CHAPTER 5

A Mathematical Trivium

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The standard of mathematical culture is falling; both undergraduate and graduate students educated at our colleges, including the Mechanics and Mathematics Department of Moscow State University, are becoming no less ignorant than the professors and teachers. What is the reason for this abnormal phenomenon? Following the general principle of propagation of knowledge, under normal conditions students usually know their subject better than their professors: new knowledge prevails not because old men learn it, but because new a generation enters the field already knowing it.

Among the many causes of this abnormal situation I would like to single out those for which we are responsible ourselves, so that we can try to correct what is within our reach. One such cause, I believe, is our system of examinations, which is specially designed for the systematic production of rejects, that is, pseudo-scientists who learn mathematics like Marxism: they cram themselves with formulae and rote-learning of answers to the most frequent examination questions.

How can the standard of training of a mathematician be measured? Neither a list of courses nor their syllabi determine the standard. The only way to determine what we have actually taught our students is to list the problems which they should be able to solve as a result of our instruction.

I am not talking about some difficult problems, but rather about those simple questions which form the absolutely essential minimum. There need not necessarily be many of these problems, but we must insist that the students are able to solve them. I. E. Tamm used to tell the story that having fallen into the hands of the Makhnovtsy[1] during the Russian Civil War, he said under interrogation that he was a student at the physics and mathematics department. He saved his life only because he could solve a problem in the theory of infinite series he was immediately given to check if he is telling the truth. Our students should be prepared for such ordeals!

Throughout the world a mathematics exam consists of presenting written solution of problems. The written character of the test is everywhere considered just as much a necessary attribute of a democratic society as the choice between several candidates in an election. In fact, in an oral examination the student is completely defenceless. While conducting examinations in the Division of Differential Equations at the Mechanics and Mathematics Department of Moscow State University,

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[1] *Editors' remark*: Nestor Makhno was the commander of an independent anarchist army in Ukraine that led a guerrilla campaign during the Russian Civil War.

I overheard examiners at a nearby table failing students who gave immaculate answers (which perhaps exceeded the level of comprehension of the examiner). Also, there have been cases where they examiners have failed a student on purpose (sometimes I was able to save the situation by entering the examination room).

Written work is a document and the examiner has to be more objective in grading it (particularly if the work to be graded is anonymous, as it should be).

There is another important advantage of written examinations: the problems are preserved and can be published or passed on to the students of the next year to help them to prepare for their exams. In addition, these problems determine both the level of the course and the quality of the teacher who compiled them. The teacher's strong and weak points can be seen at once, and specialists can immediately assess the teacher both in what he wants to teach the students and how he succeeded in doing this.

Incidentally, in France the problems in the *Concours général*, common to the entire country and roughly equivalent to our Olympiad, are compiled by teachers sending their problems to Paris where the best are chosen. The Ministry obtains objective data about the standard of its teachers by comparing first the problem sets, and second the results of their students. However, in our situation teachers are assessed, as we all know, according to such indicators as their appearance, quickness of speech, and ideological "correctness".

It is not surprising that other countries are unwilling to recognize our diplomas (in future I think this will even extend to diplomas in mathematics). Assessments obtained from oral exams that leave no records cannot be objectively compared with anything else and have an extremely vague and relative value, wholly depending on standards of teaching and the requirements established in a particular college. With the same syllabi and grades the knowledge and ability of the graduates may vary (in an appropriate sense) by a factor of ten. In addition, an oral exam can be far more easily falsified (this has even happened with us at the Mechanics and Mathematics Department of Moscow State University, where, as a blind teacher once said, a good mark must be given to a student whose report is "very close to the textbook", even if he cannot answer a single question).

The essence and the shortcomings of our system of mathematical education have been brilliantly described by Richard Feynman in his memoirs (*Surely you're joking, Mr. Feynman* (Norton, New York 1984), in the chapter on physics education in Brazil, a Russian translation of which was published in *Uspekhi Fizicheskikh Nauk* **148**:3 (1986)).

