Vector Fields and the Cohomology Ring of Toric Varieties

Kiumars Kaveh
Department of Mathematics
University of British Columbia

March 6, 2005

Abstract. Let X be a smooth complex projective variety with a holomorphic vector field with isolated zero set Z. From the results of Carrell and Lieberman ([3] and [4]) there exists a filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$, the ring of $\mathbb{C}$-valued functions on Z, such that $\mathrm{Gr}A(Z) \cong H^*(X, \mathbb{C})$ as graded algebras. In this note, for a smooth projective toric variety and a vector field generated by the action of a 1-parameter subgroup of the torus, we work out this filtration. Our main result is an explicit connection between this filtration and the polytope algebra of X.

Key words: Toric variety, torus action, cohomology ring, simple polytope, polytope algebra.
Subject Classification: Primary 14M25; Secondary 52B20.

Contents

1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$ with a holomorphic vector field $\mathcal{V}$ such that Zero($\mathcal{V}$) is non-trivial and isolated. In [3] and [4], using the Koszul complex of the vector field $\mathcal{V}$, Carrell and Lieberman prove that the coordinate ring $A(Z)$ of the zero scheme Z of $\mathcal{V}$ admits a filtration $F_0 \subset F_1 \subset \cdots$ such that the associated graded Gr$A(Z)$ is isomorphic to $H^*(X, \mathbb{C})$ as graded algebra. In this paper, for a smooth projective toric variety, we work out this filtration. Our main result is a natural isomorphism between Gr$A(Z)$ and Brion's description of the polytope algebra (see [1]). We also give direct proofs that the usual relations in the cohomology of a toric variety hold in Gr$A(Z)$.

For the vector field $\mathcal{V}$, in the toric case, we take the generating vector field of a 1-parameter subgroup γ, in general position, of the torus T, so that the fixed point set Z of γ is the same as the fixed point set of T. Any lattice polytope Δ normal to the fan of X, determines a line bundle L_Δ on X. We show that $c_1(L_\Delta)$, the first Chern class of this line bundle, under the isomorphism $H^*(X, \mathbb{C}) \cong \mathrm{Gr}A(Z)$, is represented by the the function $f_\Delta \in A(Z)$ given by the simple formula:

$$f_\Delta(z) = \langle \gamma, v_z \rangle \quad \forall z \in Z,$$

where v_z is the vertex of Δ corresponding to a fixed point z. Multiplication by f_Δ is the Lefschetz operator in $H^*(X, \mathbb{C}) \cong \mathrm{Gr}A(Z)$. From these functions f_Δ we obtain the functions f_ρ in Gr$A(Z)$ corresponding to D_ρ, the cohomology classes of the orbit closures of codimension 1. It is known that these span $H^2(X, \mathbb{C})$ as a vector space and generate $H^*(X, \mathbb{C})$ as an algebra. Using this, we then construct the filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$ (Theorem 4.3).

From [1], $H^*(X, \mathbb{C})$ can be realized as a quotient of the algebra of continuous functions on the fan of X whose restriction to each cone is a polynomial. Let p be a continuous function on the fan whose restriction to each cone of maximal dimension is a homogeneous polynomial of degree k, representing a cohomology class in $H^{2k}(X, \mathbb{C})$. We show that p corresponds to the function $f \in F_k A(Z)$ defined by

$$f(z) = p_{|\sigma_z}(\gamma),$$

where σ_z is the cone of maximal dimension corresponding to a fixed point z (Theorem 5.2).

This paper is motivated in part by a comment of T. Oda. In [8, p. 417], Oda briefly comments about how to explain the results of Carrell-Lieberman in the toric case: as Khovanskii has shown in [6], composition of γ and the moment map of the toric variety X defines a Morse function on X whose critical points are the fixed points (see Remark 4.8). Since the number of critical points of index i is the i-th Betti number, Oda suggests that the grading on the fixed point set induced by the Morse index is the grading in Carrell-Lieberman and hence gives the cohomology algebra. It is not difficult to see that this is not necessarily correct [1] (also see Example 6.1).

The present note is closely related to the work of V. Puppe [10] which gives a similar filtration in the topological setting.

In Section 2, we discuss Carrell-Lieberman results on the connection between the zeros of holomorphic vector fields and cohomology. In Section 3 we discuss the classical results on the cohomology of toric varieties. In Section 4, we work out the Carrell-Lieberman filtration in the toric case. The main result of the paper, that is the connection between the polytope algebra and the Carrell-Lieberman filtration is discussed in Section 5. In Section 6 we see two examples in dimension 2.

Acknowledgment: I would like to express my gratitude to Prof. James Carrell for suggesting the problem to me and helpful discussions. Also, I would like to thank Prof. Kalle Karu and Dr. Vladlen Timorin for helpful discussions. In particular, during the discussion with Timorin we found the relation between $\mathrm{Gr}A(Z)$ and the description of cohomology in terms of the conewise continuous functions. I am also very thankful to the referee for numerous suggestions and pointing out some crucial errors in the first version of the paper.

---

[1] In fact, if F_i denotes the space of functions on Z consisting of the constant functions and the functions supported on the fixed points of index less than or equal to $2i$, then for $i \leq j$ we have $F_iF_j \subset F_i$. This means that in the corresponding graded algebra, the multiplication is zero which is certainly not the case in the cohomology algebra.

