CRASH COURSE ON FLOWS

Let $M$ be a manifold.

A vector field $X$ on $M$ is a map that associates to each point $m \in M$ a tangent vector in $T_m M$, denoted $X|_m$ or $X(m)$, that is smooth in the following sense. In local coordinates $x^1, \ldots, x^n$, a vector field has the form $X = \sum a^j(x) \frac{\partial}{\partial x^j}$; we require that the functions $x \mapsto a^j(x)$ be smooth.

A flow on $M$ is a smooth one parameter group of diffeomorphisms $\psi_t: M \to M$. This means that $\psi_0 =$identity and $\psi_{t+s} = \psi_t \circ \psi_s$ for all $t$ and $s$ in $\mathbb{R}$ (so that $t \mapsto \psi_t$ is a group homomorphism from $\mathbb{R}$ to $\text{Diff}(M)$, the group of diffeomorphisms of $M$), and that $(t,m) \mapsto \psi_t(m)$ is smooth as a map from $\mathbb{R} \times M$ to $M$.

Its trajectories, (or flow lines, or integral curves) are the curves $t \mapsto \psi_t(m)$. The manifold $M$ decomposes into a disjoint union of trajectories. Moreover, if $\gamma_1(t)$ and $\gamma_2(t)$ are trajectories that both pass through a point $p$, then there exists an $s$ such that $\gamma_2(t) = \gamma_1(t+s)$ for all $t \in \mathbb{R}$. Hence, the velocity vectors of $\gamma_1$ and $\gamma_2$ at $p$ coincide.

Its velocity field is the vector field $X$ that is tangent to the trajectories at all points. That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time $t_0$, which is a tangent vector to $M$ at the point $p = \psi_{t_0}(m)$, is the vector $X(p)$. We express this as

$$\frac{d}{dt}\psi_t = X \circ \psi_t.$$

Conversely, any vector field $X$ on $M$ generates a local flow. This means the following. Let $X$ be a vector field. Then there exists an open subset $A \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map $\psi: A \subset \mathbb{R} \times M$ such that the following holds. Write $A = \{(t,x) \mid a_x < t < b_x\}$ and $\psi_t(x) = \psi(t,x)$.

1. $\psi_0 =$identity.
2. $\frac{d}{dt}\psi_t = X \circ \psi_t$.
3. For each $x \in M$, if $\gamma: (a,b) \to M$ satisfies the differential equation $\dot{\gamma}(t) = X(\gamma(t))$ with initial condition $\gamma(0) = x$, then $(a,b) \subset (a_x, b_x)$ and $\gamma(t) = \psi_t(x)$ for all $t$. 
Moreover, \( \psi_{t+s}(x) = \psi_t(\psi_s(x)) \) whenever these are defined. Finally, if \( X \) is compactly supported, then \( A = \mathbb{R} \times M \), so that \( X \) generates a (globally defined) flow. Good references are chapter 8 of “Introduction to differential topology” by Bröcker and Jänich and chapter 5 of “A comprehensive introduction to differential geometry”, volume I, by Michael Spivak.

A time dependent vector field parametrized by the interval \([0, 1]\) is a family of vector fields \( X_t \), for \( t \in [0, 1] \), that is smooth in the following sense. In local coordinates it has the form \( X_t = \sum a^j(t, x) \frac{\partial}{\partial x^j} \); we require \( a^j \) to be smooth functions of \((t, x^1, \ldots, x^n)\).

An isotopy (or time dependent flow) of \( M \) is a family of diffeomorphisms \( \psi_t : M \rightarrow M \), for \( t \in [0, 1] \), such that \( \psi_0 = \text{identity} \) and \((t, m) \mapsto \psi_t(m)\) is smooth as a map from \([0, 1] \times M\) to \( M \).

An isotopy \( \psi_t \) determines a unique time dependent vector field \( X_t \) such that

\[
\frac{d}{dt} \psi_t = X_t \circ \psi_t.
\]

That is, the velocity vector of the curve \( t \mapsto \psi_t(m) \) at time \( t \), which is a tangent vector to \( M \) at the point \( p = \psi_t(m) \), is the vector \( X_t(p) \).

A time dependent vector field \( X_t \) on \( M \) determines a vector field \( \bar{X} \) on \([0, 1] \times M\) by \( \bar{X}(t, m) = \frac{\partial}{\partial t} \oplus X_t(m)\). In this way one can treat time dependent vector fields and flows through ordinary vector fields and flows.

In particular, a time dependent vector field \( X_t \), \( t \in [0, 1] \), generates a “local isotopy” \( \psi_t(x) = \psi(t, x) \). If \( X_t \) is compactly supported then \( \psi_t(x) \) is defined for all \((t, x) \in [0, 1] \times M\). If \( X_t(m) = 0 \) for all \( t \in [0, 1] \) then there exists an open neighborhood \( U \) of \( m \) such that \( \psi_t : U \rightarrow M \) is defined for all \( t \in [0, 1] \).
The *Lie derivative* of a $k$-form $\alpha$ in the direction of a vector field $X$ is

$$L_X \alpha = \left. \frac{d}{dt} \right|_{t=0} \psi_t^* \alpha$$

where $\psi_t$ is the flow generated by $X$.

We have

$$L_X (\alpha \wedge \beta) = (L_X \alpha) \wedge \beta + \alpha \wedge (L_X \beta)$$

and

$$L_X (d\alpha) = d(L_X \alpha).$$

These follow from $\psi^*(\alpha \wedge \beta) = \psi^* \alpha \wedge \psi^* \beta$ and $\psi^* d\alpha = d\psi^* \alpha$.

**Cartan formula:**

$$L_X \alpha = \iota_X d\alpha + d\iota_X \alpha$$

where $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is

$$(\iota_X \alpha)(u_1, \ldots, u_{k-1}) = \alpha(X, u_1, \ldots, u_{k-1}).$$

(Outline of proof: it is true for functions. If it is true for $\alpha$ and $\beta$ then it is true for $\alpha \wedge \beta$ and for $d\alpha$.)

Let $\alpha_t$ be a time dependent $k$-form and $X_t$ a time dependent vector field that generates an isotopy $\psi_t$. Then

$$\frac{d}{dt} \psi_t^* \alpha_t = \psi_t^* \left( \frac{d\alpha_t}{dt} + L_{X_t} \alpha_t \right).$$

Outline of proof: if it is true for $\alpha$ and for $\beta$ then it is true for $\alpha \wedge \beta$ and for $d\alpha$. Hence, it is enough to prove it for functions.

The left hand side, applied to a time dependent function $f_t$ and evaluated at $m \in M$, is the limit as $t \to t_0$ of the difference quotient

$$\frac{f_t(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

This difference quotient is equal to

$$\left( \frac{f_t - f_{t_0}}{t - t_0} \right)(\psi_t(m)) + \frac{f_{t_0}(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

The limit as $t \to t_0$ of the first summand is

$$\left. \frac{df_t}{dt} \right|_{t=t_0} \psi_{t_0}^* (m).$$
The limit as $t \to t_0$ of the second summand is the derivative of $f_{t_0}$ along the tangent vector
\[
\left. \frac{d}{dt} \right|_{t=t_0} \psi_t(m) = X_{t_0}(\psi_{t_0}(m));
\]
this derivative is
\[
(X_{t_0} f_{t_0}) (\psi_{t_0}(m)) = \left( \psi_{t_0}^* (L_{X_{t_0} f_{t_0}}) \right)(m).
\]