In Feynman's words, these students understand nothing, but never ask questions, so that they appear to understand everything. If anybody begins to ask questions, he is quickly put in his place, as he is wasting the time of the lecturer dictating the lecture and of students copying it down. The result is that no one can apply anything they have been taught to even a single example. The examinations (dogmatic like ours: state the definition, state the theorem) are always successfully passed. The students reach a state of "self-propagating pseudo-education" and can teach future generations in the same way. But all this activity is completely meaningless, and in fact our output of specialists is to a significant extent craft, illusion, and sham: these so-called specialists cannot solve simplest problems, and do not possess even rudiments of their profession.

Thus, *to put an end to this spurious enhancement of the results, we must specify not a list of theorems, but a collection of problems which students should be able to solve. These lists of problems must be published annually (I think there should be ten problems for each one-semester course).* Then we shall see what we really teach our students and how successful we are. And in order for students to learn how to use their knowledge, *all examinations must be written examinations.*

Naturally the list of problems will vary from college to college and from year to year. Then one can compare levels of different teachers and of students graduating at different years. A student who takes much more than five minutes to calculate the mean value of $\sin^{100} x$ with 10% accuracy did not maste mathematics even if he has studied non-standard analysis, universal algebras, supermanifolds, or embedding theorems.

The compilation of model problems is a laborious job, but I think it must be done. As an attempt I give below a list of one hundred problems forming a mathematical minimum for a physics student. Model problems (unlike syllabi) are not uniquely defined, and many will probably not agree with me. Nonetheless I believe that it is necessary to start defining mathematical standards using written examinations and model problems. One wants to hope that in the future students will receive model problems for each course at the beginning of each semester, while oral examinations with cramming of rote learning will become a thing of the past.

(1) Given the graph of a function, sketch the graph of the derivative and the graph of the anti-derivative of this function.

(2) Find the limit
$$\lim_{x\to 0} \frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}.$$

(3) Find the critical values and critical points of the mapping $z \mapsto z^2 + 2\bar{z}$. (Sketch the answer.)

(4) Calculate the 100th derivative of the function
$$\frac{x^2+1}{x^3-x}.$$

(5) Calculate the 100th derivative of the function
$$\frac{1}{x^2+3x+2}$$
at $x = 0$ with 10% relative error.

(6) In the (x, y)-plane sketch the curve given parametrically by
$$x = 2t - 4t^3, \quad y = t^2 - 3t^4.$$

(7) How many normals to an ellipse can be drawn from a given point of the plane? Find the region in which the number of normals is maximal.

(8) How many maxima, minima, and saddle points does the function
$$x^4 + y^4 + z^4 + u^4 + v^4$$
have on the surface
$$x + \cdots + v = 0, \quad x^2 + \cdots + v^2 = 1, \quad x^3 + ... + v^3 = C?$$

(9) Does every positive polynomial in two real variables attain its lower bound in the plane?

(10) Investigate the asymptotic behaviour of the solutions y of the equation $x^5 + x^2y^2 = y^6$ that tend to zero as $x \to 0$.

(11) Investigate the convergence of the integral
$$\iint\limits_{-\infty}^{+\infty} \frac{dx\,dy}{1+x^4y^4}.$$

(12) Find the flux of the vector field $\vec{r}/r^3$ through the surface
$$(x-1)^2 + y^2 + z^2 = 2.$$

(13) Calculate
$$\int_1^{10} x^x\,dx$$
with 5% relative error.

(14) Calculate
$$\int_{-\infty}^{\infty} (x^4 + 4x + 4)^{-100}\,dx$$
with at most 10% relative error.

(15) Calculate
$$\int_{-\infty}^{\infty} \cos(100(x^4 - x))\,dx$$
with 10% relative error.

(16) What fraction of the volume of a 5-dimensional cube is the volume of the inscribed ball? What fraction is it of a 10-dimensional cube?