2 Zeros of Holomorphic Vector Fields and Cohomology

Let X be a smooth projective variety over $\mathbb{C}$. The purpose of this section is to give a brief survey of the results of Carrell and Lieberman on the connection between the zeros of vector fields and cohomology. We start with the Koszul complex of a holomorphic vector field $\mathcal{V}$ on X. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. The vector field $\mathcal{V}$ defines a derivation $\mathcal{V} : \mathcal{O}_X \to \mathcal{O}_X$, which extends to give a contraction operator $i(\mathcal{V}) : \Omega^p \to \Omega^{p-1}$ on the sheaves of holomorphic p-forms on X such that $i(\mathcal{V})^2 = 0$. In addition, for all $\phi, \omega \in \Omega^*$,

$$i(\mathcal{V})(\phi \wedge \omega) = i(V)\phi \wedge \omega + (-1)^p \phi \wedge i(\mathcal{V})\omega$$

if $\phi \in \Omega^p$. Thus we get a complex K^* of sheaves

$$0 \to \Omega^n \to \Omega^{n-1} \to \cdots \to \Omega^1 \to \mathcal{O}_X \to 0$$

where $n = \dim X$, and, in turn, a spectral sequence whose first term is $E_1^{-p,q} = H^q(X, \Omega^p)$ with first differential $i(\mathcal{V})$. In [3] Carrell and Lieberman prove that if $\mathcal{V}$ has zeros, then every differential in this spectral sequence is zero. Consequently $E_1 = E_\infty$, and we obtain a $\mathbb{C}$-algebra isomorphism

$$\bigoplus_s H^{q+s}(X, \Omega^q) \cong \bigoplus_s F_s H^q(K^*)/F_{s-1} H^q(K^*),$$

Here $H^q(K^*)$ denotes the hypercohomology of this Koszul complex and $F_\bullet$ is its canonical filtration.

They also prove that when $\mathcal{V}$ has isolated zeros the hypercohomology groups $H^q(K^*)$ vanish for $q > 0$ (see [3]). Zero($\mathcal{V}$) can be viewed as the scheme Z defined by the sheaf of ideals $i(\mathcal{V})\Omega^1 \subset \mathcal{O}_X$, so when this scheme is finite (and non trivial), we get the following result:

Theorem 2.1 ([2], Theorem 5.4). *Suppose X admits a holomorphic vector field $\mathcal{V}$ with Zero($\mathcal{V}$) isolated but non-trivial, then $H^p(X, \Omega^q) = \{0\}$ for all $p \neq q$ (hence $H^p(X, \Omega^p) = H^{2p}(X, \mathbb{C})$). Moreover, the coordinate ring $A(Z)$ of the zero scheme Z of $\mathcal{V}$ admits an increasing filtration $F_\bullet = F_\bullet A(Z)$ such that*

(*i*) *$F_i F_j \subset F_{i+j}$; and*

(ii) $H^*(X,\mathbb{C}) = \bigoplus_{i\geq 0} H^{2i}(X,\mathbb{C}) \cong \bigoplus_{i\geq 0} \mathrm{Gr}_i(A(Z)) = \mathrm{Gr}A(Z)$,

where the displayed summands are isomorphic over $\mathbb{C}$*. Here*

$$\mathrm{Gr}_i(A(Z)) := F_iA(Z)/F_{i-1}A(Z).$$

For the rest of this section we assume that the zero set Z is isolated (but non-empty). Let $E \to X$ be a holomorphic vector bundle and $\mathcal{E}$ its sheaf of holomorphic sections. One says that E is $\mathcal{V}$-equivariant if the derivation $\mathcal{V}$ of $\mathcal{O}_x$ lifts to $\mathcal{E}$. That is, there exists a $\mathbb{C}$-linear sheaf homomorphism $\tilde{\mathcal{V}} : \mathcal{E} \to \mathcal{E}$ such that if $\sigma \in \mathcal{E}_x$ and $f \in \mathcal{O}_{X,x}$ then

$$\tilde{\mathcal{V}}(f\sigma) = \mathcal{V}(f)\sigma + f\tilde{\mathcal{V}}(\sigma).$$

Hence $\tilde{\mathcal{V}}$ defines an $\mathcal{O}_Z$-linear map $\tilde{\mathcal{V}}_Z : \mathcal{E}_Z \to \mathcal{E}_Z$, where $\mathcal{E}_Z = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$.

Let us recall the Chern-Weil construction. Let

$$c(\mathcal{E}) \in H^1(X, \mathrm{Hom}(\mathcal{E},\mathcal{E}) \otimes \Omega^1)$$

denote the Atiyah-Chern class of $\mathcal{E}$, and let $p : \mathrm{Hom}(\mathcal{E},\mathcal{E})^{\otimes l} \to \mathcal{O}_X$ be any $\mathcal{O}_X$-linear map. Then $p(c(\mathcal{E}))$ is a well-defined element of $H^l(X,\Omega^l)$. On the other hand, p also defines a map $p_Z : \mathrm{Hom}(\mathcal{E}_Z,\mathcal{E}_Z)^{\otimes l} \to \mathcal{O}_Z$. This means that $p_Z(\tilde{\mathcal{V}}_Z^{\otimes l})$ gives a well-defined global section of $\mathcal{O}_Z$, that is, $p_Z(\tilde{\mathcal{V}}_Z^{\otimes l}) \in A(Z)$. We have:

Theorem 2.2 ([2], Theorem 5.5). *If p has degree l, then $p(\tilde{\mathcal{V}}_Z^{\otimes l}) \in F_lA(Z)$, and in the associated graded, i.e. in $\mathrm{Gr}_lA(Z)$, $p(\tilde{\mathcal{V}}_Z^{\otimes l})$ corresponds to $p(c(\mathcal{E})) \in H^l(X,\Omega^l) = H^{2l}(X,\mathbb{C})$, where $c(\mathcal{E})$ denotes the Atiyah-Chern class of $\mathcal{E}$.*

Later, we will use the above theorem to identify a function in $A(Z)$ corresponding to the Chern class of a line bundle on a toric variety. As the vector field $\mathcal{V}$ we take the generating vector field of a 1-parameter subgroup, in general position, of the torus.

