# An Introduction to Polarizations

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## 1. Introduction

In the prequantization construction the Hilbert space consists of all square integrable sections of the prequantum line bundle $(L, \pi, M)$, however this construction fails to satisfy the desired completeness condition. Part of the problem is that quantum states depend on only half the variables of the classical system yet as constructed the prequantum states depend on all the variables of the classical system. Polarizations are the geometric object that are used to decrease the dependency to $n$ variables.

To loosely summarize this idea, given a symplectic manifold $(M, \omega)$ of dimension $2n$ we will first choose $n$ directions in $M$ by way of a special distribution $\mathcal{P} \in TM_{\mathbb{C}}$ called a polarization. Then we say a section of the prequantum line bundle $\psi$ is polarized if it is constant along all the vector fields $\xi$ of $\mathcal{P}$, so

$$\nabla_\xi \psi = 0.$$

We remark that in general it is not sufficient to take the quantization space to be the $L^2$ integrable polarized sections, it still must be modified in some way. In what follows we define polarizations, consider some special kinds, namely real and Kähler polarizations, and briefly discuss some examples related to geometric quantization.

## 2. Definitions

Given a smooth manifold $M$ a distribution $D$ is a subbundle of $TM$. Extending this definition to $TM_{\mathbb{C}}$, we are now ready to define polarizations.

**Definition** [3] Let $(M, \omega)$ be a symplectic manifold, then a complex polarization is a distribution $\mathcal{P}$ of $TM_{\mathbb{C}}$ satisfying the following conditions

1. $\mathcal{P}$ is Lagrangian.
2. If $\eta, \xi$ are vector fields in $\mathcal{P}$ then $[\eta, \xi]$ is a vector field in $\mathcal{P}$. This will be abbreviated to $[\mathcal{P}, \mathcal{P}] \subset \mathcal{P}$ and we say $\mathcal{P}$ is involutive.
3. $\dim(\mathcal{P}_x \cap \overline{\mathcal{P}}_x \cap TM)$ is constant for all $x \in M$. [1]

It is not difficult to check that if $\mathcal{P}$ is a polarization then $\overline{\mathcal{P}}$ is also a polarization. The involutivity condition is equivalent to $\mathcal{P}$ being integrable by the Frobenius Criterion.

[1] In some of the literature this condition is omitted in the initial definitions, although ultimately it is needed in practice, here we include it as in [3].

The third condition is sometimes omitted in the defintion, see [2],[4] but it is usful to include it. For each polarization $\mathcal{P}$ we have $D = \mathcal{P} \cap \overline{\mathcal{P}} \cap TM$ and $E = \mathcal{P} \oplus \overline{\mathcal{P}} \cap TM$ are subsets of $TM$. The third condition ensures these are also distributions of $TM$. Notice that $\mathcal{P} \cap \overline{\mathcal{P}}$ is always involutive however $E$ may not be. We wish to study the polarizations $\mathcal{P}$ when $E$ is involutive.

**Definition** [3] Given a symplectic manifold $(M, \omega)$, a polarization $\mathcal{P}$ is strongly integrable if

1. $E$ is involutive
2. The spaces of integrable manifolds $M/D$ and $M/E$ are differentiable manifolds.
3. The canonical projection $\pi_{DE} : M/D \to M/E$ is a submersion.

What is important is if $\mathcal{P}$ is a strongly integrable polarization then there are local coordinates on $M$, $\{x_1, \ldots, x_{n-k}; y_1, \ldots, y_{n-k}; u_1, \ldots, u_k; v_1, \ldots, v_k\}$ such that $\mathcal{P}$ is spanned by $\{\frac{\partial}{\partial x_i}\}_{i=1}^{n-k}$ and $\{\frac{\partial}{\partial \overline{z_j}}\}_{j=1}^{k}$ where $z_k = u_k + iv_k$ [4].

We now consider examples of special types of strongly integrable polarizations.

## 2. Real and Kähler Polarizations

Given a symplectic manifold $(M, \omega)$ a polarization $\mathcal{P}$ of $M$ is real if $\mathcal{P} = \overline{\mathcal{P}}$.

**Proposition 0.1** *[3] Let $(M, \omega)$ be a symplectic manifold and suppose the polarization $\mathcal{P}$ is real. Let $D = \mathcal{P} \cap TM$, then $D$ is a Lagrangian distribution of $TM$. Conversely, if $D$ is a Lagrangian distribution of $TM$, then $D_{\mathbb{C}}$ is a real polarization.*

In particular there are coordinates $\{x_1, \ldots, x_n; y_1, \ldots y_n\}$ of $M$ such that $\mathcal{P}$ is spanned by $\{\frac{\partial}{\partial x_i}\}_{i=1}^{n}$.

A polarization $\mathcal{P}$ of a symplectic manifold $(M, \omega)$ is pseudo-Kählerian if $\mathcal{P} \cap \overline{\mathcal{P}} = 0$. For any polarization we can define a Hermitian form on $\mathcal{P}$ by

$$h(u, v) := i\omega(u, \overline{v}).$$

Then $\mathrm{Ker}(h) = \mathcal{P} \cap \overline{\mathcal{P}}$, so if $\mathcal{P}$ is pseudo-Kählerian then $h$ is non-degenerate. If $h$ is positive definite on $\mathcal{P}$ we say that $\mathcal{P}$ is a Kälher polarization [4].