(17) Find the distance between the center of gravity of a uniform 100-dimensional solid half-ball of radius 1 and the center of the sphere with 10% relative error.

(18) Calculate
$$\int \cdots \int e^{-\sum_{1\le i\le j\le n} x_i x_j}\,dx_1 \cdots dx_n.$$

(19) Investigate the path of a light ray in a plane medium with refractive index $n(y) = y^4 - y^2 + 1$, using Snell's law $n(y)\sin\alpha = \text{const}$, where α is the angle made by the ray with the y-axis.

(20) Find the derivative of the solution of the equation $\ddot{x} = x + A\dot{x}^2$, with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$, with respect to the parameter A for $A = 0$.

(21) Find the derivative of the solution of the equation $\ddot{x} = \dot{x}^2 + x^3$ with initial conditions $x(0) = 0$, $\dot{x}(0) = A$ with respect to A for $A = 0$.

(22) Investigate the boundary of the domain of stability ($\max \operatorname{Re}\lambda_j < 0$) in the space of coefficients of the equation $\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$.

(23) Solve the quasi-homogeneous equation
$$\frac{dy}{dx} = x + \frac{x^3}{y}.$$

(24) Solve the quasi-homogeneous equation
$$\ddot{x} = x^5 + x^2\dot{x}.$$

(25) Can an asymptotically stable equilibrium position become unstable in the Lyapunov sense under linearization?

(26) Investigate the behaviour as $t \to +\infty$ of solutions of the systems

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 2\sin y - y - x, \end{cases} \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = 2x - x^3 - x^2 - \varepsilon y, \end{cases}$$

where $\varepsilon \ll 1$.

(27) Sketch the images of the solutions of the equation

$$\ddot{x} = F(x) - k\dot{x}, \quad F = -dU/dx,$$

in the (x, E)-plane, where $E = \dot{x}^2 + U(x)$, near non-degenerate critical points of the potential U.

(28) Sketch the phase portrait and investigate how it depends on variation of the small complex parameter ε:

$$\dot{z} = \varepsilon z - (1+i)z|z|^2 + \bar{z}^4.$$

(29) A charge moves with velocity 1 in a plane under the action of a strong magnetic field $B(x, y)$ perpendicular to the plane. To which side will the center of the Larmor circle drift? Calculate the velocity of this drift (to a first approximation). [Mathematically, this concerns the curves of curvature NB as $N \to \infty$.]

(30) Find the sum of the indexes of the singular points other than zero of the vector field $z\bar{z}^2 + z^4 + 2\bar{z}^4$.

(31) Find the index of the singular point 0 of the vector field with components $(x^4 + y^4 + z^4, x^3y - xy^3, xyz^2)$.

(32) Find the index of the singular point 0 of the vector field

$$\operatorname{grad}(xy + yz + zx).$$

(33) Find the linking coefficient of the phase trajectories of the equation of small oscillations $\ddot{x} = -4x, \ddot{y} = -9y$ on a level surface of the total energy.

(34) Investigate the singular points on the curve $y = x^3$ in the projective plane.

(35) Sketch the geodesics on the surface

$$(x^2 + y^2 - 2)^2 + z^2 = 1.$$

(36) Sketch the evolvent of the cubic parabola $y = x^3$. (The evolvent is the locus of the points $\vec{r}(s) + (c - s)\dot{\vec{r}}(s)$, where s is the arc-length of the curve $\vec{r}(s)$ and c is a constant).

(37) Prove that in Euclidean space the surfaces

$$((A - \lambda E)^{-1}x, x) = 1$$

passing through the point x and corresponding to different values of λ are pairwise orthogonal. (A is a symmetric operator without multiple eigenvalues.)

