

AN INTRODUCTION TO THE HOFER METRIC

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1. INTRODUCTION

In this paper, we introduce a bi-invariant metric on the space of compactly supported Hamiltonian symplectomorphisms of a symplectic manifold (M, ω) . In Hofer's original work, the non-degeneracy of this metric was proved in the special case $M = \mathbb{R}^{2n}$. By invoking a generalization of Gromov's non-squeezing theorem, due to Lalonde and McDuff, we shall show that Hofer's metric is indeed a metric for arbitrary manifolds. The exposition presented herein follows closely that of McDuff and Salamon [2].

2. THE HOFER METRIC

Let (M, ω) be a $2n$ -dimensional symplectic manifold without boundary. We denote by $Ham^c(M)$ the class of symplectomorphisms that are the time-one maps of the flows generated by time-dependent, compactly supported, Hamiltonian functions. Given a path $\{\phi_t\}_{0 \leq t \leq 1} \subset Ham^c(M)$, we may define its length to be

$$\mathcal{L}(\{\phi_t\}) := \int_0^1 \sup_{z \in M} H_t(z) - \inf_{z \in M} H_t(z) dt,$$

where $\{H_t\}$ is a Hamiltonian function generating the flow from ϕ_0 to ϕ_1 . Note that the length of a path does not change if the Hamiltonian is altered by an additive constant.

We now define the Hofer metric on $Ham^c(M)$ by declaring the distance between two symplectomorphisms to be the infimum of the lengths of all paths connecting them:

$$\rho(\varphi, \psi) = \inf_{\substack{\{\phi_t\} \subset Ham^c(M) \\ \phi_0 = \varphi, \phi_1 = \psi}} \mathcal{L}(\{\phi_t\}).$$

Theorem 1. *The function ρ defined above is indeed a bi-invariant metric on $Ham^c(M)$. Namely, for all symplectomorphisms φ , ψ , and χ , we have*

1. $\rho(\varphi, \psi) = \rho(\psi, \varphi)$ (symmetry)
2. $\rho(\varphi, \psi) \leq \rho(\varphi, \chi) + \rho(\chi, \psi)$ (triangle inequality)
3. $\rho(\varphi \circ \chi, \psi \circ \chi) = \rho(\varphi, \psi) = \rho(\chi \circ \varphi, \chi \circ \psi)$ (bi-invariance)
4. $\rho(\varphi, \psi) = 0$ if and only if $\varphi = \psi$ (non-degeneracy)

Proof. The fact that ρ is symmetric, is bi-invariant, and satisfies the triangle inequality follows directly from definitions.

In addition, we have $\rho(\psi, \psi) = \rho(id, id) = 0$, for any ψ , using the generating function $H_t = 0$ for the trivial flow. It only remains to show that ρ is non-degenerate; in light of the bi-invariance property, this fact is established via the following proposition.

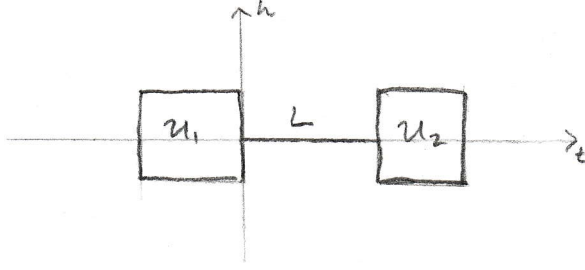


Figure 1: The configuration of U_1 , U_2 and L .

□

Proposition 2. *If $\phi \neq id$, then $\rho(\phi, id) > 0$.*

Proof. As a preliminary step, we note that there exists an embedded open ball $B(r) \subset M$ of radius r such that $\phi(B(r)) \cap B(r) = \emptyset$: if $\phi(x) \neq x$, we can separate these two points by neighbourhoods and use a coordinate chart to embed a sufficiently small ball in M which has the required property.

Our strategy for the proof is as follows: given any $\epsilon > 0$, we shall show that it is possible to construct a symplectic embedding of the standard $2n + 2$ -dimensional open ball $B^{2n+2}(r)$ of radius r , into some cylinder $B^2(R) \times M$, with $\pi R^2 \leq \frac{\pi r^2}{2} + \rho(\phi, id) + \epsilon$. The non-squeezing theorem asserts that $\pi r^2 \leq \pi R^2$, and hence choosing ϵ sufficiently small will imply that $\rho(\phi, id)$ is strictly positive.

We shall construct the embedding $B^{2n+2}(r) \hookrightarrow B^2(R) \times M$ in two stages.

