Irrational and Algebraic Numbers, IVT, Upper and Lower Bounds

Original Notes adopted from October 30, 2001 (Week 8)

Is $\sqrt[3]{4}$ irrational?
$\sqrt[3]{4} = m/n \Rightarrow 4 = m^3/n^3 \Rightarrow 4n^3 = m^3$

If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}, n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$, then

$$m^3 = p_1^{3\alpha_1} p_2^{3\alpha_2} \cdots p_s^{3\alpha_s}, \quad n^3 = q_1^{3\beta_1} q_2^{3\beta_2} \cdots q_t^{3\beta_t}, \quad 2^2 q_1^{3\beta_1} q_2^{3\beta_2} \cdots q_t^{3\beta_t} = p_1^{3\alpha_1} \cdots p_s^{3\alpha_s}.$$

2^2 needs to occur on the right side.

On the right side, the power of 2 occurring is a multiple of 3. On the left side, it is not a multiple of 3 (the power of 2 is $\equiv 2 \pmod 3$). But this is a contradiction, since prime factorizations are unique.

Theorem. For k a natural number & L a natural number, $\sqrt[k]{L}$ is rational only if $\sqrt[k]{L}$ is an integer.

Proof: Suppose $\sqrt[k]{L} = m/n$. Then $L = m^k/n^k$, so $Ln^k = m^k$.

If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ and $n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$, then

$$m^k = p_1^{k\alpha_1} p_2^{k\alpha_2} \cdots p_s^{k\alpha_s}, \quad n^k = q_1^{k\beta_1} q_2^{k\beta_2} \cdots q_t^{k\beta_t},$$

so

$$Lq_1^{k\beta_1} \cdots q_t^{k\beta t} = p_1^{k\alpha_1} \cdots p_s^{k\alpha_s}.$$

Write L as a product of primes: $L = r_1^{\phi_1} r_2^{\phi_2} \cdots r_u^{\phi_u}$,then

$$(r_1^{\phi_1} r_2^{\phi_2} \cdots r_u^{\phi_u}) q_1^{k\beta_1} \cdots .. q_t^{k\beta_t} = p_1^{k\alpha_1} \cdots p_s^{k\alpha_s}$$

Fix any i between 1 & u. Then r_i must be equal to one of the ps, say $r_i = p_j$. On the right side, it occurs to a power that is a multiple of k. On the left side, it occurs to the power ϕ_i + multiple of k (just ϕ_i if r_i is not among the qs).
These powers must be equal (by unique factorization), so ϕ_i + (multiple of k) is a multiple of k, and so ϕ_i is a multiple of k. Let $\phi_i = k\delta_i$.
Then

$$L = r_1^{\phi_1} r_2^{\phi_2} \cdots r_u^{\phi_u} = r_1^{k\delta_1} r_2^{k\delta_2} \cdots r_u^{k\delta_u} = (r_1^{\delta_1} r_2^{\delta_2} \cdots r_u^{\delta_u})^k.$$

Thus $\sqrt[k]{L} = r_1^{\delta_1} r_2^{\delta_2} \cdots r_u^{\delta_n}$ is an integer.

Eg. $\sqrt[17]{1/2} = 1/\sqrt[17]{2}$
If $\sqrt[17]{1/2} = m/n$, then $\sqrt[17]{2} = n/m$. Contradiction.

Eg. $\sqrt[3]{2/7}$
Suppose $\sqrt[3]{2/7} = m/n$, then $2/7 = m^3/n^3, 2n^3 = 7m^3$
2 occurs to a power a multiple of 3. On the left, to 1 plus a multiple of 3. Contradicts uniqueness of prime factorization.

Definition. A real number is *algebraic* if there exists a (non-zero) polynomial with integer coefficients that has it as a root.

Eg. $\sqrt{2}$ is algebraic: it's a root of $x^2 - 2 = 0$.
7/12 is algebraic: it's a root of $12x - 7 = 0$.
In fact, any rational $m/n, n \neq 0$ is a root of $nx - m = 0$ and therefore is algebraic.

Definition. A real number is *transcendental* if it isn't algebraic, i.e. it is not the root of any polynomial with integer coeffients (not all of which are 0).

e.g. e, π are transcendental.

Recall **Intermediate Value Theorem:** If f is a continuous function such that $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Eg. $f(x) = x^2 - 2$
$f(1) < 0$ and $f(3) > 0 \therefore$ there exists $c \in (1, 3)$ such that $f(c) = 0$.
$c = \sqrt{2}$ is irrational.
Thus any proof of Intermediate Value Theorem requires some property of the set of real numbers that doesn't hold for set of rational numbers.
$\mathbb{Q}$ = set of rational numbers
$\mathbb{R}$ = set of real numbers.
The basic property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$: Every subset of R other than $\emptyset$ with an upper bound has a least upper bound.

Assume we have $\mathbb{R}$ (We'll develop it formally later).

Definition. For $S \subset \mathbb{R}, c$ is an upper bound of S if $x \leqslant c$ for all $x \in S$.

Examples: 1) $S = \{x : x < 10\}$; upperbounds include 17, 25, 10 (which is the least upper bound)
2) $S = \{x : x^2 < 7\}$; upperbounds include 25, 14, $\sqrt{7}$
3) $S = [3, 7] \cup [-4, 6] \cup [12, 99] \cup \{107\}$; upperbounds: 254, $100\sqrt{2}$, 107 (lub)
4) $S = \{x : x^2 < 2\}$; upperbounds: 3,2, $\sqrt{2}$ (lub).
5) $S = \{x \in \mathbb{Q} : x^2 < 2\}$, $\sqrt{2}$ (lub).

If the entire number system were $\mathbb{Q}$, then $\{x : x \in \mathbb{Q}, x^2 < 2\}$ wouldn't have a least upper bound (although it has many upper bounds).

Definition. : A *least upper bound* for a set of S is a number c such that
1) c is an upper bound for S (i.e., $x \leqslant c$ for all $x \in S$)
2) if d is an upper bound of S, then $d \geqslant c$.

Eg. $S = \mathbb{N} = \{1, 2, 3, \cdots\}$. $\mathbb{N}$ has no upper bound.

Completeness Property of R Every subset of $\mathbb{R}$ (other than $\emptyset$) that has an upper bound has a least upper bound.

Theorem. (Intermediate Value Theorem) If f is continuous on [a,b], $f(a) < 0$, and $f(b) > 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof: Let $S = \{x \in [a, b] : f(t) < 0 \text{ for all } t \in [a, x]\}$. Then S has b as an upper bound.
S is not empty since $a \in S$.
Completeness implies that S has a least upper bound, say c.
Claim: $f(c) = 0$.
Suppose $f(c) < 0$.
Since f is continuous, there is some interval $(c - \delta, c + \delta)$ such that $f(x) < 0$ when $x \in (c - \delta, c + \delta)$.
Then $c + \delta/2 \in S$, because $f(t) < 0$ for all $t < c + \delta/2$.
But $c + \delta/2 > c$, contradicting c being an upper bound of S.
Therefore we can't have $f(c) < 0$.

Now suppose $f(c) > 0$, then $f(x) > 0\ \forall x \in (c - \epsilon, c + \epsilon)$ for some $\epsilon > 0$, and so $c - \epsilon/2$ is an upper bound for S. If $x \in S$ and $x > c - \epsilon/2$, then $f(c - \epsilon/2) < 0$.
So $c - \epsilon/2$ is an upperbound for S.
But $c - \epsilon/2 < c$, contradicting c being the least upper bound of S.
Thus we can't have $f(c) > 0$.

Therefore, $f(c) = 0$.