

Some aspects of the Geodesic flow

Pablo C.

Abstract

This is a presentation for Yael's course on Symplectic Geometry. We discuss here the context in which the geodesic flow can be understood using techniques from Symplectic Geometry.

1 Introduction

We begin defining our object of study.

Definition 1.1. *Given a closed manifold \mathcal{M} with Riemannian metric g , the co-geodesic flow is defined as the Hamiltonian flow ϕ^t on the cotangent bundle $(T^*\mathcal{M}, \Theta)$ (Θ will always denote the canonical form on $T^*\mathcal{M}$) where the corresponding lagrangian is $L = \frac{1}{2}g$*

Suppose that the Legendre transform is given by $(x, v) \mapsto (x, p)$. Then we know

- $H = \langle p, v \rangle - L(x, v)$
- p is given as the fiber derivative of L . Namely, consider $f(v) = L(x, v)$ for x fixed and then $p(x, v)$ is given by $p = D_v f$

In our very particular case, we can easily calculate the transform using Euler's lemma for homogeneous functions (L is just a quadratic form here); get we

$$\langle p, v \rangle = \langle D_v f, v \rangle = 2f(v) = 2L = g(v, v)$$

This implies

- the Legendre transform identifies the fibers of $\mathcal{T}\mathcal{M}$ and $\mathcal{T}^*\mathcal{M}$ using the metric $(v \mapsto g(v, \cdot))$.
- $H = 2L - L = L$.

This shows that the co-geodesic flow induces a flow (which we will continue denoting ϕ^t) on $\mathcal{T}\mathcal{M}$ such that $\phi^t = (\gamma, \dot{\gamma})$ and γ satisfies the variational equation for the action defined by $L = H$.

We have that¹ $\frac{d}{dt} \frac{\partial L}{\partial v} = 0$, so the norm of $\dot{\gamma}$ is constant. We will restrict ourselves to $\|v\| = 1$ (ie we'll work in the unit tangent bundle of \mathcal{M}).

Definition 1.2. *The flow ϕ^t on $\mathcal{T}\mathcal{M}$ is called the geodesic flow, and the curves γ are called geodesics.*

Notice that in particular, the generating vector field (called the *geodesic vector field*) of the flow in $\mathcal{T}\mathcal{M}$ is given by

$$G(X) = \left. \frac{\partial}{\partial t} \right|_{t=0} \phi^t(X) = \left. \frac{\partial}{\partial t} \right|_{t=0} (\gamma, \dot{\gamma})$$

More can be said about the level sets of the Lagrangian (\sim metric). We know that the level sets of the Hamilton equation are preserved by any Hamiltonian flow, but in general, different level sets carry different dynamics. However in our special case, we have

Proposition 1.1. *Suppose that Γ_a and Γ_b are two level sets of the metric g . Then, up to reparametrization Γ_a and Γ_b are isomorphic.*

Proof. Notice that $\left\| \sqrt{\frac{b}{a}} v \right\|^2 = \frac{b}{a} \|v\|^2$. So

$$v \in \Gamma_a \Leftrightarrow \sqrt{\frac{b}{a}} v \in \Gamma_b$$

Since Γ_a and Γ_b are invariant by the flow, we are done. □

¹Conservation of energy!

Now we want to see how the geodesic flow looks like in the tangent bundle of \mathcal{M} . Recall (cf. appendix) that the Riemannian connexion induces an splitting of $\mathcal{T}\mathcal{M}$

$$X \in \mathcal{T}\mathcal{M} \Rightarrow \mathcal{T}_X\mathcal{T}\mathcal{M} = \mathcal{T}_{X_h}\mathcal{T}\mathcal{M} \oplus \mathcal{T}_{X_v}\mathcal{T}\mathcal{M}$$

If $X = (x, v)$ each of the previous subspaces is (canonically) isomorphic to $\mathcal{T}_x\mathcal{M}$, and we'll identify corresponding vectors with their images in $\mathcal{T}_x\mathcal{M}$.