Stage 1: Let L denote the unit interval $[0, 1] \times \{0\}$ in the plane, and let U_1 and U_2 be two squares, each of area $\frac{\pi r^2}{2}$, as in Figure 1. Note that for any small neighbourhood U of $U_1 \cup U_2 \cup L$, we can symplectically embed a standard two-dimensional ball $B^2(r)$ into U (with respect to the standard symplectic form on \mathbb{R}^2), since U is symplectomorphic to a slightly larger ball in the plane.

Let $Z = (U_1 \times B(r)) \cup (U_2 \times B(r)) \cup (L \times B(r)) \subset \mathbb{R}^2 \times M$. We may construct a symplectic embedding

$$B^{2n+2}(r) \hookrightarrow \mathcal{N}(Z)$$

of a $2n + 2$ -dimensional ball into any small neighbourhood $\mathcal{N}(Z)$ of Z , by applying the embedding in the previous paragraph to the first two coordinates of $B^{2n+2}(r)$, and the identification $B^{2n}(r) \rightarrow B(r) \subset M$ to the last $2n$ coordinates.

Stage 2: Fix some $\epsilon > 0$, and choose a path $\{\phi_t\}$, with $\phi_0 = id$ and $\phi_1 = \phi$, and corresponding Hamiltonian function $\{H_t\}$, such that

$$\mathcal{L}(\{\phi_t\}) = \int_0^1 \sup_{z \in M} H_t(z) - \inf_{z \in M} H_t(z) dt \leq \rho(\phi, id) + \epsilon/2.$$

Let $A = \{(h, t); t \in [0, 1], \inf_M H_t \leq h \leq \sup_M H_t\} \subset \mathbb{R}^2$ be the area spanned by the range of values of $\{H_t\}$ as t varies. By assumption, $area(A) \leq \rho(\phi, id) + \epsilon/2$.

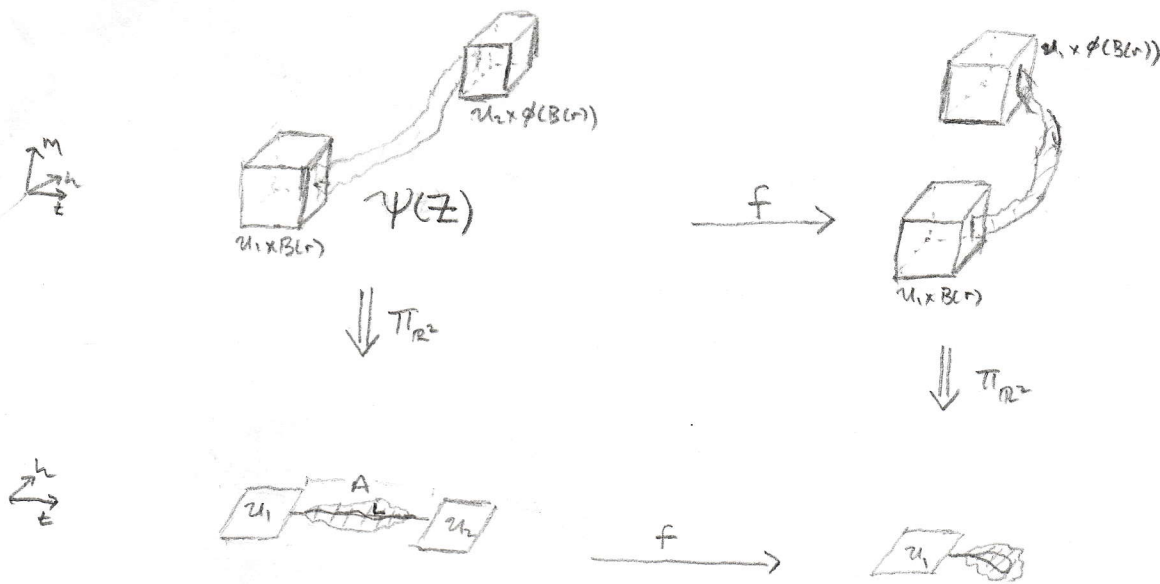


Figure 2: The effect of rotation.

Now consider the map Ψ defined by

$$\Psi(h, t, z) = (h + H_t(z), t, \phi_t(z)), \quad \text{for } h \in \mathbb{R}, t \in [0, 1], z \in M$$

A priori, Ψ is only defined on $\mathbb{R} \times [0, 1] \times M$. Note, however, that without loss of generality, we may assume $\{H_t\}$ is identically zero in small neighbourhoods of 0 and 1 (since this can be achieved by reparametrizing $\{H_t\}$, which does not affect the length of $\{\phi_t\}$), and hence we may smoothly extend $\{H_t\}$ by zero for all t outside $[0, 1]$. Equivalently, we may extend $\{\phi_t\}$, by setting $\phi_t = id$ for $t \leq 0$ and $\phi_t = \phi$ for $t \geq 1$. Thus, we may assume Ψ is defined on all of $\mathbb{R} \times \mathbb{R} \times M$. A direct computation reveals that it is a symplectomorphism, if we choose the obvious symplectic form $\Omega = dh \wedge dt + \omega$ on $\mathbb{R} \times \mathbb{R} \times M$.