3 Preliminaries on the Cohomology of Toric Varieties

Let T be the algebraic torus $(\mathbb{C}^*)^d$. As usual, N denotes the lattice of 1-parameter subgroups of T, $N_{\mathbb{R}}$ the real vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$, M the dual lattice of N which is the lattice of characters of T and, $M_{\mathbb{R}}$ the real vector space $M \otimes_{\mathbb{Z}} \mathbb{R}$. A vector $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \cong N$ corresponds to the 1-parameter subgroup $t^n = (t^{n_1}, \ldots, t^{n_d})$. Similarly, a covector $m = (m_1, \ldots, m_d) \in (\mathbb{Z}^d)^* \cong M$ corresponds to the character $x^m = x_1^{m_1} \cdots x_d^{m_d}$. We use $\langle \ , \ \rangle : N \times M \to \mathbb{Z}$ for the natural pairing between N and M.

Let X be a d-dimensional smooth projective toric variety. Let $\Sigma \subset N_{\mathbb{R}}$ be the simplicial fan corresponding to X. We denote by $\Sigma(i)$ the set of all i-dimensional cones in Σ. For each $\rho \in \Sigma(1)$, let ξ_ρ be the primitive vector along ρ, i.e. the smallest integral vector on ρ.

There is a 1-1 correspondence between the orbits of dimension i in X and the cones in $\Sigma(d-i)$. The fixed points of T correspond to the cones in $\Sigma(d)$. In a smooth toric variety all the orbit closures are smooth, the cohomology class dual to the closure of the orbit corresponding to $\rho \in \Sigma(1)$ is denoted by $D_\rho \in H^2(X, \mathbb{C})$. It is well-known that the cohomology algebra of a toric variety is generated by the classes D_ρ. More precisely, we have:

Theorem 3.1 (see [5], p.106). *Let X be a smooth projective toric variety. Then $H^*(X, \mathbb{C}) = \mathbb{Z}[D_\rho, \rho \in \Sigma(1)]/I$, where I is the ideal generated by all*

(i) $D_{\rho_1} \cdot \ldots \cdot D_{\rho_k}, \quad \forall \rho_1, \ldots, \rho_k$ *not in a cone of* Σ*; and*

(ii) $\sum_{\rho \in \Sigma(1)} \langle \xi_\rho, u \rangle D_\rho, \quad \forall u \in M.$

Now, let $\Delta \subset M_{\mathbb{R}}$ be a simple rational polytope normal to the fan Σ. The polytope Δ defines a diagonal representation $\pi : T \to GL(V)$ where $\dim_{\mathbb{C}}(V) =$ the number of lattice points in Δ. If the mutual differences of the lattice points in Δ generate M then we get an embedding of X in $\mathbb{P}(V)$ as the closure of the orbit of $(1 : \ldots : 1)$. In the rest of the paper, we assume that the above condition holds for Δ.

The set of faces of dimension i in Δ is denoted by $\Delta(i)$. There is a 1-1 correspondence between the faces in $\Delta(i)$ and the cones in $\Sigma(d-i)$ which in turn correspond to the orbits of dimension i in X. Hence the fixed points of T on X correspond to the vertices of Δ.

The support function $l_\Delta : N_{\mathbb{R}} \to \mathbb{R}$ is defined by: $l_\Delta(\xi) = \max_{x \in \Delta} \langle \xi, x \rangle$.

Let L_Δ be the line bundle on X obtained by restricting the dual of the universal subbundle on $\mathbb{P}(V)$ to X. We will need the following classical theorem which tells us how the first Chern class $c_1(L_\Delta)$ is represented as a linear combination of the classes D_ρ.

Theorem 3.2. *With notation as above we have*

$$c_1(L_\Delta) = \sum_{\rho\in\Sigma(1)} l_\Delta(\xi_\rho) D_\rho.$$

4 The filtration on $A(Z)$ in the toric case

As before, let X be a smooth projective toric variety with fan Σ and a lattice polytope Δ normal to the fan which gives rise to a representation $\pi : T \to GL(V)$ and a T-equivariant embedding of X in $\mathbb{P}(V)$, for a vector space V over $\mathbb{C}$. Let $\gamma \in N$ be a 1-parameter subgroup of T. We can choose γ so that the set of fixed points of γ is the same as the set of fixed points of T. We denote the set of fixed points by Z.

In this section, we construct a filtration $F_0 \subset F_1 \subset \cdots$ for $A(Z)$ such that $H^*(X,\mathbb{C}) \cong \mathrm{Gr}A(Z)$.

Notation: In the following, z denotes a fixed point, σ_z the corresponding d-dimensional cone in Σ and v_z the corresponding vertex in Δ. A 1-dimensional cone in Σ is denoted by ρ and the corresponding facet of Δ by F_ρ.

From Theorem 2.1 applied to the generating vector field of γ, there exists a filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$, the ring of $\mathbb{C}$ valued functions on Z, so that $H^*(X,\mathbb{C}) \cong \bigoplus_{i=0}^{\infty} F_{i+1}/F_i$, as graded algebras. In particular, we have $H^2(X,\mathbb{C}) \cong F_1/F_0$. The subspace $F_0A(Z)$ is just the set of constant functions. To determine the image of $H^2(X,\mathbb{C})$ in $\mathrm{Gr}A(Z)$ we need to determine F_1. We start by finding the representatives in F_1 for the Chern classes of the line bundles.

The 1-parameter subgroup $\gamma : \mathbb{C}^* \to T$ acts on V via π and hence the action of γ on X lifts to an action of γ on the line bundle L_Δ. Thus the generating vector field of γ has a lift to L_Δ. If we view L_Δ as $\{(x,l) \in X \times V \mid x = [l]\}$ then the action of γ on L_Δ is given by:

$$\gamma(t)\cdot(x,l) = (\pi(t^\gamma)x, \pi(t^\gamma)l).$$

Now, from Theorem 2.2 we have:

Proposition 4.1. *Under the isomorphism $F_1/F_0 \cong H^2(X,\mathbb{C})$, the first Chern class $c_1(L_\Delta)$ is represented by the function f_Δ defined by:*

$$f_\Delta(z) = \langle \gamma, v_z\rangle, \quad \forall z \in Z,$$

where v_z is the vertex of Δ corresponding to the fixed point z.

