ON THE LOCAL CONSTANCY OF CHARACTERS

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ABSTRACT. The character of an irreducible admissible representation of a p-adic reductive group is known to be a constant function in some neighborhood of any regular semisimple element γ in the group. Under certain mild restrictions on γ, we give an explicit description of a neighborhood of γ on which the character is constant.

INTRODUCTION

Let k be a p-adic field of characteristic zero, and let $\mathbf{G}$ be a connected reductive algebraic group defined over k. We denote by G the group of k-rational points of $\mathbf{G}$, and by $\mathfrak{g}$ the Lie algebra of G. Let π be an irreducible admissible representation of G, and let Θ_π be the (distribution) character of π. In [7] Harish-Chandra showed that Θ_π can be represented by a function (also denoted by Θ_π) which is locally integrable on G and locally constant on the set G^{reg} of regular semisimple elements in G. Thus for any $\gamma \in G^{reg}$ there exists *some* neighborhood of γ on which the character is constant. In [8, Theorem 2, p. 483], R. Howe gave an elementary proof of Harish-Chandra's result for general linear groups. In this paper we give a precise version of local constancy (near compact regular semisimple tame elements) for all reductive groups. The outline of the approach given here follows the elementary argument of Howe.

Let $\mathfrak{g}_{x,r}$ (resp. $G_{x,|r|}$) be the Moy-Prasad lattices [10] in $\mathfrak{g}$ (resp. open compact subgroups of G), normalized as in [9, §1.2]. Let G_{cpt} denote the set of compact elements in G. For a maximal k-torus T, let T_r denote its filtration subgroups (section 0.3). Let $\rho(\pi)$ denote the depth of π [10, §5].

Fix a regular semisimple element γ and let $\mathbf{T} := C_{\mathbf{G}}(\gamma)^\circ$ be the connected component of its centralizer; $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$. We assume that it splits over some tamely ramified finite Galois extension E of k. Let T denote the group of k-rational points of $\mathbf{T}$. When $\gamma \in T \cap G_{cpt}$ we attach to it the nonnegative rational number $s(\gamma)$. Using the filtration subgroups T_r and the parameter $s(\gamma)$, we characterize a neighborhood of γ on which the character Θ_π is constant. Whether or not this neighborhood of constancy is maximal is not addressed here.

The main result of this paper is the following (Theorem 20).

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Theorem. *Let $r = \max\{s(\gamma), \rho(\pi)\} + s(\gamma)$. The character Θ_π is constant on the set ${}^G(\gamma T_{r+})$.*

We now give a brief sketch of the proof. Let K be any open compact subgroup of G. Decompose Θ_π into a countable sum of 'partial trace' operators Θ_d, according to the irreducible representations d of K (see section 3). For $G = GL_n$, Howe proved [8, p. 499] the following key fact. If X is a compact subset of G^{reg}, then Θ_d vanishes on X for all d not in a certain finite set F (which depends only on X). It follows (see proof of Theorem 20), that $\Theta_\pi(f) = \int_X (\sum_{d\in F} \Theta_d)(x) f(x) dx$ for all $f \in C_c^\infty(X)$. Hence Θ_π is represented on X by the locally constant function $\sum_{d\in F} \Theta_d$.

The main part of this paper is concerned with formulating an analogue of the above key fact for reductive groups (see Corollary 18).

The rational number $s(\gamma)$, defined in section 1, is used (Corollary 18) to make a precise choice of a set X and a subgroup K. Corollary 18 characterizes a finite set F of representations, such that for all d not in F, Θ_d vanishes on X (see Remark 14 for the significance of this fact). Thus the representations $d \in F$ are those which play a role in understanding the character Θ_π near γ. The proof of this corollary relies on a special case (Corollary 13), in which we only consider 1-dimensional d. Such representations have an explicit description in terms of cosets in the lie algebra $\mathfrak{g}$. In section 2 we develop the technical tools, using Moy-Prasad lattices, to handle these cosets. Once we have a characterization of the set F, we can make precise statements about the neighbourhood of constancy of the character near γ (Theorem 20).