(38) Calculate the integral of the Gaussian curvature of the surface

$$z^4 + (x^2 + y^2 - 1)(2x^2 + 3y^2 - 1) = 0.$$

(39) Calculate the Gauss integral

$$\iint \frac{(d\vec{A}, d\vec{B}, \vec{A} - \vec{B})}{|\vec{A} - \vec{B}|^3},$$

where $\vec{A}$ runs along the curve $x = \cos\alpha$, $y = \sin\alpha$, $z = 0$, and $\vec{B}$ along the curve $x = 2\cos^2\beta$, $y = \frac{1}{2}\sin\beta$, $z = \sin 2\beta$.

(40) Find the parallel displacement of a vector pointing north at Leningrad (latitude 60°) from West to East along a closed parallel.

(41) Find the geodesic curvature of the line $y = 1$ in the upper half-plane with the Lobachevskii–Poincaré metric

$$ds^2 = (dx^2 + dy^2)/y^2.$$

(42) Do the medians of a triangle meet in a single point in the Lobachevskii plane? What about the altitudes?

(43) Find the Betti numbers of the surface $x_1^2 + \cdots + x_k^2 - y_1^2 - \cdots - y_l^2 = 1$ and the set $x_1^2 + \cdots + x_k^2 \le 1 + y_1^2 + \cdots + y_l^2$ in the $(k + l)$-dimensional vector space.

(44) Find the Betti numbers of the surface $x^2 + y^2 = 1 + z^2$ in the three-dimensional projective space. The same for the surfaces $z = xy$, $z = x^2$, $z^2 = x^2 + y^2$.

(45) Find the self-intersection index of the surface $x^4 + y^4 = 1$ in the projective plane $\mathbb{CP}^2$.

(46) Map the interior of the unit disc conformally onto the first quadrant.

(47) Map the exterior of a disc conformally onto the exterior of a given ellipse.

(48) Map the half-plane without a segment perpendicular to its boundary conformally onto the halfplane.

(49) Calculate

$$\oint_{|z|=2} \frac{dz}{\sqrt{1 + z^{10}}}.$$

(50) Calculate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + x^2}\, dx.$$

(51) Calculate the integral

$$\int_{-\infty}^{\infty} e^{ikx}\, \frac{1 - e^x}{1 + e^x}\, dx.$$

(52) Calculate the first term of the asymptotic expansion as $k \to \infty$ of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx} dx}{\sqrt{1 + x^{2n}}}.$$

(53) Investigate the singular points of the differential form $dt = dx/y$ on the compact Riemann surface $y^2/2 + U(x) = E$, where U is a polynomial and E is not a critical value.

(54) Let $\ddot{x} = 3x - x^3 - 1$. In which of the potential wells is the period of oscillation greater (in the more shallow one or in the deeper one) for equal values of the total energy?

(55) Investigate topologically the Riemann surface of the function

$$w = \arctan z.$$

(56) How many handles does the Riemann surface of the function

$$w = \sqrt{1 + z^n}$$

Have?

(57) Find the dimension of the solution space of the problem $\partial u/\partial \overline{z} = \delta(z-i)$ for $\operatorname{Im} z \geq 0$, $\operatorname{Im} u(z) = 0$ for $\operatorname{Im} z = 0$, $u \to 0$ as $z \to \infty$.

(58) Find the dimension of the solution space of the problem $\partial u/\partial \overline{z} = a\delta(z-i) + b\delta(z+i)$ for $|z| \leq 2$, $\operatorname{Im} z = 0$ for $|z| = 2$.

(59) Investigate the existence and uniqueness of the solution of the problem $yu_x = xu_y$, $u|_{x=1} = \cos y$ in a neighbourhood of the point $(1, y_0)$.

(60) Is there a solution of the Cauchy problem

$$x(x^2+y^2)\frac{\partial u}{\partial x} + y^3\frac{\partial u}{\partial y} = 0, \quad u|_{y=0} = 1,$$

in a neighbourhood of the point $(x_0, 0)$ of the x-axis? Is it unique?

(61) What is the largest value of t for which the solution of the problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \sin x, \quad u|_{t=0} = 0,$$

can be extended to the interval $[0, t)$?

