An Introduction to Poisson Manifolds

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1 Preliminaries

The purpose of this paper is to introduce Poisson manifolds, and to prove the Splitting Theorem; the result of which will show that Poisson manifolds are in fact foliations with symplectic leaves. The presentation is based on that of Weinstein's paper, [3]. We begin with some basic notions.

Definition 1.1. A *Poisson structure* on a manifold P is a Lie algebra structure on $C^\infty(P)$, called a *Poisson bracket*, which satisfies Leibniz' rule in both entries. That is, P is equipped with an antisymmetric, bilinear map $\{\cdot,\cdot\} : C^\infty(P) \to C^\infty(P)$ that satisfies Jacobi's identity, and for any $f, g, h \in C^\infty(P)$

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \tag{1}$$

with a similar equation being valid in the other entry.

Note that for a fixed $h \in C^\infty(P)$, (1) implies that $\{\cdot, h\}$ is in fact a derivation on the ring $C^\infty(P)$; *i.e.* there exists a vector field $\xi_h \in \mathrm{Vect}(P)$ such that for any $f \in C^\infty(P)$, $\xi_h f = \{f, h\}$. This vector field is called the *Hamiltonian vector field* of h. If one of the functions in the bracket is constant, then the Leibniz rule demands that the bracket be equal to zero. Thus, there is a well-defined bundle map $B : T^*P \to TP : df|_p \mapsto \xi_f|_p$, or equivalently, there is a smooth contravariant antisymmetric 2-tensor w on P, called the *cosymplectic structure*, such that for any $f, g \in C^\infty(P)$, $\langle (df, dg)|w\rangle = \{f, g\}$.

There are a couple properties of the Poisson bracket worth mentioning. First, given a point $p \in P$ and local coordinates $(x^1, ..., x^n)$ about p, for any point q in the corresponding coordinate neighbourhood of p, $\{x^i, x^j\}(q) = \langle (dx^i, dx^j)|w\rangle(q) = w_{ij}|_q$, and so the bracket is locally determined by the bracket of the coordinate projections. Second, by the Jacobi identity, $\xi_{\{f,g\}} = -[\xi_f, \xi_g]$ for any $f, g \in C^\infty(P)$, and so the map $f \mapsto \xi_f$ is an anti-Lie algebra homomorphism from $C^\infty(P)$ to $\mathrm{Vect}(P)$.

Definition 1.2. A smooth map $\varphi : P_1 \to P_2$ between two Poisson manifolds P_1 and P_2 with Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, respectively, is called a *Poisson mapping* if for any $f, g \in C^\infty(P_2)$, $\{f \circ \varphi, g \circ \varphi\}_1 = \{f, g\}_2 \circ \varphi$.

Definition 1.3. A *Poisson submanifold* Q of a Poisson manifold P is a submanifold equipped with a Poisson structure such that the inclusion map is a Poisson mapping.

Definition 1.4. Given two Poisson manifolds, P_1 and P_2, we define the product $P_1 \times P_2$ to be the usual product of smooth manifolds equipped with a Poisson structure $\{\cdot,\cdot\}$ such that the projection maps $\pi_i : P_1 \times P_2 \to P_i$ are Poisson mappings, and

$$\{f \circ \pi_1, g \circ \pi_2\} = 0$$

for all $f \in C^\infty(P_1)$ and $g \in C^\infty(P_2)$. This last condition ensures that $C^\infty(P_1)$ and $C^\infty(P_2)$ can be mapped via bijective Lie algebra homomorphisms onto commuting subalgebras of $C^\infty(P_1 \times P_2)$.

Definition 1.5. The *rank* of a Poisson manifold at a point $p \in P$ is the rank of the map $B|_p : T_p^*P \to T_pP$, or equivalently, the rank of the cosymplectic structure at p.

Remark 1.6. If the rank of the Poisson structure of a Poisson manifold P is equal to the dimension of the manifold at every point, then P has a natural symplectic structure. Define a symplectic form Ω on P as $\Omega(\xi|_p, \zeta|_p) = w(B^{-1}(\xi|_p), B^{-1}(\zeta|_p))$ for any $p \in P$, noting that B would have full rank at every point, and hence be an invertible bundle map. The Poisson bracket and the Hamiltonian vector fields will coincide with those defined in the usual sense on symplectic manifolds.

2 The Splitting Theorem

Theorem 2.1. *Let P be a Poisson manifold, and let $p \in P$. Then there exist a neighbourhood $U \subseteq P$ of p and a diffeomorphic Poisson mapping $\varphi = \varphi_S \times \varphi_N : U \to S \times N$ where S is a symplectic manifold and N is a Poisson manifold with rank zero at $\varphi_N(p)$.*

Proof. If P has rank zero at p, then U is diffeomorphic to $\{p\} \times U$. $\{p\}$ is a symplectic manifold, and U, as a Poisson submanifold of P, has rank zero at p, and we are done.

If P has rank greater than zero at p, then there exist functions $f_1, g_1 \in C^\infty(P)$ such that $\{f_1, g_1\}(p) = \xi_{g_1} f_1(p) \neq 0$. Hence, $\xi_{g_1} \neq 0$, and so we can apply the "straightening-out lemma" to this vector field in a neighbourhood $U_1 \subseteq P$ of p (see [2]) to get $\xi_{g_1}|_{U_1} = \frac{\partial}{\partial h_1}|_{U_1}$ for some $h_1 \in C^\infty(P)$. Hence, $\{h_1, g_1\} = 1$.

