POISSON GEOMETRY

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1. INTRODUCTION

This document is intended as a brief introduction to the study of Poisson manifolds, with a focus on their local structure. Following the first half of Weinstein’s paper [3], we discuss the basic properties of Poisson manifolds, state the splitting theorem, and introduce the topic of linearization. Some of our exposition is also based on notes by Fernandes and Mărcuț [1], which provide a more detailed look at this material.

2. BASIC DEFINITIONS AND PROPERTIES

Definition. A Poisson structure on a manifold $M$ is a Lie bracket $\{,\}$ on $C^\infty(M)$ that satisfies the Leibniz property

$$\{fg, h\} = f\{g, h\} + \{f, h\}g$$

for all $f, g, h \in C^\infty(M)$. The pair $(M, \{,\})$ is called a Poisson manifold.

It follows from the Leibniz property and antisymmetry that $\{,\}$ is a derivation in each argument. In particular, for every $h \in C^\infty(M)$, the operator

$$X_h := \{\cdot, h\}$$

is a vector field, called the Hamiltonian vector field generated by $h$. (Warning: some authors instead define $X_h = \{h, \cdot\}$.)

Lemma 1. For all $f, g \in C^\infty(M)$,

$$X_{\{f, g\}} = [X_g, X_f]$$

(i.e., the map $C^\infty(M) \to \mathfrak{X}(M), f \mapsto X_f$, is a Lie algebra antihomomorphism).

Proof. For any $h \in C^\infty(M)$,

$$X_{\{f, g\}}(h) = \{h, \{f, g\}\}$$

$$= -\{g, \{h, f\}\} - \{f, \{g, h\}\} \quad \text{(Jacobi identity)}$$

$$= \{\{h, f\}, g\} - \{\{h, g\}, f\} \quad \text{(antisymmetry)}$$

$$= X_g(X_f(h)) - X_f(X_g(h)) = [X_g, X_f](h). \quad \square$$

Given a Poisson structure $\{,\}$ on $M$, there is a corresponding bivector field $\pi \in \mathfrak{X}^2(M) := \Gamma(\Lambda^2 TM)$ such that

$$\pi(df, dg) = \{f, g\}. \quad (1)$$

Viewing $\pi$ as a map $T^*M \times T^*M \to \mathbb{R}$, we obtain (by contraction) a map $\pi^\sharp : T^*M \to TM$. Note that (according to our sign conventions) we have $\pi^\sharp(dh) = -X_h$ for all $h \in C^\infty(M)$. 
Remark. Given an arbitrary bivector field $\pi$, the bracket defined by (1) may not satisfy the Jacobi identity. One can show that $\pi$ defines a Poisson bracket if and only if the Schouten bracket $[\pi, \pi]$ is zero. In this case, we will also refer to $\pi$ as a Poisson structure.

In local coordinates $(x_1, \ldots, x_n)$ on $M$, we can write

\[ \{f, g\} = \sum_{i,j=1}^{n} \left\{ x_i, x_j \right\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \]

and

\[ \pi = \sum_{i<j} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \text{where} \quad \pi_{ij}(x) = \left\{ x_i, x_j \right\}. \]

Thus the Poisson structure is completely determined by the components $\pi_{ij}(x) = \left\{ x_i, x_j \right\}$.

**Example 1** (Classical bracket). Let $M = \mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. Then

\[ \{f, g\} := \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right) \] (2)

is a Poisson structure, corresponding to the bivector field $\pi := \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$.

**Example 2** (Symplectic manifolds). On any symplectic manifold $(M, \omega)$ there is a Poisson bracket such that the corresponding $X_f$ are the usual symplectic Hamiltonian vector fields (i.e., they satisfy $df = \iota_{X_f} \omega$). In local symplectic coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, this bracket is just given by (2).

### 3. SYMPLECTIC FOLIATION

**Definition.** The rank of a Poisson structure $\pi$ at $x \in M$ is the rank of the (linear) map $\pi^x : T^*_x M \to T_x M$. (In local coordinates, this is given by the rank of the matrix $(\pi_{ij}(x))$.)

**Definition.** A Poisson structure $\pi$ on $M$ is called nondegenerate (or symplectic) if its rank is equal to $\dim M$ everywhere.

**Remark.** Example 2 gives a bijective correspondence between symplectic forms $\omega$ and non-degenerate Poisson structures $\pi$, justifying the above nomenclature.

**Theorem 1.** If a Poisson structure $\pi$ has constant rank on $M$, then $\text{Image } \pi^x$ is an integrable distribution which gives rise to a foliation of $M$ into symplectic leaves.

**Outline.** Since $\pi$ has constant rank, $\text{Image } \pi^x$ is a subbundle of $TM$, i.e., a distribution. By definition of $\pi^x$, this distribution is spanned by Hamiltonian vector fields. Since the bracket of Hamiltonian vector fields is again Hamiltonian (by Lemma 1), we see that $\text{Image } \pi^x$ is involutive, and therefore integrable (by the Frobenius theorem).

Showing that the leaves of the corresponding foliation are symplectic requires the technical notion of an induced Poisson structure; we will omit the details here.  

**Remark.** For a general Poisson structure, one obtains a singular foliation of $M$ into symplectic leaves (i.e., the leaves may have different dimensions).