(62) Find all solutions of the equation $y\,\partial u/\partial x - \sin x\,\partial u/\partial y = u^2$ in a neighbourhood of the point $(0, 0)$.

(63) Is there a solution of the Cauchy problem $y\,\partial u/\partial x + \sin x \partial u/\partial y = y$, $u|_{x=0} = y^4$ on the whole (x, y) plane? Is it unique?

(64) Does the Cauchy problem $u|_{y=x^2} = 1$, $(\nabla u)^2 = 1$ have a smooth solution in the domain $y \geq x^2$? In the domain $y \leq x^2$?

(65) Find the mean value of the function $\ln r$ on the circle $(x-a)^2+(y-b)^2 = R^2$ (of the function $1/r$ on the sphere).

(66) Solve the Dirichlet problem

$$\begin{aligned} \Delta u = 0 &\text{ for } \quad x^2+y^2 < 1; \\ u = 1 \quad &\text{ for } \quad x^2+y^2 = 1,\ y > 0; \\ u = -1 &\text{ for } \quad x^2+y^2 = 1,\ y < 0. \end{aligned}$$

(67) What is the dimension of the space of solutions of the problem

$$\Delta u = 0 \text{ for } x^2+y^2 > 1, \quad \partial u/\partial n = 0 \text{ for } x^2+y^2 = 1$$

that are continuous on $x^2+y^2 \geq 1$?

(68) Find

$$\inf \iint_{x^2+y^2 \leq 1} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy$$

for C^∞-functions u that vanish at 0 and are equal to 1 on $x^2+y^2 = 1$.

(69) Prove that the solid angle based on a given closed contour is a function of the vertex of the angle that is harmonic outside of the contour.

(70) Calculate the mean value of the solid angle by which the disc $x^2+y^2 \leq 1$ lying in the plane $z=0$ is seen from points of the sphere $x^2+y^2+(z-2)^2=1$.

(71) Calculate the charge density on the conducting boundary $x^2+y^2+z^2 = 1$ of a cavity in which a charge $q = 1$ is placed at distance r from the center.

(72) Calculate to the first order in ε the effect that the flattening of the earth ($\varepsilon \approx 1/300$) has on the gravitational field of the earth at the distance of the moon (assuming the earth to be homogeneous).

(73) Find (to the first order in ε) the influence of the imperfection of an almost spherical capacitor $R = 1 + \varepsilon f(\phi, \theta)$ on its capacity.

(74) Sketch the graph of $u(x, 1)$, if $0 \le x \le 1$,

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \qquad u|_{t=0} = x^2, \quad u|_{x^2=x} = x^2.$$

(75) On account of the annual fluctuation of temperature the ground at a given town freezes to a depth of 2 metres. To what depth would it freeze on account of the daily fluctuation of the same amplitude?

(76) Investigate the behaviour at $t \to \infty$ of the solution of the problem

$$u_t + (u \sin x)_x = \varepsilon u_{xx}, \qquad u|_{t=0} \equiv 1, \quad \varepsilon \ll 1.$$

(77) Find the eigenvalues and their multiplicities of the Laplace operator $\Delta = \operatorname{div}\operatorname{grad}$ on a sphere of radius R in Euclidean space of dimension n.

(78) Solve the Cauchy problem

$$\frac{\partial^2 A}{\partial t^2} = 9\frac{\partial^2 A}{\partial x^2} - 2B, \qquad \frac{\partial^2 B}{\partial t^2} = 6\frac{\partial^2 B}{\partial x^2} - 2A,$$

$$A|_{t=0} = \cos x, \quad B|_{t=0} = 0, \quad \left.\frac{\partial A}{\partial t}\right|_{t=0} = \left.\frac{\partial B}{\partial t}\right|_{t=0} = 0.$$

(79) How many solutions does the boundary-value problem

$$u_{xx} + \lambda u = \sin x, \quad u(0) = u(\pi) = 0$$

have?

