KdV Tables

For $L = \partial^n + u_{n-2}(x)\partial^{n-2} + \ldots + u_0(x)$ a linear differential operator, the (k,n)-KdV flow is defined as $\dfrac{dL}{dt} = [Q_+^k, L]$, where Q is the nth root of L and Q_+^k is the differential part of Q^k. Below are tables showing Q_+^k and $[Q_+^k, L]$ for small values of n, k.

Q_+^k for L of order n

k \ n	1	2	3	4	5	6
1	∂^1	∂^1	∂^1	∂^1	∂^1	∂^1
2	∂^2	$\partial^2 + u_0[x]\,\partial^0$	$\partial^2 + \frac{1}{3}\,(2\,u_1[x])\,\partial^0$	$\partial^2 + \frac{1}{2}\,u_2[x]\,\partial^0$	$\partial^2 + \frac{1}{5}\,(2\,u_3[x])\,\partial^0$	$\partial^2 + \frac{1}{3}\,u_4[x]\,\partial^0$
3	∂^3	$\partial^3 + \frac{1}{2}\,(3\,u_0[x])\,\partial^1 + \frac{1}{4}\,(3\,u_0'[x])\,\partial^0$	$\partial^3 + u_1[x]\,\partial^1 + u_0[x]\,\partial^0$	$\partial^3 + \frac{1}{4}\,(3\,u_2[x])\,\partial^1 + \frac{1}{8}\,(3\,(2\,u_1[x] - u_2'[x]))\,\partial^0$	$\partial^3 + \frac{1}{5}\,(3\,u_3[x])\,\partial^1 + \frac{1}{5}\,(3\,(u_2[x] - u_3'[x]))\,\partial^0$	$\partial^3 + \frac{1}{2}\,u_4[x]\,\partial^1 + \frac{1}{4}\,(2\,u_3[x] - 3\,u_4'[x])\,\partial^0$
4	∂^4	$\partial^4 + (2\,u_0[x])\,\partial^2 + (2\,u_0'[x])\,\partial^1 + \left(u_0[x]^2 + u_0''[x]\right)\partial^0$	$\partial^4 + \frac{1}{3}\,(4\,u_1[x])\,\partial^2 + \frac{1}{3}\,(2\,(2\,u_0[x] + u_1'[x]))\,\partial^1 + \frac{1}{9}\left(2\left(u_1[x]^2 + 3\,u_0'[x] + u_1''[x]\right)\right)\partial^0$	$\partial^4 + u_2[x]\,\partial^2 + u_1[x]\,\partial^1 + u_0[x]\,\partial^0$	$\partial^4 + \frac{1}{5}\,(4\,u_3[x])\,\partial^2 + \frac{1}{5}\,(2\,(2\,u_2[x] - u_3'[x]))\,\partial^1 + \frac{1}{25}\left(-2\left(-10\,u_1[x] + u_3[x]^2 + 5\,u_2'[x]\right)\right)\partial^0$	$\partial^4 + \frac{1}{3}\,(2\,u_4[x])\,\partial^2 + \frac{1}{3}\,(2\,(u_3[x] - u_4'[x]))\,\partial^1 + \frac{1}{9}\left(6\,u_2[x] - u_4[x]^2 - 6\,u_3'[x] + u_4''[x]\right)\partial^0$
5	∂^5	$\partial^5 + \frac{1}{2}\,(5\,u_0[x])\,\partial^3 + \frac{1}{4}\,(15\,u_0'[x])\,\partial^2 + \frac{1}{8}\left(5\left(3\,u_0[x]^2 + 5\,u_0''[x]\right)\right)\partial^1 + \frac{1}{8}\left(5\left(2\,u_0[x]\,u_0'[x] + u_0^{(3)}[x]\right)\right)\partial^0$	$\partial^5 + \frac{1}{3}\,(5\,u_1[x])\,\partial^3 + \frac{1}{3}\,(5\,(u_0[x] + u_1'[x]))\,\partial^2 + \frac{1}{9}\left(5\left(u_1[x]^2 + 3\,u_0'[x] + 2\,u_1''[x]\right)\right)\partial^1 + \frac{1}{9}\,(10\,(u_0[x]\,u_1[x] + u_0''[x]))\,\partial^0$	$\partial^5 + \frac{1}{4}\,(5\,u_2[x])\,\partial^3 + \frac{1}{8}\,(5\,(2\,u_1[x] + u_2'[x]))\,\partial^2 + \frac{1}{32}\left(5\left(8\,u_0[x] + u_2[x]^2 + 4\,u_1'[x] + 2\,u_2''[x]\right)\right)\partial^1 + \frac{1}{32}\left(5\left(2\,u_1[x]\,u_2[x] + 4\,u_0'[x] - u_2[x]\,u_2'[x] + 2\,u_1''[x] - u_2^{(3)}[x]\right)\right)\partial^0$	$\partial^5 + u_3[x]\,\partial^3 + u_2[x]\,\partial^2 + u_1[x]\,\partial^1 + u_0[x]\,\partial^0$	$\partial^5 + \frac{1}{6}\,(5\,u_4[x])\,\partial^3 + \frac{1}{12}\,(5\,(2\,u_3[x] - u_4'[x]))\,\partial^2 + \frac{1}{72}\left(-5\left(-12\,u_2[x] + u_4[x]^2 + 6\,u_3'[x] + u_4''[x]\right)\right)\partial^1 + \frac{1}{144}\left(5\left(24\,u_1[x] - 4\,u_3[x]\,u_4[x] - 12\,u_2'[x] + 6\,u_4[x]\,u_4'[x] - 2\,u_3''[x] + 9\,u_4^{(3)}[x]\right)\right)\partial^0$
6	∂^6	$\partial^6 + (3\,u_0[x])\,\partial^4 + (6\,u_0'[x])\,\partial^3 + \left(3\,u_0[x]^2 + 7\,u_0''[x]\right)\partial^2 + \frac{1}{8}\left(42\,u_0[x]\,u_0'[x] + 29\,u_0^{(3)}[x]\right)\partial^1 + \frac{1}{16}\left(10\,u_0[x]^3 + 11\,u_0'[x]^2 + 36\,u_0[x]\,u_0''[x] + 10\,u_0^{(4)}[x]\right)\partial^0$	$\partial^6 + (2\,u_1[x])\,\partial^4 + (2\,u_0[x] + 3\,u_1'[x])\,\partial^3 + \left(u_1[x]^2 + 3\,(u_0'[x] + u_1''[x])\right)\partial^2 + \left(2\,u_0[x]\,u_1[x] + u_1[x]\,u_1'[x] + 3\,u_0''[x] + u_1^{(3)}[x]\right)\partial^1 + \frac{1}{27}\left(45\,u_0[x]^2 + 2\,u_1[x]^3 - 18\,u_0[x]\,u_1'[x] + 21\,u_1[x]\,(3\,u_0'[x] - 2\,u_1''[x]) - 3\left(6\,u_1'[x]^2 - 9\,u_0^{(3)}[x] + 2\,u_1^{(4)}[x]\right)\right)\partial^0$	$\partial^6 + \frac{1}{2}\,(3\,u_2[x])\,\partial^4 + \frac{1}{2}\,(3\,(u_1[x] + u_2'[x]))\,\partial^3 + \frac{1}{8}\left(12\,u_0[x] + 3\,u_2[x]^2 + 12\,u_1'[x] + 10\,u_2''[x]\right)\partial^2 + \frac{1}{4}\,(3\,u_1[x]\,u_2[x] + 6\,u_0'[x] + 5\,u_1''[x])\,\partial^1 + \frac{1}{16}\left(6\,u_1[x]^2 + 12\,u_0[x]\,u_2[x] - u_2[x]^3 - 6\,u_1[x]\,u_2'[x] - 2\,u_2'[x]^2 + 20\,u_0''[x] - 4\,u_2[x]\,u_2''[x] - 2\,u_2^{(4)}[x]\right)\partial^0$	$\partial^6 + \frac{1}{5}\,(6\,u_3[x])\,\partial^4 + \frac{1}{5}\,(3\,(2\,u_2[x] + u_3'[x]))\,\partial^3 + \frac{1}{25}\left(30\,u_1[x] + 3\,u_3[x]^2 + 15\,u_2'[x] + 10\,u_3''[x]\right)\partial^2 + \frac{1}{25}\left(30\,u_0[x] + 6\,u_2[x]\,u_3[x] + 15\,u_1'[x] - 3\,u_3[x]\,u_3'[x] + 10\,u_2''[x] - 5\,u_3^{(3)}[x]\right)\partial^1 + \frac{1}{125}\left(15\,u_2[x]^2 + 30\,u_1[x]\,u_3[x] - 4\,u_3[x]^3 + 75\,u_0'[x] - 15\,u_3[x]\,u_2'[x] - 30\,u_2[x]\,u_3'[x] - 5\,u_3'[x]^2 + 50\,u_1''[x] - 15\,u_3[x]\,u_3''[x] - 25\,u_2^{(3)}[x] - 5\,u_3^{(4)}[x]\right)\partial^0$	$\partial^6 + u_4[x]\,\partial^4 + u_3[x]\,\partial^3 + u_2[x]\,\partial^2 + u_1[x]\,\partial^1 + u_0[x]\,\partial^0$