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Notation and Conventions

Let k be a p-adic field (a finite extension of some $\mathbb{Q}_p$) with residue field $\mathbb{F}_{p^n}$. Let ν be a valuation on k normalized such that $\nu(k^\times) = \mathbb{Z}$.

For any algebraic extension field E of k, ν extends uniquely to a valuation (also denoted ν) of E.

We denote the ring of integers in E by R_E (write R for R_k), and the prime ideal in R_E by $\wp_E$ (write $\wp$ for $\wp_k$).

Let $\mathbf{G}$ be a connected reductive group defined over k, and $\mathbf{G}(E)$ the group of E-rational points of $\mathbf{G}$. We denote by G the group of k-rational points of $\mathbf{G}$. Denote the Lie algebras of $\mathbf{G}$ and $\mathbf{G}(E)$ by $\boldsymbol{\mathfrak{g}}$ and $\boldsymbol{\mathfrak{g}}(E)$, respectively. Write $\mathfrak{g}$ for the Lie algebra of k-rational points of $\boldsymbol{\mathfrak{g}}$.

Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}$. There are different notions of nilpotency, but since we assume that $\mathrm{char}(k) = 0$, these notions all coincide.

Let Ad (resp. ad) denote the adjoint representation of $\mathbf{G}$ (resp. $\boldsymbol{\mathfrak{g}}$) on its Lie algebra $\boldsymbol{\mathfrak{g}}$. For elements $g \in G$ and $X \in \mathfrak{g}$ (resp. $x \in G$) we will sometimes write ${}^{g}X$ (resp. ${}^{g}x$) instead of $\mathrm{Ad}(g)X$ (resp. gxg^{-1}). For a subset S of $\mathfrak{g}$ (resp. G) let ${}^{G}S$ denote the set $\{{}^{g}s \mid g \in G \text{ and } s \in S\}$.

Let n denote the (absolute) rank of $\mathbf{G}$. We say that an element $g \in G$ is *regular semisimple* if the coefficient of t^n in $\det(t-1+\mathrm{Ad}(g))$ is nonzero. We denote the set of regular semisimple elements in G by G^{reg}. Similarly we say that an element $X \in \mathfrak{g}$ is *regular semisimple* if the coefficient of t^n in $\det(t-\mathrm{ad}(X))$ is nonzero. We denote the set of regular semisimple elements in $\mathfrak{g}$ by $\mathfrak{g}^{reg}$. Let G_{cpt} denotes the set of compact elements in G. For a subset S of G we will sometimes write S_{cpt} for $S \cap G_{cpt}$.

For a subset S of $\mathfrak{g}$ (resp. G) let $[S]$ denote the characteristic function of S on $\mathfrak{g}$ (resp. G).

For any compact group K, let $K^{\wedge}$ denote the set of equivalence classes of irreducible, continuous representations of K.

Let π be an irreducible admissible representation of G. We denote by Θ_π the character of π thought of as a locally constant function on the set G^{reg}. Let $\rho(\pi)$ denote the depth of π [10, §5].

0. Preliminaries

0.1. **Apartments and buildings.** For a finite extension E of k, let $\mathcal{B}(\mathbf{G}, E)$ denote the extended Bruhat-Tits building of $\mathbf{G}$ over E; write $\mathcal{B}(G)$ for $\mathcal{B}(\mathbf{G}, k)$. It is known (e.g. [13]) that if E is a tamely ramified finite Galois extension of k then $\mathcal{B}(\mathbf{G}, k)$ can be embedded into $\mathcal{B}(\mathbf{G}, E)$ and its image is equal to the set of Galois fixed points in $\mathcal{B}(\mathbf{G}, E)$. If $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$ that splits over E, let $\mathcal{A}(\mathbf{T}, E)$ be the corresponding apartment over E. Let $\mathbf{X}^*(\mathbf{T}, E)$ (resp. $\mathbf{X}_*(\mathbf{T}, E)$) denote the group of E-rational characters (resp. cocharacters) of $\mathbf{T}$.