Proof. In Theorem 2.2, take E to be L_Δ and p be the identity polynomial. The derivation $\tilde{\mathcal{V}}$ is just the derivation given by the $\mathbb{G}_m$-action of γ on L_Δ. Let z be a fixed point and $(z,l) \in (L_\Delta)_z$ a point in the fiber of z. We have:

$$\begin{aligned} \gamma(t)\cdot(z,l) &= (z, \pi(t^\gamma)l), \\ &= (z, \langle\gamma, v_z\rangle l). \end{aligned}$$

and hence $f_\Delta(z) = \langle\gamma, v_z\rangle$. □

Next, we wish to determine the images of the classes $D_\rho, \rho \in \Sigma(1)$, in F_1/F_0. Fix a 1-dimensional cone ρ in $\Sigma(1)$. Let F_ρ be the facet of Δ orthogonal to ρ.

Let us assume that we can move the facet F_ρ of Δ parallelly to obtain a new integral polytope Δ' (Figure 1). The polytope Δ' is still normal to the fan Σ. If the facet can not be moved, we can replace Δ with $k\Delta$, for a big enough integer k, to make this moving of a facet possible. Replacing Δ with $k\Delta$ does not affect the formula we are going to obtain for D_ρ. Let ${F'}_\rho$ denote the facet of Δ' obtained by moving F_ρ. The maximum of the function $\langle\xi_\rho,\cdot\rangle$ on Δ and Δ' is obtained on the facets F_ρ and ${F'}_\rho$ respectively. For support functions of these polytopes we can write:

$$l_\Delta(\xi_\rho) = \langle\xi_\rho, \text{ some point in } F_\rho\rangle,$$

$$l_{\Delta'}(\xi_\rho) = \langle\xi_\rho, \text{ some point in } {F'}_\rho\rangle$$

$$l_\Delta(\xi_{\rho'}) = l_{\Delta'}(\xi_{\rho'}), \quad \forall\rho' \neq \rho.$$

We also have:

$$c_1(L_\Delta) = l_\Delta(\xi_\rho)D_\rho + \sum_{\rho'\in\Sigma(1),\rho'\neq\rho} l_\Delta(\xi_{\rho'})D_{\rho'},$$

$$c_1(L_{\Delta'}) = l_{\Delta'}(\xi_\rho)D_\rho + \sum_{\rho'\in\Sigma(1),\rho'\neq\rho} l_{\Delta'}(\xi_{\rho'})D_{\rho'}.$$

Hence

$$c_1(L_\Delta) - c_1(L_{\Delta'}) = (l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho))D_\rho.$$

So

$$D_\rho = \frac{c_1(L_\Delta) - c_1(L_{\Delta'})}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}.$$

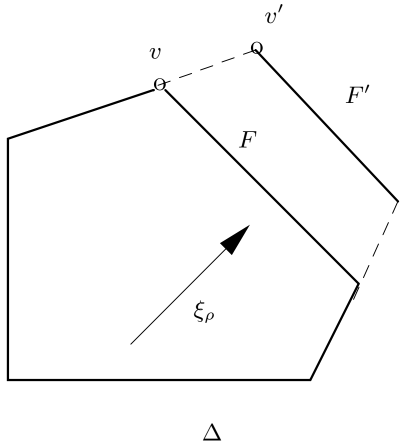

Figure 1: Moving a facet F_ρ

Now, let z be a torus fixed point, σ_z the corresponding d-dimensional cone, and v_z and ${v'}_z$ the corresponding vertices in Δ and Δ' respectively. From Proposition 4.1, D_ρ corresponds to the function $f_\rho \in F_1 A(Z)$ given by:

$$\begin{aligned} f_\rho(z) &= \frac{f_\Delta(z) - f_{\Delta'}(z)}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}, \\ &= \frac{\langle \gamma, v_z - {v'}_z \rangle}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}. \end{aligned}$$

If $v_z \notin F_\rho$ then $v_z = {v'}_z$ and hence $f_\rho(z) = 0$. If $v_z \in F_\rho$ then $l_\Delta(\xi_\rho) = \langle \xi_\rho, v_z \rangle$ and $l_{\Delta'}(\xi_\rho) = \langle \xi_\rho, {v'}_z \rangle$. We obtain that:

$$f_\rho(z) = \begin{cases} \frac{\langle \gamma, v_z - {v'}_z \rangle}{\langle \xi_\rho, v_z - {v'}_z \rangle} & \text{if } v_z \in F_\rho \\ 0 \quad \text{if } v_z \notin F_\rho \end{cases}$$

Since Δ is a simple polytope, there are d edges at the vertex v_z. If $v_z \in F_\rho$, then there is only one edge e at v_z which does not belong to F_ρ. The vector $v_z - {v'}_z$, in fact, is along this edge. Note that the above formula for $f_\rho(z)$ does not depend on the length of the vector $v_z - {v'}_z$ (i.e. how much we move the facet F_ρ to obtain the new polytope Δ'). Let $u_{\sigma_z,\rho}$ be the vector along the edge e normalized such that $\langle u_{\sigma_z,\rho}, \xi_\rho \rangle = 1$. Then we have:

Proposition 4.2. *With notation as above, the cohomology class D_ρ is represented by the function f_ρ in $F_1A(Z)$ defined by*

$$f_\rho(z) = \begin{cases} \langle \gamma, u_{\sigma_z,\rho} \rangle & \text{if } v_z \in F_\rho \\ 0 \quad \text{if } v_z \notin F_\rho \end{cases}$$

Since $H^2(X, \mathbb{C})$ is spanned by the classes $D_\rho, \rho \in \Sigma(1)$ and $H^*(X, \mathbb{C})$ is generated in degree 2, from Theorem 2.2 we obtain:

Theorem 4.3. $F_1A(Z)/F_0A(Z) = \mathrm{Span}_{\mathbb{C}}\langle f_\rho, \rho \in \Sigma(1)\rangle$. *Moreover,* $F_iA(Z) =$ *all polynomials of degree* $\leq i$ *in the* f_ρ*.*

One can prove directly that the functions $f_\rho, \rho \in \Sigma(1)$, satisfy the relations in the statement of Theorem 3.1. More precisely:

Theorem 4.4. *The functions $f_\rho, \rho \in \Sigma(1)$, satisfy the following relations:*

(i) $f_{\rho_1} \cdot \ldots \cdot f_{\rho_k} = 0, \quad \forall \rho_1, \ldots, \rho_k$ *not in a cone of* Σ*; and*

(ii) $\sum_{\rho \in \Sigma(1)} \langle \xi_\rho, u \rangle f_\rho =$ *some constant function on* Z*,* $\quad \forall u \in M$.

Proof. (i) is straight forward because every f_ρ is non-zero only at z such that the corresponding vertex lies in the facet F_ρ corresponding to ρ. Now, if $\rho_1, \ldots, \rho_k$ are not in a cone of Σ, it means that the intersection of the corresponding facets F_{ρ_i} is empty, i.e. the product of the f_{ρ_i} is zero.

For (ii), let z be a torus fixed point and, σ_z and v_z the corresponding d-dimensional cone and vertex respectively. Let A be the $d \times d$ matrix whose rows are vectors ξ_ρ and let B be the $d \times d$ matrix whose columns are vectors $u_{\sigma_z,\rho}$, where ρ is an edge of σ_z. Since the cone at the vertex v_z, which is generated by the vectors $u_{\sigma_z,\rho}$, is dual to the cone σ_z, we get $AB = \mathrm{id}$. Now, we have

$$\begin{aligned} \sum_{\rho \in \Sigma(1)} \langle \xi_\rho, u \rangle f_\rho(z) &= \sum_{\rho \text{ an edge of } \sigma_z} \langle \xi_\rho, u \rangle f_\rho(z) \\ &= \sum_{\rho \text{ an edge of } \sigma_z} \langle \xi_\rho, u \rangle \langle \gamma, u_{\sigma_z,\rho} \rangle \\ &= (A \cdot u)^t \cdot (\gamma \cdot B)^t \\ &= \langle \gamma, u \rangle, \end{aligned}$$

where $\cdot$ means product of matrices and γ is regarded as a row vector and u is regarded as a column vector. So we proved that the expression (ii) is equal to $\langle \gamma, u \rangle$ which is independent of z and hence is a constant function on Z. $\square$

Remark 4.5. One can introduce a finite subset $\mathcal{Z}$ of the affine space $\mathbb{A}^n$ such that $\mathcal{Z}$ is isomorphic to Z, and the natural grading on the coordinate ring $A(\mathcal{Z})$ induced from the grading on $\mathbb{A}^n$ coincides with the above filtration $F_\bullet$ given by the f_ρ. Define the function $\Theta : Z \to \mathbb{R}^{\Sigma(1)} \subset \mathbb{C}^{\Sigma(1)}$ by

$$\Theta(z)_\rho = f_\rho(z),$$

and let $\mathcal{Z} = \Theta(Z)$.

Proposition 4.6. *With the grading on $A(\mathcal{Z})$ as above,* $\mathrm{Gr}A(\mathcal{Z}) \cong H^*(X, \mathbb{C})$ *as graded algebras.*

Proof. Immediate. □

Remark 4.7 (Lefschetz operator). A lattice polytope Δ normal to the fan of X gives rise to an embedding of X in a projective space. The Lefschetz operator in $H^*(X, \mathbb{C}) \cong \mathrm{Gr}A(Z)$ corresponding to this embedding is given by the multiplication by the function f_Δ (see Proposition 4.1).

Remark 4.8. Let $\mu : X \to M_\mathbb{R}$ be the moment map of the toric variety and, as before, $\gamma \in N$ a 1-parameter subgroup in general position. In [6] Khovanskii shows that the composition of γ and μ defines a Morse function on X whose critical points are the fixed points of X. The Morse index of a fixed point corresponding to a vertex v_z is twice the number of edges at v_z on which the linear function γ is decreasing. Returning to the definition of the functions f_ρ (Proposition 4.2), the linear function γ is decreasing on the edge e at v_z if and only if $f_\rho(z) < 0$. That is, the Morse index of a fixed point z is equal to twice the number of negative coordinates of the point $\Theta(z) \in \mathbb{R}^{\Sigma(1)}$. Since the number of critical points of index $2i$ is the $2i$-th Betti number of X, we conclude the non-trivial relation that: the number of points in $\mathcal{Z}$ exactly i of their coordinates are negative is equal to $\dim \mathrm{Gr}_i A(\mathcal{Z})$.

5 Relation with the Polytope Algebra

Consider the abelian group generated by all the convex polytopes in a vector space subject to the relation

$$[P \cup Q] + [P \cap Q] - [P] - [Q] = 0,$$

whenever P, Q and $P \cup Q$ are convex polytopes. This group can be equipped with a ring structure, product of two polytopes being their Minkowski sum.

The ring we obtain is McMullen's *polytope algebra* (see [7, p. 86]). The polytope algebra plays an important role in the study of finitely additive measures on the convex polytopes. To each simplicial polytope Δ, one can associate a subalgebra of the polytope algebra generated by all the polytopes whose facets are parallel to the facets of Δ. It is called the *polytope algebra* of Δ. For an integrally simple polytope Δ, its polytope algebra coincides with the cohomology algebra of the corresponding toric variety X. There is a description of the polytope algebra of Δ as a quotient of the algebra of differential operators (see [9], and for more details [11]).