**Example 3.** Consider $\mathbb{R}^{2n+s}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n, c_1, \ldots, c_s)$, and with the Poisson bracket bracket given by (2). (Thus, functions of $(c_1, \ldots, c_s)$ have zero bracket with all of $C^\infty(M)$.) In this case, the symplectic leaves are the submanifolds on which all of the $c_i$ are constant, i.e., they are copies of $\mathbb{R}^{2n}$. Clearly $\mathbb{R}^{2n+s}$ is a union of such submanifolds.
4. Splitting

The product of Poisson manifolds can be given a Poisson structure. Suppose $M_1$ and $M_2$ have Poisson structures specified in local coordinates by relations $\{x_i, x_j\} = \pi_{ij}^1(x)$ and $\{y_i, y_j\} = \pi_{ij}^2(y)$ respectively. If we additionally specify that $\{x_i, y_j\} = 0$, then these relations together define a Poisson structure on $M_1 \times M_2$.

**Theorem 2** (Splitting). Let $M$ be a Poisson manifold, and let $x \in M$. Then there exists a neighbourhood $U$ of $x$ and an isomorphism $\phi = \phi_S \times \phi_P : U \to S \times P$ where

- $S$ is symplectic;
- $P$ is Poisson, with rank 0 at $\phi_p(x)$.

Furthermore, the factors $S$ and $P$ are unique up to local isomorphism.

**Remark.** One possible representative for the factor $S$ is the symplectic leaf through $x$.

This theorem reduces the local study of Poisson manifolds to the case where the rank at a point is equal to 0. This case will be studied in further detail in Section 6.

5. Linear Poisson structures

Fix a finite-dimensional Lie algebra $\mathfrak{g}$. For any $f \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$, the differential $df|_\mu$ is an element of $\mathfrak{g}^{**} \cong \mathfrak{g}$. Making this identification, we can define a Poisson structure on $\mathfrak{g}^*$, called the **Lie-Poisson structure**, by

$$\{f, g\}(\mu) = \langle \mu, [df|_\mu, dg|_\mu] \rangle, \quad f, g \in C^\infty(M)$$

(where $\langle \cdot, \cdot \rangle$ is the pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$).

We will describe the Lie-Poisson structure in coordinates. Let $x_1, \ldots, x_r$ be a basis for $\mathfrak{g}$, with corresponding structure constants $c_{ij}^k$ (so that $[x_i, x_j] = \sum_k c_{ij}^k x_k$). Abusing notation, we can also view the $x_i$ as coordinates on $\mathfrak{g}^*$ (via the identification of $\mathfrak{g}^{**}$ with $\mathfrak{g}$). Then the components of the Lie-Poisson structure are just the linear functions

$$\{x_i, x_j\} = \sum_k c_{ij}^k x_k.$$

The following fact is worth mentioning, although we will not use it later.

**Fact.** The symplectic leaves in $\mathfrak{g}^*$ are the coadjoint orbits.

6. Linear approximation

Let $M$ be a manifold with Poisson structure $\pi$. Suppose that $\pi$ has rank 0 at a point $p \in M$. We will show that we can provide $T_p M$ with a Lie-Poisson structure.

First note that

$$T^*_p M \cong \mathfrak{m}^2_p / \mathfrak{m}_p,$$

where $\mathfrak{m}_p \subseteq C^\infty(M)$ is the ideal of functions vanishing at $p$.

**Claim.** $\mathfrak{m}^2_p$ is a Lie ideal of $\mathfrak{m}_p$.

**Proof.** Let $f, g, h \in \mathfrak{m}_p$, and write

$$\{f, gh\} = \{f, g\} h + g \{f, h\}.$$

The brackets $\{f, g\}$ and $\{f, h\}$ belong to $\mathfrak{m}_p$ (i.e., vanish at $p$) since the rank of the Poisson structure at $p$ is 0. Hence $\{f, gh\} \in \mathfrak{m}^2_p$. \(\square\)
Therefore $\mathfrak{g}_p := T^*_p M$ is a Lie algebra, which means that $\mathfrak{g}_p^* = T_p M$ can be given the Lie-Poisson structure. We call this the **linear approximation** to the Poisson structure at $p$.

The linear approximation has a simple description in coordinates. Suppose $x_1, \ldots, x_r$ are coordinates on $M$ which vanish at $p$. Since $\pi_{ij} = \{x_i, x_j\}$ vanishes at 0 for all $i,j$, we can write the Taylor expansion

$$
\pi_{ij}(x) = \sum_k c^k_{ij} x_k + O(x^2), \quad \text{where } c^k_{ij} := \frac{\partial \pi_{ij}}{\partial x_k}(0).
$$

Then the $c^k_{ij}$ are the structure constants of $\mathfrak{g}_p$ (as the notation suggests), and the components of the linear approximation Poisson structure are just given by

$$
\sum_k c^k_{ij} x_k
$$

(i.e., by removing the higher order terms).

**7. Linearization**

The notion of linear approximation leads to the question of linearization: When is the linear approximation at a point isomorphic to the original Poisson structure? The following example shows that this is not always the case.

**Example 4.** One can define a nontrivial Poisson structure on $\mathbb{R}^3$ by

$$
\{x_1, x_2\} = |x|^2 x_3
$$

$$
\{x_2, x_3\} = |x|^2 x_1
$$

$$
\{x_3, x_1\} = |x|^2 x_2,
$$

but the linear approximation to this structure at the origin is trivial (i.e., the zero bracket).

**Definition.** A Lie algebra $\mathfrak{g}$ is said to be formally/analytically/C\(^\infty\) nondegenerate if any Poisson manifold whose linear approximation at a point $p$ is isomorphic to $\mathfrak{g}^*$ is itself isomorphic to $\mathfrak{g}^*$ at $p$, via a formal/analytic/C\(^\infty\) local isomorphism.

The classification of nondegenerate Lie algebras is a difficult problem, but certain results are known. For instance, Weinstein proves (in Theorem 6.1 of [3]) that any semisimple Lie algebra is formally nondegenerate. A more recent survey of the linearization problem can be found in [2].

**References**

1. Rui Loja Fernandes and Ioan Marcut, *Lectures on Poisson geometry*.