Note that if we view U_1, U_2 and L from step 1 as living in the (t, h) plane, $\Psi|_{U_1 \times M}$ acts as the identity, since the t -coordinate is negative there. Similarly, $\Psi|_{U_2 \times M} = (id, id, \phi)$, and $\Psi(L \times M) \subset A \times M$. Hence, given a neighbourhood $\mathcal{N}(Z)$ of Z , Ψ provides an embedding

$$\Psi : \mathcal{N}(Z) \hookrightarrow \mathcal{N}(U_1 \cup U_2 \cup A) \times M$$

into some neighbourhood of $U_1 \cup U_2 \cup A$.

Now, we shall “push” the set $U_2 \times \phi(B(r))$ on top of $U_1 \times B(r)$. Choose some cutoff function $\beta : M \rightarrow \mathbb{R}$ that vanishes on $B(r)$ and is identically 1 on a small neighbourhood of $\phi(B(r))$, and let $f : \mathbb{R} \times \mathbb{R} \times M \rightarrow \mathbb{R} \times \mathbb{R} \times M$ act by rotating the first two coordinates through an angle of $\beta\pi$; this transformation clearly preserves the symplectic structure, and will be injective on a sufficiently small neighbourhood of $\Psi(Z)$. In particular, $f(U_2 \times \phi(B(r))) = U_1 \times \phi(B(r))$. Moreover, since we assumed that $\{H_t\}$ is zero in neighbourhoods of 0 and 1, the projection of the image of $\Psi(U_1 \cup A \cup U_2)$ under f is the union of the square U_1 and a “teardrop” shape of area less than that of A . See Figure 2.

In other words, if $Y = \pi_{\mathbb{R}^2}(f \circ \Psi(Z))$ is the projection on to the first two components of the image of Z , we have an embedding

$$\mathcal{N}(Z) \hookrightarrow \mathcal{N}(Y) \times M,$$

for $\mathcal{N}(Z)$ and $\mathcal{N}(Y)$ any sufficiently small neighbourhoods of Z and Y respectively. Also, by the above considerations, it follows that

$$\text{area}(Y) \leq \text{area}(U_1) + \text{area}(A) \leq \frac{\pi r^2}{2} + \rho(\phi, id) + \epsilon/2.$$

Hence, we can symplectically embed a sufficiently small neighbourhood of Y into a slightly larger ball - say, of radius R , where

$$\pi R^2 \leq \frac{\pi r^2}{2} + \rho(\phi, id) + \epsilon,$$

which yields an embedding

$$\mathcal{N}(Y) \times M \hookrightarrow B^2(R) \times M.$$

Fixing an appropriately small neighbourhood of Z , and invoking step 1, we thus arrive at our desired embedding

$$B^{2n+2}(r) \hookrightarrow B^2(R) \times M.$$

The non-squeezing theorem [1](Theorem 1.4) asserts that $r \leq R$, so

$$\rho(\phi, id) \geq \frac{\pi r^2}{2} - \epsilon.$$

As ϵ was arbitrary, choosing ϵ sufficiently small implies that $\rho(\phi, id)$ is strictly positive as required. □

Roughly speaking, the reference to the non-squeezing theorem is not altogether unexpected. If one defines the *displacement energy* $e(A)$ of a subset $A \subset \mathbb{R}^{2n}$ to be

$$e(A) := \sup_{K \subset A \text{ compact}} \left\{ \inf_{\substack{\phi \in \text{Ham}^c(M) \\ \phi(K) \cap K = \emptyset}} \rho(\phi, id) \right\},$$

a slight modification of the argument presented in this paper proves that $e(A)$ is a relative symplectic capacity on \mathbb{R}^{2n} (which implies Proposition 2 in the case $M = \mathbb{R}^{2n}$). Of course, one needs the non-squeezing theorem in order to exhibit this result; nevertheless, the fact that a symplectic capacity is lurking nearby might suggest that the non-squeezing theorem should play a role in the proof presented here.

References

- [1] Lalonde, F., McDuff, D. "The Geometry of Symplectic Energy". *The Annals of Mathematics*, v. 141. pp. 349-371 (1995).
- [2] McDuff, D., Salamon, D. *Introduction to Symplectic Topology* Oxford University Press, Oxford (1998).