In [1], Brion gives a description of the polytope algebra of a polytope as a quotient of the algebra of *continuous conewise polynomial functions*. Let $\Sigma \subset N_{\mathbb{R}}$ be the normal fan of the polytope Δ. Let R be the algebra of all continuous functions on $N_{\mathbb{R}}$ which restricted to each cone of Σ are given by a polynomial. Let I be the ideal of R generated by all the linear functions on $N_{\mathbb{R}}$. Then the polytope algebra of Δ is isomorphic to R/I.

There is a good set of generators for R parameterized by the set of 1-dimensional cones $\Sigma(1)$. For each $\rho \in \Sigma(1)$, define $g_\rho : N_{\mathbb{R}} \to \mathbb{R}$ as a conewise linear function, supported on the cones containing ρ, as follows:

(i) $g_\rho = 0$ on any cone not containing ρ; and

(ii) for a d-dimensional cone σ containing ρ, the function g_ρ restricted to σ is the unique linear function defined by $g_\rho(x) = 0$ for $x \in \rho' \neq \rho, \rho' \in \Sigma(1)$ and $g_\rho(\xi_\rho) = 1$.

One can show that the g_ρ are a set of generators for R. Moreover, by sending g_ρ to D_ρ, we get an isomorphism between R/I and $H^*(X, \mathbb{C})$. In particular, the images of the g_ρ in R/I satisfy the relations in Theorem 3.1.

In what follows, we show how this description of the cohomology is related to the Gr$A(Z)$ description. Let γ be a 1-parameter subgroup in general position. Take $p \in R$. For a cone σ of maximum dimension, let p_σ denote the restriction of p to σ. Then p_σ is a polynomial on σ. Since the vector space spanned by σ is all of $N_{\mathbb{R}}$, the function p_σ can be considered as a polynomial on all of $N_{\mathbb{R}}$. Thus it makes sense to evaluate p_σ at γ. We define a homomorphism $\Phi : R \to A(Z)$ by $\Phi(p) = f$ where the function f is defined by

$$f(z) = p_{\sigma_z}(\gamma) \quad z \in Z,$$

here σ_z is the cone corresponding to the fixed point z.

Proposition 5.1. *We can write* $R = \bigoplus_{k=0}^{\infty} R_k$ *where* R_k *is the subspace of all conewise polynomial functions on* $N_{\mathbb{R}}$ *whose restriction to each cone is a homogeneous polynomial of degree* k.

Proof. Let $p \in R$ and let p_k denote the function on $N_{\mathbb{R}}$ whose restriction to each cone σ is the degree k part of p_σ. We need to prove that p_k belongs to R, that is, p_k is continuous. Let σ and τ be two adjacent cones and let $v \in \sigma \cap \tau$. Since p is continuous, we have

$$p_\sigma(tv) = p_\tau(tv), \quad \forall t \in \mathbb{R} \quad (*)$$

Let $p_{\sigma,k}(v)$ (respectively $p_{\tau,k}(v)$) denote the coefficient of t^k in $p_\sigma(tv)$ (respectively $p_\tau(tv)$). From (*), we have

$$p_{\sigma,k}(v) = p_{\tau,k}(v).$$

But $p_{\sigma,k} = p_{k|\sigma}$ and $p_{\tau,k} = p_{k|\tau}$. Thus $p_{k|\sigma}$ and $p_{k|\tau}$ agree at the intersection of σ and τ and hence p_k is continuous, that is, $p_k \in R$. ☐

Let $F_\bullet = F_0 \subset F_1 \subset \cdots$ be the filtration in the Carrell-Lieberman theorem (Theorem 2.1) where the vector field $\mathcal{V}$ is the generating vector field of a 1-parameter subgroup γ in general position. The following theorem gives an explicit connection between the grading in R by the R_k and the Carrell-Lieberman filtration $F_\bullet$.

Theorem 5.2. *(i)* $\Phi(g_\rho) = f_\rho$;

(ii) $\Phi(\oplus_{i=0}^{k} R_i) = F_k$.

Proof. (i) Let $\rho \in \Sigma(1)$ and let σ be a d-dimensional cone containing ρ. Since σ is simplicial the set $\{\xi_{\rho'} \mid \rho' \in \Sigma(1), \rho' \subset \sigma\}$ form a basis for $N_{\mathbb{R}}$. Consider the linear function l defined by $l(\xi_\rho) = 1$ and $l(\xi_{\rho'}) = 0, \rho' \subset \sigma$ and $\rho' \neq \rho$. Let A be the $d \times d$ matrix whose rows are vectors ξ_ρ and B be the $d \times d$ matrix whose columns are vectors $u_{\sigma,\rho'}$, where ρ' is an edge of σ. Let v be the vertex of Δ corresponding to σ. The cone at v is dual to σ and hence we have $AB = \mathrm{id}$. View γ as a row vector. Then γ in the basis $\xi_{\rho'}, \rho' \subset \sigma$ is $\gamma A^{-1} = \gamma B$. Thus, one sees that $l(\gamma)$ is equal to the ρ-th component of γB. But this is the same as $f_\rho(z)$.

(ii) Each $p \in \oplus_{i=0}^{k} R_i$ can be written as a polynomial of degree $\leq k$ in the g_ρ. Also, each $f \in F_k$ can be written as a polynomial of degree $\leq k$ in the f_ρ. Now, (ii) follows from (i). ☐

Now, let us define an algebra homomorphism $\Psi : R \to \mathrm{Gr}A(Z) = \bigoplus_{i=1}^{\infty} F_i/F_{i-1}$ as follows. For $p \in R_k$, let $\Psi(p) = \Phi(p) \in F_k/F_{k-1}$, and extend the definition of Ψ to all of R by linearity.