It is known in the tame case [1, §1.9] that there is a Galois equivariant embedding of $\mathcal{B}(\mathbf{T}, E)$ into $\mathcal{B}(\mathbf{G}, E)$, which in turn induces an embedding of $\mathcal{B}(\mathbf{T}, k)$ into $\mathcal{B}(\mathbf{G}, k)$. Such embeddings are only unique modulo translations by elements of $\mathbf{X}_*(\mathbf{T}, k) \otimes \mathbb{R}$, however their images are all the same and are equal to the set $\mathcal{A}(\mathbf{T}, E) \cap \mathcal{B}(\mathbf{G}, k)$. From now on we fix a T-equivariant embedding $i : \mathcal{B}(\mathbf{T}, k) \longrightarrow \mathcal{B}(\mathbf{G}, k)$, and use it to regard $\mathcal{B}(\mathbf{T}, k)$ as a subset of $\mathcal{B}(\mathbf{G}, k)$; write x for $i(x)$.

Notation. We write $\mathcal{A}(\mathbf{T}, k)$ for $\mathcal{A}(\mathbf{T}, E) \cap \mathcal{B}(\mathbf{G}, k)$. This is well defined independent of the choice of E [15]. Moreover, $\mathcal{A}(\mathbf{T}, k)$ is the set of Galois fixed points in $\mathcal{A}(\mathbf{T}, E)$.

We remark that the image of $\mathcal{B}(\mathbf{T}, E)$ in $\mathcal{B}(\mathbf{G}, E)$ is the apartment $\mathcal{A}(\mathbf{T}, E)$, while the image of $\mathcal{B}(\mathbf{T}, k)$ in $\mathcal{B}(\mathbf{G}, k)$ is the set $\mathcal{A}(\mathbf{T}, k)$.

0.2. **Moy-Prasad filtrations.** Regarding $\mathbf{G}$ as a group over E, Moy and Prasad (see [10] and [11]) define lattices in $\mathfrak{g}(E)$ and subgroups of $\mathbf{G}(E)$.

We can and will normalize (with respect to the normalized valuation ν) the indexing $(x, r) \in \mathcal{B}(\mathbf{G}, E) \times \mathbb{R}$ of these lattices and subgroups as in [9, §1.2]. We will denote the (normalized) lattices by $\mathfrak{g}(E)_{x,r}$, and the (normalized) subgroups by $\mathbf{G}(E)_{x,|r|}$.

If ϖ_E is a uniformizing element of E, and $e = e(E/k)$ is the ramification index of E over k, then these normalized lattices (resp. subgroups) satisfy ϖ_E $\mathfrak{g}(E)_{x,r} = \mathfrak{g}(E)_{x,r+\frac{1}{e}}$. Write $\mathfrak{g}_{x,r}$ (resp. $G_{x,|r|}$) for $\mathfrak{g}(k)_{x,r}$ (resp. $\mathbf{G}(k)_{x,|r|}$).

The above normalization was chosen to have the following property [1, 1.4.1]: when E is a tamely ramified Galois extension of k and $x \in \mathcal{B}(\mathbf{G}, k) \subset \mathcal{B}(\mathbf{G}, E)$, we have

$$(1) \qquad \mathfrak{g}_{x,r} = \mathfrak{g}(E)_{x,r} \cap \mathfrak{g}, \quad \text{and (for } r > 0) \;\; G_{x,r} = \mathbf{G}(E)_{x,r} \cap G.$$

We will also use the following notation. Let $r \in \mathbf{R}$ and $x \in \mathcal{B}(G)$.

- $\mathfrak{g}_{x,r+} = \cup_{s>r} \mathfrak{g}_{x,s}$ and $G_{x,|r|+} = \cup_{s>|r|} G_{x,s}$.
- $G_r = \cup_{x \in \mathcal{B}(G)} G_{x,r}$ and $G_{r+} = \cup_{s>r} G_s$ for $r \geq 0$.

The lattices $\mathfrak{g}_{x,r+}$ (resp. groups $G_{x,|r|+}$) have analogous properties to those of $\mathfrak{g}_{x,r}$ (resp. $G_{x,|r|}$). The set G_0 is the set of compact elements G_{cpt}. We remark that $G_{cpt} \subset \mathbf{G}(E)_{cpt} \cap G$, and in general they need not be equal [3, §2.2.3].