The following proposition explains the name Kähler polarizations. First recall that a Kähler manifold is a symplectic manifold $(M, \omega)$ with a compatible complex structure $J$. Then we have distributions of $TM_{\mathbb{C}}$,

$$T_{(1,0)} = \{v \in T_x M^{\mathbb{C}} | J_x(v) = iv\} \quad \text{and} \quad T_{(0,1)} = \{v \in T_x M^{\mathbb{C}} | J_x(v) = -iv\}.$$

**Proposition 0.2** *Given a symplectic manifold $(M, \omega)$ and a complex structure $J$, $\mathcal{P} = T_{(0,1)}$ and $\overline{\mathcal{P}} = T_{(1,0)}$ are Kähler polarizations. Conversely if $(M, \omega)$ has a Kähler polarization then there exists a compatible complex structure $J$ on $(M, \omega)$.*

Supposing $M$ is $2n$-dimensional we know $\dim {T_{(1,0)}}_x = \dim {T_{(0,1)}}_x = n$. Since $\omega$ is compatible with $J$ if $u, v \in {T_{(1,0)}}_x$ then

$$\omega(u,v) = \omega(Ju, Jv) = \omega(iu, iv) = -\omega(u,v),$$

so $\omega(u,v) = 0$ and $T_{(1,0)}$ is Lagrangian, similarly for $T_{(0,1)}$. The Newlander-Nirenberg Theorem says $J$ is integrable if and only if $T_{(1,0)}$ is involutive.

Suppose $\mathcal{P}$ is Kähler, then $TM_{\mathbb{C}} = \mathcal{P} \oplus \overline{\mathcal{P}}$, so we for all $v_x \in T_xM \subset \mathbb{T}_x M_{\mathbb{C}}$ can write $v_x = w_x + w_x^{'}$ where $w_x \in \mathcal{P}_x$ and $w_x^{'} \in \overline{\mathcal{P}}_x$. Now we define $J : TM \to TM$ by $J_x(v_x) = -iw_x + iw_x^{'}$. This is compatible with $\omega$.

$$\begin{aligned}
\omega(Jv, Ju) &= \omega(Jw + Jw', Jz + Jz') \\
&= \omega(-iw + iw', -iz + iz') = \omega(-w' + w, -z + z') \\
&= \omega(-w', -z) + \omega(+w, +z') + \omega(-w', z') + \omega(w, -z) \\
&= \omega(v, u)
\end{aligned}$$

The Riemannian metric $g(u,v) = \omega(u, Jv)$ is positive definite since the Hermitian form $h$ is. Finally by the definition of $J$, $\overline{\mathcal{P}} = T_{(1,0)}$, then integrability of $J$ follows from Newlander-Niremberg and involutivity of $\overline{\mathcal{P}}$. $\square$

## 4. Examples

Let us consider two simple examples in relation to geometric quantization. Take $M = T^*Q$, with canonical basis $\{q_i, p_j\}$ and standard symplectic form $\omega = \sum dq_i \wedge dp_i$. Take the polarization $\mathcal{P}$ to be the span of $\{\frac{\partial}{\partial p_i}\}_{i=1}^n$, ie. the vertical vector fields. The polarized sections $\psi$ are the sections for which $\frac{\partial \psi}{\partial p_i} = 0$, so those which are constant along the fibers. This is the Schrödinger representation of $(T^*M, \omega)$. If $Q = \mathbb{R}^n$ we could alternatively take $\mathcal{P}$ to be spanned by $\{\frac{\partial}{\partial q_i}\}_{i=1}^n$. Then we would obtain what is called the momentum representation [3].

For the next example let $M = T^*Q$ and take the basis $z_j, \overline{z_j}$ where $z_j = p_j + iq_j$. Then the standard symplectic form becomes $\omega = \frac{1}{2} d\overline{z_j} \wedge dz_j$ and the complex structure is defined by $Jz_i = iz_i$, $J\overline{z_i} = -i\overline{z_j}$. Choosing the Kähler polarization corresponding to $J$, ie. $\mathcal{P}$ spanned by $\{\frac{\partial}{\partial \overline{z_j}}\}_{j=1}^n$ the polarized sections must satisfy $\frac{\partial \phi}{\partial \overline{z_j}} = 0$ so they are the holomorphic sections. This representation is called the *holomorphic* or *Bargmann-Fock* representation. If instead we took $\overline{\mathcal{P}}$ we would obtain the anti-holomorphic sections [3].

# References


[1] Cannas da Silva, A. *Lectures on Symplectic Geometry* Lecture Notes in Mathematics 1764, Springer-Verlag, Berlin, 2001.

[2] Guillemin, V., Ginzburg V., Karshon, Y., *Moment Maps, Cobordisms, and Hamiltonian Group Actions* American Mathematical Society, Providence, 1997.

[3] Echeverria-Enriquez, A., Munoz-Lecanda A. C., Roman-Roy N. and Victoria-Monge, C. " Mathematical Foundations of Geometric Quantization", EXTRACTA MATH., [13], 135, (1931). http://www.citebase.org/abstract?id=oai:arXiv.org:math-ph/9904008.

[4] Kirillov A.A., " Geometric Quantization ", in *Encyclopaedia of Mathematical Sciences* (vol. 4) *Dynamical Systems (IV)* Springer, Berlin, (1985), 137-172.