Fundamental Theorem of Algebra

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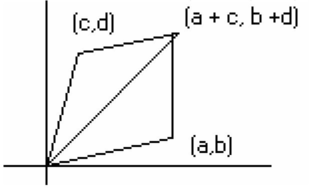
a + bi + c + di = (a + c) + (b + d) i



Triangle Inequality

For z_1 , $z_2 \in$ C,

$|z_1 + z_2| \leq |z_1| + |z_2|$

(Or $| z_1 + z_2 + \dots z_n | \leq | z_1 | + | z_1 | + \dots + |z_n|$

Fundamental Theorem of Algebra: Every polynomial with complex coeffients other than constant polynomials has a complex proof.

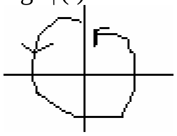
$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_1 z + a_0$, $a_j \in$ C, $n \in$ N. $a_n \neq 0$. n degree of polynomial.

Definition: A closed curve in the plane is a continuous function from $[0, 2\pi]$ into C such its values at 0 and 2 π are the same.

Eg. A Function

$\phi(t) = f(t) + i\, g(t)$, f,g into functions mapping $[0, 2\pi]$ into R & $f(0) = f(2\pi)$, $g(0) = g(2\pi)$.

Eg. $\phi(t) = \cos t + \sin t$, $t \in (0, 2\pi)$



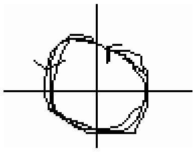
Winding number is 1.

Eg. $\phi(t) = \cos 3t + i \sin 3t$, $t \in [0, 2\pi]$

PICTURE

Winding number is 3

Definition: If ϕ is a closed curve in C that doesn't go through (0,0), its winding number about (0,0) is the total number of times a vector from (0,0) to point on curve winds around (0,0) as t goes from 0 to 2π.



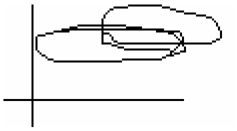
Eg. $\phi(t) = 27 \cos 4t + 27 i \sin 4t$, $t \in [0,2\pi]$

Winding number is 4

Eg. $\phi(t) = \sin 4t + i \cos 4t$.

Winding number is –4.

Winding number is 0.



Idea of Proof of Fundamental Theorem of Algebra.

Suppose p, non-constant polynomial & $p(z) \neq 0 \ \forall z$ (We will get a contradiction)
For each R>0, let ϕ_R be curve: $\phi_R(t) = Rcost + Risint$, $t \in (0, 2\pi)$
Let $4_R(t) = p(\phi_R(t) \Rightarrow 4_R(t)$ closed curve too.
No root for $p \Rightarrow 4_R$ doesn't go through (0,0).
We'll show if (1) R is large enough, winding number of 4_R is the degree of p.
Also if (2)R small enough, winding number of 4_R is 0.
But the winding number of 4_R is a continuous function of R, & assumes integer values. $\therefore$ Its constant, contradiction.
To simplify a little, can assume $a_n= 1$ (divide through).

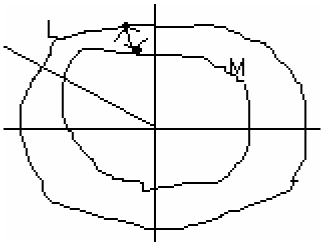
Detailed Proof of Fundamental Theorem of Algebra.

$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} +.... a_1z + a_0$
For each R> 0 , let ϕ_R be curve R, radius.
$\phi_R(t) = Rcost + Rsint$, $t \in (0, 2\pi)$, Winding number =1.
Let $4\phi_R(t) = p(\phi_R(t))$ – Take image
$4_R(t)$ (closed curve too. No root for $p \Rightarrow 4_R$ not through (0,0))
Let $q(z) = z^n$
Let $L_R(z) = q(\phi_R(t))$
$= (Rcost + iRsint)^n$
$= R^n (cosnt + isint)$.

Winding number of $L_R(t)$ is n
Suppose $p(\phi_R(t)) = 0$, for $t =t_0$
$p(\phi_R(t_0))= 0$.
No roots means doesn't to go through origin.

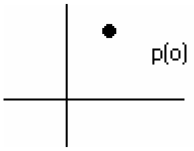
Lemma: If L(t) & M(t) are closed curves not through (0,0) & $|L(t) – M(t)| < |L(t)| \ \forall t$, then L(t) & M(t) have the same winding number.