Theorem 5.3. *Ψ induces an isomorphism between R/I and $\mathrm{Gr}A(Z)$.*

Proof. It follows from the definition that Ψ is an algebra homomorphism. Since R is generated by the g_ρ and $\mathrm{Gr}A(Z)$ is generated by the images of the f_ρ, from Theorem 5.2(i) it follows that Ψ is surjective. To prove the theorem we need to show that $\ker(\Psi) = I$. Let l be a (global) linear function on $N_{\mathbb{R}}$. Then $\Phi(l) = f$ is a constant function on Z defined by $f(z) = l(\gamma), \forall z \in Z$. Thus $\Phi(l)$ belongs to F_0, that is $\Psi(l)$, as an element of F_1/F_0, is zero. Thus, l belongs to $\ker(\Psi)$. Since I is the ideal of R generated by the (global) linear functions, we see that $I \subset \ker(\Psi)$. But both of R/I and $\mathrm{Gr}A(Z)$ are isomorphic, as graded algebras, to $H^*(X, \mathbb{C})$. Hence the dimensions of the graded pieces of R/I and $\mathrm{Gr}A(Z)$ are the same. Since $I \subset \ker(\Psi)$, by comparing dimensions, we conclude that $I = \ker(\Psi)$ and the theorem is proved. $\square$

6 Examples

In this section we consider two examples in dimension 2, namely, $\mathbb{C}P^2$ and the Hirzebruch surface $\mathbb{F}_a$. For each example, we compute the functions f_ρ and the finite affine set $\mathcal{Z}$.

Example 6.1 ($\mathbb{C}P^2$). The fan of $\mathbb{C}P^2$, and a polytope normal to it, is shown in Figure 2. There are 3 one dimensional cones denoted by ρ_1, ρ_2 and ρ_3 along the primitive vectors $\xi_1 = (1, 0), \xi_2 = (0, 1)$ and $\xi_3 = (-1, -1)$. There are 3 two dimensional cones σ_1, σ_2 and σ_3 corresponding to the three fixed points z_1, z_2, z_3. To each cone σ_i, there corresponds a vertex of the normal polytope and two vectors $u_{\sigma_i,\rho}$ at this vertex along the edges. For σ_1, these vectors are $\{(1, 0), (0, 1)\}$, for σ_2 they are $\{(-1, 0), (-1, 1)\}$ and finally, for σ_3 they are $\{(0, -1), (1, -1)\}$.

Let $\gamma = (\gamma_1, \gamma_2)$ be a 1-parameter subgroup. From the definition of the

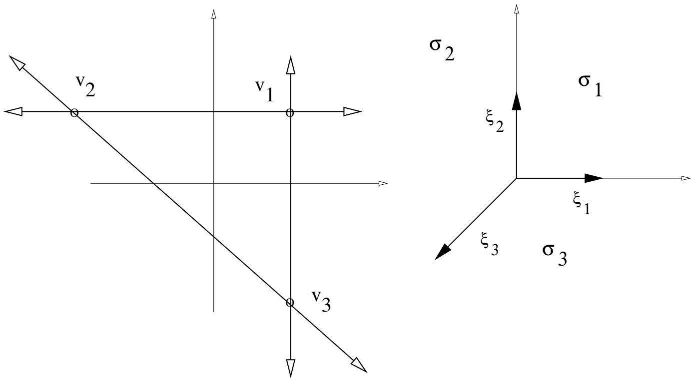

Figure 2: Fan of $\mathbb{C}P^2$ (right) and a polytope normal to the fan together with the vectors $u_{\sigma_i,\rho}$ (left).

functions f_ρ (Proposition 4.2), we get the following table for their values:

	z_1	z_2	z_3
f_1	γ_2	$\gamma_2 - \gamma_1$	0
f_2	γ_1	0	$\gamma_1 - \gamma_2$
f_3	0	$-\gamma_1$	$-\gamma_2$

and hence, $\mathcal{Z} = \{(\gamma_2, \gamma_1, 0), (\gamma_2 - \gamma_1, 0, -\gamma_1), (0, \gamma_1 - \gamma_2, -\gamma_2)\} \subset \mathbb{C}^3$. Note that the points in $\mathcal{Z}$ lie on the same line parallel to $(1, 1, 1)$. One can see that $\mathrm{Gr}_i A(\mathcal{Z}) \cong \mathbb{C}, 0 \leq i \leq 2$ and $\mathrm{Gr}_i A(\mathcal{Z}) = \{0\}, i > 2$. If x is a non-zero element of $\mathrm{Gr}_1 A(\mathcal{Z})$ then, $H^*(\mathbb{C}P^2, \mathbb{C}) \cong \mathrm{Gr}A(\mathcal{Z}) \cong \mathbb{C}[x]/\langle x^3 \rangle$.

The above calculation can be carried out in general for $\mathbb{C}P^n$. One can show that all the points in the set $\mathcal{Z}$ lie on the same line parallel to $(1, \ldots, 1)$, and $\mathrm{Gr}_i \cong \mathbb{C}$ for $0 \leq i \leq n$ and $\mathrm{Gr}_i \cong 0$ for $i > n$ and thus $H^*(\mathbb{C}P^n, \mathbb{C}) \cong \mathrm{Gr}A(\mathcal{Z}) \cong \mathbb{C}[x]/\langle x^{n+1} \rangle$. In fact, the associate graded algebra of the coordinate ring of any set of $n+1$ points lying on the same line (in the affine space) gives the cohomology algebra of $\mathbb{C}P^n$.