Proposition 1.2. *The symplectic 2-form Θ on $\mathcal{T}\mathcal{M}$ is given by*

$$\Theta(\xi, \eta) = g(\xi_h, \eta_v) - g(\xi_v, \eta_h)$$

Proof. We'll show that the form Ω defined by the right hand of the equation satisfies for every $X \in \mathcal{T}\mathcal{M}$, $\xi \in \mathcal{T}_X\mathcal{T}\mathcal{M}$

$$dL_X(\xi) = \Omega_X(G(X), \xi) = \iota_{G(X)}\Omega(\xi) \quad (1.1)$$

Take a curve $z(t)$ on $\mathcal{T}\mathcal{M}$ such that $z(0) = X$, $\dot{z}(0) = \xi$ and write it as $z(t) = (\alpha(t), Z(t))$. Then

$$dL_X(\xi) = \left. \frac{\partial}{\partial t} \right|_{t=0} L(z(t)) = \left. \frac{\partial}{\partial t} \right|_{t=0} \frac{1}{2}g((\nabla_{\dot{\alpha}}Z)(0), Z(0)) = g(\xi_h, v) \quad (1.2)$$

We know calculate with Ω to get

$$\Omega_X(G(X), \xi) = g(G(X)_h, \xi_v) - g(G(X)_v, \xi_h) = g(G(X)_h, \xi_v) = g(v, \xi_h) \quad (1.3)$$

□

We'll finish with a very well know example.

Example : integrability of the geodesic flows for surfaces of revolution

Let's recall first if we have a Hamiltonian system $H : \mathcal{M}^{2n} \mapsto \mathbf{R}$, we say that the a real valued function $f : \mathcal{M} \mapsto \mathbf{R}$ is an integral of H if f is constant along trajectories of H (equivalently, $\{f, H\} = 0$). We say that the Hamiltonian systems is completely integrable if it has $n - 1$ independent² integrals.

Now suppose that $S \subset \mathbf{R}^3$ is a surface of revolution and suppose S is given by the induced metric from the ambient space.

²ie. its differentials are l.i. in $\mathcal{T}^*\mathcal{M}$

If we use polar coordinates $(r, \theta) \mapsto (f(r) \cos \theta, f(r) \sin \theta, g(z))$ we see that θ doesn't appear explicitly in $L = 1/2g$. So by Euler-Lagrange equations we have

$$P = r^2 \dot{\theta} = \text{const.} \quad (1.4)$$

The previous equation is known as Clairaut's equation, and so L and P are a complete system of integrals for the geodesic flow on S .

2 Appendix

Here we collect some facts about Riemannian geometry.

Proposition 2.1. *Let \mathcal{M} a closed Riemannian manifold with metric $\|\cdot\|$. Then for any $a > 0$ the set $\Gamma = \{v \in \mathcal{T}\mathcal{M} : \|v\|^2 = a\}$ is a (codimension 1) submanifold of $\mathcal{T}\mathcal{M}$.*

Proof. We just need to check that a is a regular value for the function $f(v) = \|v\|^2$. We'll work in coordinates, so without loss of generality $\mathcal{M} = \mathbf{R}^n$

$$D_{(x,v)}f(\alpha, \beta) = \frac{\partial f}{\partial x} \cdot \alpha + \frac{\partial f}{\partial v} \cdot \beta = g(v, \beta)$$

(the first term vanishes because $g(v, v) = a$). Now by non degeneracy of g , we get that $D_{(x,v)}f$ is surjective. \square

Also recall that in a Riemannian manifold, a connexion ∇ is compatible with the metric g if for every pair of vector fields X, Y along the differentiable curve α we have

$$\frac{d}{dt}g(X, Y) = g((\nabla_{\dot{\alpha}}X, Y)) + g(X, (\nabla_{\dot{\alpha}}Y)) \quad (2.1)$$

We have used that in the proof of Proposition 1.2.

References

- [1] *Riemannian Geometry*, Wilhelm Klingenberg, De Gruyter Studies in Mathematics
- [2] *Geodesic flows*, Gabriel Paternain, Birkhäuser Boston;
- [3] *Riemannian Manifolds With Integrable Geodesic Flows*, Andrew Miller
- [4] *Differential Geometry of Curves and Surfaces*, Manfredo Do Carmo.