THE SERIES OF RECIPROCALS OF PRIMES AND POLYNOMIALS WITH PRIME EXPONENTS

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A famous theorem of Weierstrass states that every continuous function on the closed unit interval $[0,1]$ can be uniformly approximated on the interval by polynomials.

A natural question arises as to whether it suffices to use only polynomials with some of the possible exponents. There is a very beautiful theorem that gives a complete answer to this question.

Theorem (The Müntz-Szasz Theorem). *If $\{p_n\}$ is an increasing sequence of positive numbers, then $\sum_{n=1}^{\infty} \frac{1}{p_n}$ diverges if and only if every continuous function can be uniformly approximated on $[0,1]$ by functions in the linear span of the collection $\{x^{p_n}\} \cup \{1\}$.*

A very nice discussion of the proof of this theorem can be found in [4, p. 313].

The purpose of this note is twofold. We first point out that the following is an immediate consequence of the Müntz–Szasz Theorem and Euler's famous theorem that the sum of the reciprocals of the primes diverges: Every continuous function on $[0,1]$ can be uniformly approximated by polynomials with prime exponents.

Our second purpose takes a little more space to accomplish. We offer a description of a proof of Euler's theorem that appears to be easier to check and to remember than other presentations that we have seen (although it is not very different from other proofs).

Theorem (Euler's Theorem). *If p_j denotes the j^{th} prime number, then the series $\displaystyle\sum_{j=1}^{\infty} \frac{1}{p_j}$ diverges.*

Proof. If the series converged, there would exist an M such that

$$\sum_{j=M}^{\infty} \frac{1}{p_j} < \frac{1}{2}.$$

Assume there exists such an M; we will show that this leads to a contradiction.

Note that the series

$$\sum_{k=0}^{\infty} \left(\sum_{j=M}^{\infty} \frac{1}{p_j} \right)^k$$

would then be a convergent geometric series.

For each $j < M$, the series

$$\sum_{k=0}^{\infty} \frac{1}{p_j^k}$$

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is also a convergent geometric series.

Multiplying these series together gives

$$\left(\sum_{k=0}^{\infty}\frac{1}{2^k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{5^k}\right)\cdots\left(\sum_{k=0}^{\infty}\frac{1}{p_{M-1}^k}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^k\right).$$

We will obtain a contradiction by showing that multiplying out the above expression yields a series whose terms contain all the terms of the harmonic series.

To see this we begin as follows. Suppose that the set

$$\{p_{n_1}, p_{n_2}, \ldots, p_{n_s}\}$$

is any collection of prime numbers with $p_{n_i} \geq p_M$ for all i, and suppose that

$$\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$$

are natural numbers. Then

$$\frac{1}{p_{n_1}^{\alpha_1} p_{n_2}^{\alpha_2} \ldots p_{n_s}^{\alpha_s}}$$

is a term that occurs in the expansion of

$$\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_s}.$$

If n is a natural number greater than 1, it can be written in the form

$$n = p_{m_1}^{\beta_1} p_{m_2}^{\beta_2} \cdots p_{m_t}^{\beta_t} p_{n_1}^{\alpha_1} p_{n_2}^{\alpha_2} \cdots p_{n_s}^{\alpha_s},$$

where each $p_{m_i} < p_M$ and each $p_{n_i} \geq p_M$. Then $\frac{1}{n}$ occurs as a term in the product of

$$\frac{1}{p_{m_1}^{\beta_1}}\frac{1}{p_{m_2}^{\beta_2}}\cdots\frac{1}{p_{m_t}^{\beta_t}}\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_s}.$$

Thus the expansion of

$$\left(\sum_{k=0}^{\infty}\frac{1}{2^k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{5^k}\right)\cdots\left(\sum_{k=0}^{\infty}\frac{1}{p_{M-1}^k}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^k\right)$$

contains $\frac{1}{n}$ for every natural number n. This contradicts the divergence of the harmonic series. □

A very-related proof, with the same beginning as ours, was found by Clarkson [2] (see also Apostol [1, page 18]).

Note that the above proof of Euler's Theorem does not require knowledge of the uniqueness of factorization into primes; it merely requires the trivial fact that every integer greater than 1 can be written as a product of primes.

We conclude by pointing out the following. Recall that the integers p and $p+2$ are called "twin primes" if both p and $p+2$ are prime numbers. The twin prime problem is the famous question of whether there are an infinite number of twin primes. Although this problem is unsolved, in 1919 Viggo Brun proved the remarkable theorem that the sum of the reciprocals of the twin primes converges. A beautiful exposition of Brun's proof of this theorem can be found in [3, Ch. 15]

References

[1] Tom M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York-Heidelberg, 1976.
[2] James A. Clarkson, *On the series of prime reciprocals*, Proc. Amer. Math. Soc. **17** (1966) 541.
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[4] Walter Rudin, *Real and complex analysis*, Third Edition, McGraw Hill, New York, 1986.

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