Lemma 1. *Let γ be a compact regular semisimple element, and consider the maximal k-torus $\mathbf{T} := C_{\mathbf{G}}(\gamma)^\circ$. Suppose that $\mathbf{T}$ splits over a tamely ramified finite Galois extension E of k. Then γ fixes $\mathcal{B}(\mathbf{T}, k)$ pointwise.*

Proof. Recall that γ acts on $\mathcal{A}(\mathbf{T}, E)$ by translations [14, §1]. Since γ belongs to a compact subgroup, it has a fixed point $x \in \mathcal{B}(\mathbf{G}, E)$.

If γ acts on $\mathcal{A}(\mathbf{T}, E)$ by a nontrivial translation, then for any $y \in \mathcal{A}(\mathbf{T}, E)$ there is an $n \in \mathbb{N}$ such that $d(x, y) \neq d(x, \gamma^n.y)$. This contradicts the fact that the action preserves distances. So γ must act trivially on $\mathcal{A}(\mathbf{T}, E)$. In particular, γ fixes $\mathcal{A}(\mathbf{T}, k)$, and hence $\mathcal{B}(\mathbf{T}, k)$, pointwise. $\square$

0.3. **Root decomposition.** Let $\mathbf{T}$ be a maximal k-torus in $\mathbf{G}$ that splits over a tamely ramified finite Galois extension E of k. Let $\Phi(\mathbf{T}, E)$ denote the set of roots of $\mathbf{G}$ with respect to E and $\mathbf{T}$, and let $\Psi(\mathbf{T}, E)$ denote the corresponding set of affine roots of $\mathbf{G}$ with respect to E, $\mathbf{T}$ and ν. When $\mathbf{T}$ is k-split, we also write $\Phi(\mathbf{T})$ for $\Phi(\mathbf{T}, k)$ (resp. $\Psi(\mathbf{T})$ for $\Psi(\mathbf{T}, k)$). If $\psi \in \Psi(\mathbf{T}, E)$, let $\dot{\psi} \in \Phi(\mathbf{T}, E)$ be the gradient of ψ, and let $\mathfrak{g}(E)_{\dot{\psi}} \subset \mathfrak{g}(E)$ be the root space corresponding to $\dot{\psi}$. We denote the root lattice in $\mathfrak{g}(E)_{\dot{\psi}}$ corresponding to ψ by $\mathfrak{g}(E)_\psi$ [10, 3.2].

For $x \in \mathcal{A}(\mathbf{T}, E)$ and $r \in \mathbb{R}$, let $\mathfrak{t}(E)_r := \mathfrak{t}(E) \cap \mathfrak{g}(E)_{x,r}$ and $\mathfrak{t}(E)_{r+} := \mathfrak{t}(E) \cap \mathfrak{g}(E)_{x,r+}$. Note that $\mathfrak{t}(E)_r$ and $\mathfrak{t}(E)_{r+}$ are defined independent of the choice of $x \in \mathcal{A}(\mathbf{T}, E)$. Similarly one defines the subgroups $\mathbf{T}(E)_r$ and $\mathbf{T}(E)_{r+}$ for $r \geq 0$; they have analogous properties. Note that using our conventions we will sometimes denote $\mathbf{T}(E)_0$ by $\mathbf{T}(E)_{cpt}$.

An alternative description is [9, §2.1]: for $r \in \mathbb{R}$,

$$\mathfrak{t}(E)_r = \{\Gamma \in \mathfrak{t}(E) |\ \nu(d\chi(\Gamma)) \geq r \ \text{ for all } \ \chi \in \mathbf{X}^*(\mathbf{T}, E)\}$$

and for $r > 0$,

$$\mathbf{T}(E)_r = \{t \in \mathbf{T}(E) |\ \nu(\chi(t) - 1) \geq r \ \text{ for all } \ \chi \in \mathbf{X}^*(\mathbf{T}, E)\}.$$