Next, from the Jacobi identity for the Poisson bracket we have that ξ_{g_1} and ξ_{h_1} commute (and so g_1 and h_1 are independent). If P has dimension n, then we can find $n-2$ functions $x^3, ..., x^n \in C^\infty(P)$ such that

$$\frac{\partial x^i}{\partial g_1} = \frac{\partial x^i}{\partial h_1} = 0 \tag{2}$$

for each $i = 3, ..., n$; that is, we have n (independent) coordinates $(g_1, h_1, x^3, ..., x^n)$ about p, and in fact, from (2), we have that for each $i = 3, ..., n$, x^i commutes with both g_1 and h_1. The coordinates (g_1, h_1) induce a Poisson submanifold $S_1 \subseteq U_1$ of dimension 2 (since the coordinates are independent), with bracket defined such that the projection is a Poisson mapping. The rank of S_1 everywhere is 2; hence, S_1 is symplectic. The coordinates $(x^3, ..., x^n)$ induce a Poisson submanifold $N_1 \subseteq U_1$ as well, with bracket determined by the cosymplectic structure defined by $(w_1)_{i-2,j-2}|_q = \{x^i, x^j\}(q)$ for all $q \in N_1$. We have shown that the subalgebras of $C^\infty(U_1)$ corresponding to $C^\infty(S_1)$ and $C^\infty(N_1)$ commute, and so $U_1 = S_1 \times N_1$ is a well-defined product of Poisson manifolds.

We apply the same procedure above to N_1, and so on, going through the procedure a finite number of times (say m times) after which we have a resulting neighbourhood U of p such that $U = S_1 \times ... \times S_m \times N_m$, with local coordinates $(g_1, ..., g_m, h_1, ..., h_m, y^1, ..., y^{n-2m})$ satisfying

$$\{g_i, g_j\} = \{h_i, h_j\} = \{g_i, y^j\} = \{h_i, y^j\} = 0$$

for all i, j, and $\{g_i, h_j\} = \delta_{ij}$, and $\{y^i, y^j\}(q) = w_{ij}(q)$ for all $q \in N_m$. $S = S_1 \times ... \times S_m$ is a symplectic manifold, and $N = N_m$ is a Poisson manifold with Poisson bracket determined by w, which we claim has rank zero at $\varphi_N(\phi_N(p))$ for large enough m. To justify this, we need only show that an m exists such that the rank of the Poisson bracket of N is zero at $\varphi_N(p)$. But if $m = n$ then the Poisson bracket becomes trivial (since the cosymplectic structure would have rank less than two; *i.e.* 0). Similar to the above argument, U is a well-defined product of Poisson manifolds, and the theorem is proved. $\square$

Remark 2.2. S and N as described in the above theorem are in fact unique up to local Poisson diffeomorphism (the proof of which shall be omitted – see [3]).

Examining the proof of the splitting theorem more closely, we have the following. Define a relation on the points of a Poisson manifold P: set $p \sim q$ if there exists a piecewise smooth curve from p to q, the smooth segments of which are trajectories of Hamiltonian vector fields. Then $\sim$ is clearly an equivalence relation. Take a point $p \in P$, and given the neighbourhood U about p as described in the hypothesis of the splitting theorem, all Hamiltonian vector fields of U are tangent to S, and so for any $q \in U$, $p \sim q$ if and only if there exists a curve from p to q that never travels transversely with respect to S. Cover any equivalence class with enough such neighbourhoods, and glueing the symplectic submanifolds together, we have that the equivalence class itself is a symplectic submanifold. Thus, P is a foliation with symplectic leaves.

3 Examples

Example 3.1. Given a smooth manifold M, we can equip this with the trivial Poisson structure: $\{f, g\} = 0$ for all $f, g \in C^\infty(M)$. The rank of M is 0 everywhere, and the symplectic leaves are precisely the points of M.

Example 3.2. Let M be a symplectic manifold and N an arbitrary manifold equipped with the trivial Poisson structure. Then $M \times N$ is a Poisson manifold with symplectic leaves $\{M \times \{x\} \mid x \in N\}$.

Example 3.3. The coadjoint orbits of a Lie group G can be realised as the symplectic leaves of the Poisson manifold $\mathfrak{g}^*$. Define the Poisson bracket on $\mathfrak{g}^*$ as

$$\{f, h\}(\phi) = \langle \phi \mid [df|_\phi, dh|_\phi] \rangle$$

for $f, h \in C^\infty(\mathfrak{g}^*)$ and $\phi \in \mathfrak{g}^*$. For example, consider the Lie group SU(2). Then $\mathfrak{g}^* = \mathfrak{su}(2) \cong \mathbb{R}^3$, and the coadjoint action is in fact rotations about the origin. The corresponding orbits are concentric spheres about the origin, $\{\partial B(0, r) \mid r \geq 0\}$, each of which is symplectic.

References

[1] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, New York, 2003.

[2] Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, Inc., Houston, Texas, 1999.

[3] Alan Weinstein, *The Local Structure of Poisson Manifolds*, J. Differential Geometry **18** (1983), 523–557.