_{L^2},
\]
where the first two summands denote the $L^2$ closures of the images of the operators $d$ and $\delta$, while $H^1_{L^2}$ is the space of the $L^2$ harmonic 1-forms on $M$. Therefore, we have a natural representation of the dual space
\[
g^*_{\text{reg}} = \delta\Omega^2_{L^2} \oplus H^1_{L^2}.
\]

It turns out that the summand of the harmonic forms in the above representation corresponds to steady solutions of the Euler equation. Namely, one has the following proposition.

**Proposition 1.2.** Each harmonic 1-form on a complete manifold $M$ which belongs to $L^2 \cap L^4$ defines a steady solution of the Euler equation \((3)\) on $M$.

**Proof of Proposition 1.2.** Let $\alpha$ be a bounded $L^2$ harmonic 1-form on $M$. Let $v_\alpha$ denote the divergence-free vector field corresponding to $\alpha$, i.e., $v^\flat_\alpha = \alpha$. Since the 1-form $\alpha$ is harmonic, using Cartan’s formula gives
\[
\frac{d}{dt}\alpha = -Lv_\alpha \alpha = -\iota_{v_\alpha} d\alpha - d\iota_{v_\alpha} \alpha = -d\iota_{v_\alpha} \alpha.
\]

We claim that $\iota_{v_\alpha} \alpha \in \Omega^0_{L^2}$ and consequently $d\iota_{v_\alpha} \alpha \in \delta\Omega^0_{L^2}$. Indeed, by the definition of the vector field $v_\alpha$ we have
\[
\|\iota_{v_\alpha} \alpha\|^2_{L^2} = \int_M (\alpha(v_\alpha))^2 \, d\mu = \|\alpha\|^4_{L^4},
\]
which is finite by assumption. It follows that the 1-form $d\iota_{v_\alpha} \alpha$ must correspond to the zero coset in the quotient space $g^*_{\text{reg}} = \Omega^1_{L^2}/\delta\Omega^0_{L^2}$, which in turn implies that
\[
\frac{d}{dt}\alpha = 0 \in g^*_{\text{reg}}.
\]

Therefore, the 1-form $\alpha$ defines a steady solution of the Euler equation, which proves the proposition. \[\square\]

If $M$ is compact then the space of harmonic 1-forms is always finite-dimensional (and isomorphic to the deRham cohomology group $H^1(M)$). According to a well-known result of Dodziuk \[Do\], the hyperbolic space $\mathbb{H}^n$ carries no $L^2$ harmonic $k$-forms except for $k = n/2$, in which case it is infinite-dimensional. Therefore, there can be no $L^2$ harmonic stationary solutions of the Euler equations on $\mathbb{H}^n$ for any $n > 2$, which proves part (ii) of the theorem.

To prove part (i) we note that for $n = 2$ the space of harmonic 1-forms on $\mathbb{H}^2$ is infinite-dimensional. Moreover, it allows for the following construction. Consider the
subspace \( S \subset H^1_{L^2} \) of 1-forms which are differentials of bounded harmonic functions whose differentials are in \( L^2 \)

\[
S = \{ d\Phi \mid \Phi \text{ is harmonic on } \mathbb{H}^2 \text{ and } d\Phi \in L^2 \}.
\]

It turns out that the subspace \( S \) is already infinite-dimensional. Indeed, let us consider the Poincaré model of \( \mathbb{H}^2 \), i.e., the unit disk \( D \) with the hyperbolic metric \( \langle \cdot, \cdot \rangle_h \), which we denote by \( D_h \). It is conformally equivalent to the standard unit disk with the Euclidean metric \( \langle \cdot, \cdot \rangle_e \), denoted by \( D_e \). Bounded harmonic functions on \( D_h \) can be obtained by solving the Dirichlet problem on \( D_e \), i.e., by constructing harmonic functions \( \Phi \) on \( D \) with boundary values \( \varphi \) prescribed on \( \partial D \). First, the 1-form \( d\Phi \) is clearly harmonic:

\[
\Delta d\Phi = d\delta d\Phi = d\Delta \Phi = 0.
\]

Secondly, observe that

\[
\|d\Phi\|^2_{L^2(D_h)} = \int_D \langle d\Phi, d\Phi \rangle_h \, d\mu_h = \int_D \det(g^{ij}) \langle d\Phi, d\Phi \rangle_e \det(g_{ij}) \, d\mu_e
\]

\[
= \int_D \langle d\Phi, d\Phi \rangle_e \, d\mu_e = \|d\Phi\|^2_{L^2(D_e)},
\]

and

\[
\|d\Phi\|^4_{L^4(D_h)} = \int_D \langle d\Phi, d\Phi \rangle^4 \, d\mu_h = \int_D \det^2(g^{ij}) \langle d\Phi, d\Phi \rangle^2 \det(g_{ij}) \, d\mu_e
\]

\[
= \int_D (1 - |z|^2)^2 \langle d\Phi, d\Phi \rangle^2 \, d\mu_e(\varphi) \leq \int_D \langle d\Phi, d\Phi \rangle^2 \, d\mu_e = \|d\Phi\|^4_{L^4(D_e)},
\]

where \( \det(g_{ij}) = 1/(1 - |z|^2)^2 \) is the determinant of the hyperbolic metric.