(80) Solve the equation

$$\int_0^1 (x+y)^2 u(x)\, dx = \lambda u(y) + 1.$$

(81) Find the Green's function of the operator $d^2/dx^2 - 1$ and solve the equation

$$\int_{-\infty}^{\infty} e^{-|x-y|} u(y)\, dy = e^{-x^2}.$$

(82) For what values of the velocity c does the equation $u_t = u - u^2 + u_{xx}$ have a solution in the form of a travelling wave $u = \phi(x - ct)$, $\phi(-\infty) = 1$, $\phi(\infty) = 0$, $0 \le u \le 1$.

(83) Find solutions of the equation $u_t = u_{xxx} + uu_x$ in the form of a travelling wave $u = \phi(x - ct)$, $\phi(\pm\infty) = 0$.

(84) Find the number of positive and negative squares in the canonical form of the quadratic form $\sum_{i<j}(x_i - x_j)^2$ in n variables. The same for the form $\sum_{i<j} x_i x_j$.

(85) Find the lengths of the principal axes of the ellipsoid

$$\sum_{i \le j} x_i x_j = 1$$

(86) Through the center of a cube (tetrahedron, icosahedron) draw a line in such a way that the sum of the squares of its distances from the vertices is a) minimal, b) maximal.

(87) Find the derivatives of the lengths of the semiaxes of the ellipsoid $x^2 + y^2 + z^2 + xy + yz + zx = 1 + \varepsilon xy$ with respect to ε at $\varepsilon = 0$.

(88) List all the figures that can be obtained by intersecting the infinite-dimensional cube $|x_k| \le 1$, $k = 1, 2, \ldots$ with a two-dimensional plane?

(89) Calculate the sum of vector products $[[x, y], z] + [[y, z], x] + [[z, x], y]$,

(90) Calculate the sum of matrix commutators $[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$, where $[A,B] = AB - BA$.

(91) Find the Jordan normal form of the operator $e^{d/dt}$ in the space of quasi-polynomials $\{e^{\lambda t}p(t)\}$ where the degree of the polynomial p is less than 5, and of the operator $\mathrm{ad}_A : B \mapsto [A,B]$, in the space of $n \times n$ matrices B, where A is a diagonal matrix.

(92) Find the orders of the subgroups of the rotation group of the cube, and find its normal subgroups.

(93) Decompose the space of functions defined on the vertices of a cube into invariant subspaces irreducible with respect to the group of a) its symmetries, b) its rotations.

(94) Decompose a 5-dimensional real vector space into the irreducible invariant subspaces of the group generated by cyclic permutations of the basis vectors.

(95) Decompose the space of homogeneous polynomials of degree 5 in (x, y, z) into irreducible subspaces invariant with respect to the rotation group $SO(3)$.

(96) Each of 3600 subscribers of a telephone exchange calls it once an hour on average. What is the probability that in a given second 5 or more calls are received? Estimate the mean interval of time between two such seconds $(i, i+1)$.

(97) A particle performing a random walk on the integer points of the semi-axis $x \geq 0$ moves a distance 1 to the right with probability a, and to the left with probability b, and stands still in the remaining cases (if $x = 0$, it stands still instead of moving to the left). Determine the steady-state probability distribution, and also the expected value of x and x^2 over a long time, if the particle starts at the point 0.

(98) In the game of "Fingers", N players stand in a circle and simultaneously thrust out their right hands, each with a certain number of fingers showing. The total number of fingers shown is counted out round the circle from the leader, and the player on whom the count stops is the winner. How large must N be for a suitably chosen group of $N/10$ players to contain a winner with probability at least 0.9? How does the probability that the leader wins behave as $N \to \infty$?

(99) One player conceals a 10 or 20 copeck coin, and the other guesses its value. If he is right he gets the coin, if wrong he pays 15 copecks. Is this a fair game? What are the optimal mixed strategies for both players?

(100) Find the mathematical expectation of the area of the projection of a cube with edge of length 1 onto a plane with an isotropically distributed random direction of projection.