Arrows: Distance between points.



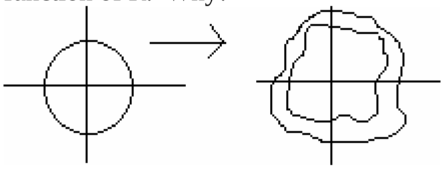
Claim: If R is large enough, then $| L_R(t) - 4_R(t)| < | L_R(t)|$ (have winding number N)
$L_R(t) = (\phi_R(t))^n$
$4_R(t) = p(\phi_R(t))$ $p(z) = z^n + a_{n-1} z^n + ... + a_0$

$| L_R(t) - 4_R(t)| = |-a_{n-1} (\phi_R(t))^{n-1} - a_{n-2} (\phi_R(t))^{n-2} + ...a_{n...} |$
$\leq | a_{n-1} (\phi_R(t)^{n-1} | + |a_{n-2} (\phi_R(t)^{n-2} |+ ....|a_0|$ (Triangle Inequality)
$= | a_{n-1} | |\phi_R(t)^{n-1}| + |a_{n-2} | |(\phi_R(t)^{n-2} |+ ....+ |a_0|$
As long as R > 1,
$=| a_{n-1} | R^{n-1} + |a_{n-2} | R^{n-2} + |a_{n-3} | R^{n-3} + ...+ |a_0| \leq| a_{n-1} | R^{n-1} + |a_{n-2} | R^{n-1} + |a_0 | R^{n-1}$ if R > 1 (bound) $=(|a_{n-1} | + |a_{n-2} | + ... + |a_0 |) R^{n-1}$ --- need to find R bigger than $< R^n$ if $R > |a_{n-1}| + |a_{n-2} | +..... + |a_0 |$ & R > 1.
$= |L_R(t)|$
Thus if R > 1 & $R > |a_{n-1}| + |a_{n-2} | +..... + |a_0 |$, then winding number of $4_R(t)$ is n (by claim and Lemma). $n \in N$, n > 0. Let p (0) = k. $k \neq 0$ (since p has no root).
Since p is continuous, $| p(\phi_R(t)) – p(0)| < |k|$ for R small enough.
$\therefore$ for such R, winding number of $4_R(t)$ is 0.



Can' t wind around origin.

Thus there exists R_1 such that Ψ_R has winding number 0, but the winding number of Ψ_R is a continuous function of R. Why?



But a continuous function on $(0, \infty)$ that has integer values must be a constant. Contradiction since it takes in values 0 and n.

$\phi_R(t) = R\cos t + iR\sin t$

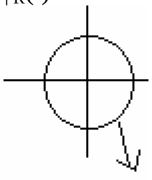
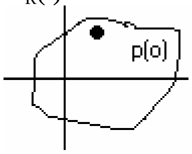


Image of that as polynomial.

$\Psi_R(t)$



If start. R big enough $\Rightarrow$ n

if star t R small $\Rightarrow$ 0

$\therefore$ Can' t be both n& 0.

Recall : If $p(z) = a_n z^n + a_{n-1} z^{n-1} + a_1 z + a_0$, & $r \in C$, $a_j \in C$, do "long division of p(z) by z –r".

Get $p(z) = (z-r)\, q(z)\, q(z) + k$ where q polynomial & $k \in C$.

Eg. $p(z) = 2z^3 + 5z^2 + z - 1$. $r = 3$.

$$
\begin{array}{r|l}
 & \overline{\quad 2z^2 + 11z + 34} \\
z - 3 & 2z^3 + 5z^2 + z - 1 \\
 & \underline{2z^3 - 6z^2} \\
 & \qquad 11z^2 \\
 & \qquad \underline{11z^2 - 33z} \\
 & \qquad\qquad 34z \\
 & \qquad\qquad \underline{34z - 102} \\
 & \qquad\qquad\qquad 101
\end{array}
$$

$2z^3 + 5z^2 + z - 1$

$$= (z-5)\underbrace{(2z^3 + 11z + 34)}_{q(z)} + \underbrace{101}_{k}$$

Factor Theorem: If $p(r) = 0$.
$p(z) = (z-r)q(z)$, some polynomial q.