Furthermore, for sufficiently smooth boundary values \( \varphi \in C^{1+\alpha}(\partial D) \) there is a uniform upper bound for its harmonic extension inside the disk:

\[
|d\Phi(x)| \leq C\|\varphi\|_{C^{1+\alpha}(\partial D)}
\]

for any \( x \in D \) and \( 0 < \alpha < 1 \), and some positive constant \( C \), see e.g. [GT]. This implies that (for sufficiently smooth \( \varphi \)) the 1-forms \( d\Phi \) define an infinite-dimensional subspace \( S \) of harmonic forms in \( L^2 \cap L^4 \), which satisfy assumptions of the proposition above. It follows that they define an infinite-dimensional space of stationary solutions of the Euler equations on the hyperbolic plane \( \mathbb{H}^2 \). This completes the proof of Theorem 1.1. \( \square \)

2. Non-unique Leray-Hopf solutions of the Navier-Stokes equations

Using the fact that suitably rescaled steady solutions of the Euler equations also solve the Navier-Stokes system the authors in [CC] obtained a type of ill-posedness result for the Leray-Hopf solutions in the hyperbolic setting.

**Theorem 2.1 (CC).** Given a vector field \( v_e = (d\Phi)^2 \) on \( \mathbb{H}^2 \) there exist infinitely many real-valued functions \( f(t) \) for which \( v_{ns} = f(t)v_e \) is a weak solution of the Navier-Stokes equations with decreasing energy (i.e., satisfying the Leray-Hopf conditions).
An immediate consequence of this result and Theorem 1.1 is the following

**Corollary 2.2.** There exist infinitely many weak Leray-Hopf solutions to the Navier-Stokes equation on $\mathbb{H}^2$. There are no non-unique Leray-Hopf harmonic solutions to the Navier-Stokes equation on $\mathbb{H}^n$ with $n \geq 3$ arising from the above construction.

**Remark 2.3.** The phenomenon of nonuniqueness of solutions to the Navier-Stokes equation in unbounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, of higher-dimensional Euclidean spaces is of similar nature, see [He]. Indeed, that construction is based on the existence of a harmonic function with gradient in $L^2$ and appropriate boundary conditions in such domains. The Green function $\Phi(x) = G(a, x)$ centered at a point $a$ outside of $\Omega$ has the decay like $|G(a, x)| \sim |x|^{1-n}$ as $x \to \infty$, so that $|d\Phi(x)|^2 \sim |x|^{2-2n}$. Thus, for $n \geq 3$ the 1-forms $d\Phi$ belong to $L^2 \cap L^4$ on $\Omega$. The corresponding divergence-free vector fields $(d\Phi)^b$ provide examples of stationary Eulerian solutions in $\Omega$ (with nontrivial boundary conditions) and can be used to construct time-dependent weak solutions $v_{ns} = f(t)(d\Phi)^2$ to the Navier-Stokes equation in $\Omega$, as in Theorem 2.1.

### 3. Appendix

To make this note self-contained we provide here some details of the construction of the weak solutions given in [CC]. It will be convenient to rewrite the Navier-Stokes equations (1) in the language of differential forms

\begin{equation}
\partial_t v^b + \nabla v^b v^b - \Delta v^b + 2r(v^b) = -dp, \quad \delta v^b = 0
\end{equation}

where $\delta v^b = -\text{div} \, v$ and $\Delta v^b = d\delta v^b + \delta dv^b$ is the Laplace-deRham operator on 1-forms.

Let $v$ be the vector field $v_{ns} = f(t)(d\Phi)^2$ on $\mathbb{H}^2$ as in Theorem 2.1. Since the 1-form $d\Phi$ is harmonic one only needs to compute the covariant derivative term and the Ricci term:

$$\nabla v_{ns} v_{ns}^b = \frac{1}{2} f^2(t) d|d\Phi|^2 \quad \text{and} \quad 2r(v_{ns}^b) = -2f(t)d\Phi.$$  

Direct computation, taking into account the fact that for $\mathbb{H}^2$ we have $r = -1$, shows that both terms can be absorbed by the pressure term, so that the pair $(v_{ns}^b, p)$, where

$$p := \left(2f(t) - f'(t)\right)\Phi - 1/2f^2(t)|d\Phi|^2$$

satisfies the equations (3).

Finally, a quick inspection shows that any differentiable function $f(t)$ satisfying

$$f^2(t) + 4 \int_0^t f^2(s) \, ds \leq f^2(0)$$

yields a vector field $v_{ns}$ which satisfies the remaining conditions in (4) required of a Leray-Hopf solution.
References


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