For \( L = \partial^n + u_{n-2}(x)\partial^{n-2} + \ldots + u_0(x) \) a linear differential operator, the \((k, n)\)-KdV flow is defined as \( \frac{dL}{dt} = [Q^k_+ L] \), where \( Q \) is the \( n^{th} \) root of \( L \) and \( Q^k_+ \) is the differential part of \( Q^k \). Below are tables showing \( Q^k_+ \) and \([Q^k_+, L]\) for small values of \( n, k \).

### \( Q^k_+ \) for \( L \) of order \( n \)

| \( k \) \( n \) \( 1 \) \( 2 \) \( 3 \) \( 4 \) \( 5 \) \( 6 \) |
| --- | --- | --- | --- | --- | --- |
| \( k \) \( n \) \( 1 \) \( 2 \) \( 3 \) \( 4 \) \( 5 \) \( 6 \) |
| \( \partial^3 + \frac{1}{2} (3 u_0(x)) \partial^2 + \frac{1}{4} (3 u_0''(x)) \partial^1 \) \( \partial^3 + u_1(x) \partial^1 + u_0(x) \partial^0 \) \( \partial^3 + \frac{1}{4} (3 u_2(x)) \partial^1 + \frac{1}{8} (3 (2 u_1(x) - u_2''(x))) \partial^0 \) \( \partial^3 + \frac{1}{5} (3 u_3(x)) \partial^1 + \frac{1}{5} (3 (u_2(x) - u_3''(x))) \partial^0 \) \( \partial^3 + \frac{1}{3} u_4(x) \partial^1 + \frac{1}{4} (2 u_3(x) - 3 u_3''(x)) \partial^0 \) |
| \( \partial^4 + (2 u_0(x)) \partial^2 + (2 u_0''(x)) \partial^1 + (u_0(x)^2 + u_0'''(x)) \partial^0 \) \( \partial^4 + \frac{1}{3} (4 u_1(x)) \partial^2 + \frac{1}{3} (2 (u_0(x) + u_1''(x))) \partial^1 + \frac{1}{9} (2 (u_1(x)^2 + 3 u_0''(x) + u_1'''(x))) \partial^0 \) \( \partial^4 + \frac{1}{5} (4 u_3(x)) \partial^2 + \frac{1}{5} (2 (2 u_2(x) - u_3''(x))) \partial^1 + \frac{1}{25} (2 (-10 u_1(x) + u_3(x)^2 + 5 u_2''(x))) \partial^0 \) |
| \( \partial^5 + \frac{1}{2} (5 u_0(x)) \partial^3 + \frac{1}{4} (5 u_1(x) + u_1''(x)) \partial^1 + \frac{1}{3} (5 (u_0(x)^2 + 3 u_0''''(x)) \partial^1 + \frac{1}{9} (10 u_0(x) u_1(x) + u_0'''(x))) \partial^0 \) \( \partial^5 + \frac{1}{4} (5 u_2(x)) \partial^3 + \partial^5 + \frac{1}{3} (5 (2 u_1(x) + u_2''(x))) \partial^2 + \frac{1}{32} (5 (8 u_0(x) + u_2(x)^2 + 4 u_2''''(x))) \partial^1 + \frac{1}{144} (5 (24 u_1(x) - 6 u_2(x) + u_3''''(x))) \partial^0 \) |
| \( \partial^6 + (3 u_0(x)) \partial^4 + \frac{1}{3} (3 u_0(x))^2 + 7 u_0'''(x) \partial^2 + \frac{1}{8} (42 u_0(x) u_0''(x) + 29 u_0(x)^3) \partial^1 + \frac{1}{16} (10 u_0(x)^3 + 11 u_0'''(x)^2 + 36 u_0(x) u_0'''(x) + 10 u_0^{(4)}(x)) \partial^0 \) \( \partial^6 + \frac{1}{2} (3 u_0(x)) \partial^4 + \partial^6 + \frac{1}{3} (3 (u_0(x) + u_1''(x))) \partial^3 + \frac{1}{2} (2 u_0(x) + 3 u_3''(x)) \partial^2 + \frac{1}{8} (12 u_0(x) + 3 u_0''''(x)) \partial^1 + \frac{1}{18} (4 u_0(x)^2 + 6 u_0''''(x) + 2 u_0'''(x)) \partial^0 \) \( \partial^6 + \frac{1}{2} (6 u_1(x)) \partial^4 + \partial^6 + \frac{1}{2} (3 u_2(x) + u_2''(x)) \partial^3 + \frac{1}{5} (2 u_0(x) + 3 u_0''''(x)) \partial^2 + \frac{1}{30} (30 u_0(x) + 3 u_0''''(x)) \partial^1 + \frac{1}{15} (15 u_0(x)^2 + 10 u_0'''(x)) \partial^0 \) |
| \( \partial^6 + u_3(x) \partial^3 + \frac{1}{2} (3 u_1(x) + u_3''(x)) \partial^2 + \frac{1}{8} (12 u_0(x) + 3 u_0''''(x)) \partial^1 + \frac{1}{18} (4 u_0(x)^2 + 6 u_0''''(x) + 2 u_0'''(x)) \partial^0 \) \( \partial^6 + u_4(x) \partial^4 + \partial^6 + u_3(x) \partial^3 + \frac{1}{2} (3 u_1(x) + u_3''(x)) \partial^2 + \frac{1}{8} (12 u_0(x) + 3 u_0''''(x)) \partial^1 + \frac{1}{18} (4 u_0(x)^2 + 6 u_0''''(x) + 2 u_0'''(x)) \partial^0 \) |
| \( \partial^6 + u_4(x) \partial^4 + u_3(x) \partial^3 + u_2(x) \partial^2 + u_1(x) \partial^1 + u_0(x) \partial^0 \) |
$[Q^k_+, L]$ for $L$ of order $n$

