MAT1344 (SYMPLECTIC GEOMETRY)
2006-07, SECOND TERM.

First half of course: Gromov non-squeezing.

Week 1. Manifolds and differential forms. Linear symplectic coordinates. Area form. Liouville volume. Symplectomorphisms. Statement of Gromov non-squeezing.

Homework 1: [Archimedes.] (Outside the poles), the standard area form on S^2 equals $d\theta \wedge dz$ in cylindrical coordinates.

Week 2. Vector fields and flows. de Rham cohomology. Ring structure. Darboux's theorem via Moser's method. Digression: embedded submanifolds.

Homework 2: Given two closed manifolds w/volume forms, if the total volumes are equal and there exists an orientation preserving diffeomorphism, then there exists a diffeomorphism that respects the volume forms.

Week 3. Singular homology. Integation on smooth cycles. CP^2, $S^2 \times S^2$. Vector bundles, tangent and cotangent bundles. Tautological one-form and canonical two-form on cotangent bundle. Riemannian metrics. Almost complex structures. Projection from metrics to compatible metrics.

Homework 3:
1. Prove: If n is even, there does not exist an orientation reversing diffeomorphism of CP^n.
2. On $S^2 \times S^2$, let $\omega_{a,b} = a\omega_{std} \oplus b\omega_{std}$. Prove: if $(S^2 \times S^2, \omega_{a,b})$ is symplectomorphic to $(S^2 \times S^2, \omega_{a',b'})$ and $a \geq b > 0$ and $a' \geq b' > 0$, then $a = a'$ and $b = b'$.

Homework 4: let M be a symplectic manifold, U an open subset, J_1 a compatible almost complex structure on an open set containing the closure of U. Show that there exists a compatible almost complex structure J_2 on M that coincides with J_1 on U.

Week 4. Flexibility of almost complex structures. Complex manifolds as almost complex manifolds. Differential forms and vector fields on complex and almost complex manifolds. Criteria for integrability. Holomorphic functions/maps/curves. Example: Fubini Studi form on CP^1. Outline of proof of Gromov non-squeezing.

Week 5. Wirtinger inequality and consequences. Moduli spaces of J-holomorphic curves. Example: curves in $\mathbb{CP}^1 \times V$ with product complex structure in the class $\mathbb{CP}^1 \times$point. Example of Gromov convergence and bubbling in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Homework 5: if (V, ω, J) is almost Kähler and $\varphi \colon \mathbb{CP}^1 \to V$ is J-holomorphic and not constant then $[\varphi]$ is nonzero in H_2.

Week 6. Definition of cusp-curve and weak convergence. Indecomposable classes. Example: $\mathbb{CP}^1 \times V$ when $\pi_2(V) = 0$, e.g. $V = \mathbb{C}^{n-1}/\lambda\mathbb{Z}^{2n-2}$. Statement of Gromov compactness. Case of indecomposable class. Evaluation map. Definition of Gromov-Witten invariant for curves through a point. Case of $\mathbb{CP}^1 \times V$.

Homework 6: Choose presentation topic.

Week 7. Proof of Gromov non-squeezing.

Second half of course: Classical mechanics, symmetries, geometric quantization.

Week 7, continued: Hamiltonian dynamics. Geometry on regular energy level. Symplectic reduction.

Week 8. Classical mechanics: kinematics; dynamics. Newton equations; Newton principle of determinacy. Conservative system. Hamilton principle of least action.

Calculus of variations; Euler Lagrange equations. Case of classical mechanics. Calculus of variations on a manifold.

Homework 7:
1. On $\mathbb{R}^6$ with $\Omega_{std} = \sum dq_i \wedge dp_i$, show

(a) The linear momentum function $(q, p) \mapsto p_i$ generates the linear motion flow $\partial/\partial q_i$.
(b) Angular momentum about the e_3 axis, $(q, p) \mapsto \langle q \times p, e_3 \rangle$, generates "rotation about the e_3 axis".

2. Prove Noether's theorem: "Symmetric implies conservation law". Fix a Hamiltonian dynamical system (M, ω, H) with $H^1(M) = 0$. Suppose that a flow Ψ_t is a symmetry: preserves ω and H. Then there exists a function f that generates this symmetry ("conjugate momentum" of the symmetry), and f is conserved under time evolution.

Week 9. Legendre transform. Lie groups. Lie group actions on manifolds. Orbits and stabilizers. Exponential map. Generating vector fields; Lie algebra actions. Conjugation action of $U(n)$ on Hermitian matrices. Orbits: $\mathcal{H}_\lambda$ = matrices with fixed eigenvalues $\lambda_1, \ldots, \lambda_n$. Special cases: $\mathbb{CP}^{n-1}$, Grassmannians, flag manifolds. Adjoint and coadjoint actions.

Week 10. Coadjoint orbits. Kirillov-Kostant symplectic form. Haar measure on compact Lie group; averaging. $\mathcal{H}_\lambda$ as coadjoint orbits. Moment maps. Torus actions on $\mathbb{C}^n$. Principal bundles and associated bundles.

Homework 8:

(a) Describe the orbit type stratification (strata and stabilizers) for the action of $\mathbb{T}^2$ on $\mathbb{C}^3$ given by $(a, b) \cdot (z_1, z_2, z_3) = (abz_1, a^{-1}bz_2, bz_3)$.
(b) Compute the moment map for this action.

Week 11. Basic forms. Marsden-Weinstein symplectic reduction. Example: Fubini-Studi form on $\mathbb{CP}^n$ (and its expression in homogeneous coordinates). Hamiltonian torus actions on compact manifolds have fixed points and have isotropic orbits; their moment maps are always invariant. Corollary: in a compact Lie group, any two maximal tori are conjugate.

Week 12. Local normal form for torus action near fixed point. Sketch proof of convexity for moment maps using point-set-topology version of "Local Global Prinzip" (local convexity implies global convexity). Poisson brackets on a symplectic manifold. Physics interpretation. Digression: Hamiltonian symplectomorphisms as a Lie group. Quantum mechanics; Dirac axioms.

Week 13. Complex Hermitian line bundles; relation to circle bundles. Connection and curvature. First Chern class. Kostant recipe for pre-quantization. Example: $\mathbb{CP}^1$; restriction to holomorphic sections gives the irreps of $SU(2)$.

Student presentation topics:

(1) Capacities and convexity.
(2) Gompf construction.
(3) Geodesic flow.
(4) Deformation quantization.
(5) Hofer metric.
(6) Linear non-squeezing for ellipsoids.
(7) Polarizations.
(8) Morse theory for moment maps.
(9) Poisson manifolds.