$[Q_+^k, L]$ for L of order n

k \ n	1	2	3	4
1	0	$u_0'[x]\,\partial^0$	$u_1'[x]\,\partial^1 + u_0'[x]\,\partial^0$	$u_2'[x]\,\partial^2 + u_1'[x]\,\partial^1 + u_0'[x]\,\partial^0$
2	0	0	$(2\,u_0'[x] - u_1''[x])\,\partial^1 + \left(-\frac{2}{3}\,u_1[x]\,u_1'[x] + u_0''[x] - \frac{2}{3}\,u_1^{(3)}[x]\right)\partial^0$	$(2\,(u_1'[x] - u_2''[x]))\,\partial^2 + \left(2\,u_0'[x] - u_2[x]\,u_2'[x] + u_1''[x] - 2\,u_2^{(3)}[x]\right)\partial^1 + \frac{1}{2}\left(-u_1[x]\,u_2'[x] + 2\,u_0''[x] - u_2[x]\,u_2''[x] - u_2^{(4)}[x]\right)\partial^0$
3	0	$\frac{1}{4}\left(6\,u_0[x]\,u_0'[x] + u_0^{(3)}[x]\right)\partial^0$	0	$\frac{1}{4}\left(12\,u_0'[x] - 3\,u_2[x]\,u_2'[x] - 6\,u_1''[x] + u_2^{(3)}[x]\right)\partial^2 + \frac{1}{4}\left(-3\,u_2[x]\,u_1'[x] - 3\,u_1[x]\,u_2'[x] + 12\,u_0''[x] - 8\,u_1^{(3)}[x] + 3\,u_2^{(4)}[x]\right)\partial^1 + \frac{1}{8}\left(u_1[x]\,(-6\,u_1'[x] + 3\,u_2''[x]) + 8\,u_0^{(3)}[x] + 3\,u_2[x]\left(2\,u_0'[x] - 2\,u_1''[x] + u_2^{(3)}[x]\right) - 6\,u_1^{(4)}[x] + 3\,u_2^{(5)}[x]\right)\partial^0$
4	0	0	$\frac{1}{3}\left(4\,u_0[x]\,u_1'[x] - 2\,u_1'[x]^2 + u_1[x]\,(4\,u_0'[x] - 2\,u_1''[x]) + 2\,u_0^{(3)}[x] - u_1^{(4)}[x]\right)\partial^1 + \frac{1}{9}\left(12\,u_0[x]\,u_0'[x] - 4\,u_1[x]^2\,u_1'[x] + 6\,u_0'[x]\,u_1'[x] - 12\,u_1'[x]\,u_1''[x] + 6\,u_1[x]\left(u_0''[x] - u_1^{(3)}[x]\right) + 3\,u_0^{(4)}[x] - 2\,u_1^{(5)}[x]\right)\partial^0$	0