Example 6.2 (Hirzebruch surface). For each $a \in \mathbb{N} \cup \{0\}$, one can construct a toric surface $\mathbb{F}_a$, called a *Hirzebruch surface* whose fan, and a normal polytope to it, is shown in Figure 3. There are 4 one dimensional cones denoted by ρ_1, ρ_2, ρ_3 and ρ_4 along the primitive vectors $\xi_1 = (1,0), \xi_2 = (0,1), \xi_3 = (-1,a)$ and $\xi_4 = (0,-1)$. There are 4 two dimensional cones denoted by σ_1 to σ_4. They correspond to the four fixed points z_1, z_2, z_3 and z_4. To each σ_i there corresponds a vertex of the normal polytope to the fan and two vectors $u_{\sigma_i,\rho}$ along the edges. For σ_1 these vectors are $\{(1,0),(0,1)\}$, for σ_2 they are $\{(-1,0),(a,1)\}$, for σ_3 they are $\{(-1,0),(-a,-1)\}$ and finally, for σ_4 they are $\{(1,0),(0,-1)\}$.

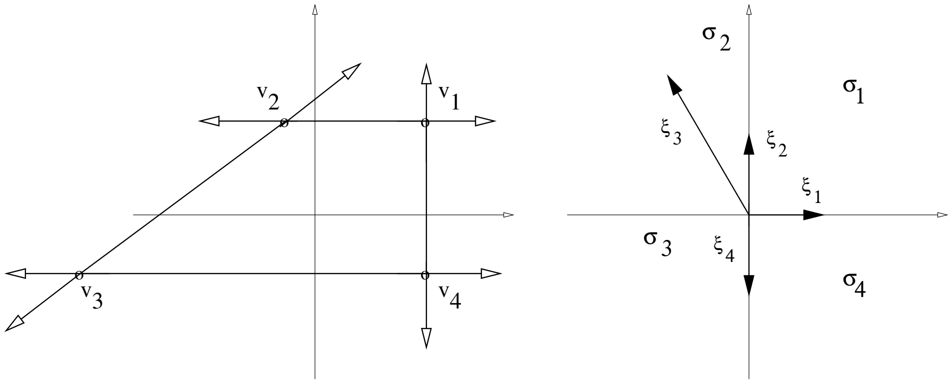

Figure 3: Fan of $\mathbb{F}_a$ (right) and a polytope normal to the fan together with the vectors $u_{\sigma_i,\rho}$ (left).

Let $\gamma = (\gamma_1, \gamma_2)$ be a 1-parameter subgroup. We get the following table for the values of the f_ρ:

	z_1	z_2	z_3	z_4
f_1	γ_1	0	0	γ_1
f_2	γ_2	$a\gamma_1 + \gamma_2$	0	0
f_3	0	$-\gamma_1$	$-\gamma_1$	0
f_4	0	0	$-a\gamma_1 - \gamma_2$	$-\gamma_2$

and hence, $\mathcal{Z} = \{(\gamma_1, \gamma_2, 0, 0), (0, a\gamma_1 + \gamma_2, -\gamma_1, 0), (0, 0, -\gamma_1, -a\gamma_1 - \gamma_2), (\gamma_1, 0, 0, -\gamma_2)\} \subset \mathbb{C}^4$. Note that the points in $\mathcal{Z}$ lie on the same 2-plane defined by $f_1 - f_3 = \gamma_1$ and $af_1 + f_2 - f_4 = a\gamma_1 + \gamma_2$. Also, no three of them are collinear. Thus, one can see that $\mathrm{Gr}_0 \cong \mathbb{C}, \mathrm{Gr}_1 \cong \mathbb{C}^2, \mathrm{Gr}_2 \cong \mathbb{C}$ and $\mathrm{Gr}_i = \{0\}, i > 2$. One can see that there are two polynomials l_1 and l_2 of degree 1 on $\mathbb{C}^4$ such that they form a basis for $\mathrm{Gr}_1 A(\mathscr{Z})$ and, ${l_1}^2 = {l_2}^2 = 0$ in $\mathrm{Gr}_2 A(\mathscr{Z})$. Hence $H^*(\mathbb{F}_a) \cong \mathrm{Gr}A(\mathscr{Z}) \cong \mathbb{C}[l_1, l_2]/\langle {l_1}^2, {l_2}^2 \rangle$. In fact, any set of 4 points lying on the same 2-plane such that no three are collinear can give the cohomology of $\mathbb{F}_a$.

References

[1] Brion, M. *The structure of the polytope algebra.* Tohoku Math. J. (2) 49 (1997), no. 1, 1–32.

[2] Carrell, J.B. *Torus actions and cohomology* in Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action, 83–158, Encyclopaedia Math. Sci., 131, Springer, Berlin, 2002.

[3] Carrell, J. B.; Lieberman, D. I. *Holomorphic vector fields and compact Kähler manifolds* . Inven. Math 21 (1973), 303–309.

[4] Carrell, J. B.; Lieberman, D. I. *Vector fields and Chern numbers.* Math. Ann. 225 (1977), no. 3, 263–273.

[5] Fulton, W. *Introduction to toric varieties.* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[6] Khovanskii, A. G. *Hyperplane sections of polyhedra, toric varieties and discrete groups in Lobachevskiĭ̆space.* Funktsional. Anal. i Prilozhen. 20 (1986), no. 1, 50–61, 96.

[7] McMullen, P. *The polytope algebra.* Adv. Math. 78 (1989), no. 1, 76–130.

[8] Oda, T. *Geometry of toric varieties.* Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 407–440, Manoj Prakashan, Madras, 1991.

[9] Pukhlikov, A. V.; Khovanskiĭ, A. G. *The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes.* Algebra i

Analiz 4 (1992), no. 4, 188–216; translation in St. Petersburg Math. J. 4 (1993), no. 4, 789–812

[10] Puppe, V, *Deformation of Algebras and Cohomology of Fixed Point Sets.* Manuscripta Mathematica 30 (1979), 119–136.

[11] Timorin, V. A. *An analogue of the Hodge-Riemann relations for simple convex polyhedra.* Uspekhi Mat. Nauk 54 (1999), no. 2(326), 113–162; translation in Russian Math. Surveys 54 (1999), no. 2, 381–426

University of British Columbia, Vancouver, B.C.
Email address: kaveh@math.ubc.ca