<table>
<thead>
<tr>
<th>$k \setminus n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$u'_0[x]$ $\sigma^0$</td>
<td>$u_1'[x]$ $\sigma^1 + u'_0[x]$ $\sigma^0$</td>
<td>$u'_2[x]$ $\sigma^2 + u'_1[x]$ $\sigma^1 + u'_0[x]$ $\sigma^0$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$(2u'_0[x] - u''_1[x])$ $\sigma^1 + (-\frac{2}{3}u_1[x] + u'_1[x] + u''_0[x] - \frac{2}{3}u_1^{(3)}[x])$ $\sigma^0$</td>
<td>$(2(u'_1[x] - u''_2[x]))$ $\sigma^2 + (2u'_0[x] - u'_2[x] + u''_1[x] - 2u_2^{(3)}[x])$ $\sigma^1 + \frac{1}{2}(-u_1[x] + u''_0[x] + 2u''_0[x] - u'_2[x] + u'_2[x] - u''_2[x])$ $\sigma^0$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2}$ ${6u_0[x] + u''_0[x] + u''_0^{(3)}[x]}$ $\sigma^0$</td>
<td>0</td>
<td>$\frac{1}{2} (12u'_0[x] - 3u_2[x] - u''_2[x]) - 6u_1''[x] + u_2^{(3)}[x])$ $\sigma^2 + (\frac{1}{4}(-3u_2[x] + u'_1[x] - 3u_1[x] + u''_0[x]) + 12u''_0[x] - 8u_2^{(3)}[x] + 3u_2^{(4)}[x])$ $\sigma^1 + \frac{1}{8}(u_1[x] - 6u''_0[x] + 3u''_2[x]) + 8u_0^{(3)}[x] + 3u_2[x] + 2u''_0[x] - 2u_1''[x] + u_2^{(3)}[x] - 6u^{(4)}[x] + 3u_2^{(5)}[x])$ $\sigma^0$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2} (4u_0[x] + u'_1[x] - 2u_1''[x] + 2u''_0[x] - 2u_1^{(4)}[x])$ $\sigma^1$ + $\frac{1}{9}(12u_0[x] + u''_0[x]) - 4u_1[x]^2 + u'_1[x] + 6u''_0[x] + 12u''_1[x] + 6u_1[x] + (u''_0[x] - u''_1[x]) + 3u_2^{(4)}[x] - 2u_1^{(5)}[x]$ $\sigma^0$</td>
<td>0</td>
</tr>
</tbody>
</table>