Since $\mathbf{G}$ splits over E, we have

$$\begin{aligned} \mathfrak{g}(E)_{x,r} &= \mathfrak{t}(E)_r \oplus \sum_{\psi\in\Psi(\mathbf{T},E),\psi(x)\geq r} \mathfrak{g}(E)_\psi \ , \\ \mathfrak{g}(E)_{x,r+} &= \mathfrak{t}(E)_{r+} \oplus \sum_{\psi\in\Psi(\mathbf{T},E),\psi(x)> r} \mathfrak{g}(E)_\psi \ . \end{aligned}$$

Let $\mathfrak{t} := \mathrm{Lie}(T)$, and define $\mathfrak{t}^\perp := (\mathrm{Ad}(\gamma)-1)\mathfrak{g}$. We have the following decomposition [7, §18]

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp \ . \tag{2}$$

We write $X = Y + Z$ with respect to this decomposition; when convenient, we also write $X_\mathfrak{t}$ for Y.

Fix $x \in \mathcal{B}(\mathbf{T},k) \subset \mathcal{B}(\mathbf{G},k)$ and $r \in \mathbb{R}$. Write $\mathfrak{t}_r$ for $\mathfrak{t}\cap\mathfrak{g}_{x,r}$ (resp. $\mathfrak{t}_{r+}$ for $\mathfrak{t}\cap\mathfrak{g}_{x,r+}$); as mentioned earlier, these definitions are independent of x. Define $\mathfrak{t}^\perp_{x,r} := \mathfrak{t}^\perp \cap \mathfrak{g}_{x,r}$ (resp. $\mathfrak{t}^\perp_{x,r+} := \mathfrak{t}^\perp \cap \mathfrak{g}_{x,r+}$). We have [1, 1.9.3],

$$\begin{aligned} \mathfrak{g}_{x,r} &= \mathfrak{t}_r \oplus \mathfrak{t}^\perp_{x,r} \ , \\ \mathfrak{g}_{x,r+} &= \mathfrak{t}_{r+} \oplus \mathfrak{t}^\perp_{x,r+} \ . \end{aligned} \tag{3}$$

0.4. **Hypotheses.** (HB) There is a nondegenerate G-invariant symmetric bilinear form B on $\mathfrak{g}$ such that we can identify $\mathfrak{g}^*_{x,r}$ with $\mathfrak{g}_{x,r}$ via the map $\Omega : \mathfrak{g} \to \mathfrak{g}^*$ defined by $\Omega(X)(Y) = B(X,Y)$.

Groups satisfying the above hypothesis are discussed in [4, §4].

Fix $r \in \mathbb{R}_{>0}$ and $x \in \mathcal{B}(\mathbf{G},k)$. For any $r \leq t \leq 2r$ the group $(G_{x,r}/G_{x,t})$ is abelian. By hypothesis (HB), there exists a ($G_{x,0}$-equivariant) isomorphism (see [1, §1.7] or [12, p.16])

$$(G_{x,r}/G_{x,t})^\wedge \ \cong \ \mathfrak{g}_{x,(-t)+}/\mathfrak{g}_{x,(-r)+} \ . \tag{4}$$

1. Regular depth

From now on let $\gamma \in G^{reg}$, and assume that the k-torus $\mathbf{T} := C_\mathbf{G}(\gamma)^\circ$ splits over a tamely ramified finite Galois extension E of k. We attach to γ the following rational number $s(\gamma)$.

Definition 2. For each $\alpha \in \Phi(\mathbf{T},E)$ let $s_\alpha(\gamma) := \nu(\alpha(\gamma)-1)$ and define $s(\gamma) := \max\{\, s_\alpha(\gamma) \mid \alpha \in \Phi(\mathbf{T},E)\,\}$.

Remark 3. Note that $s(\gamma)$ is not the same as the depth of γ (as defined in [2]). But for good elements [1, §2.2], these two notions agree.

Remark 4. A priori $s(\gamma) \in \mathbb{Q} \cup \{+\infty\}$, but since γ is regular, $\alpha(\gamma) \neq 1$ for all $\alpha \in \Phi(\mathbf{T}, E)$ and so $s(\gamma) \in \mathbb{Q}$. If γ is compact then $s(\gamma) \geq 0$. Also note that $s(\gamma z) = s(\gamma)$ for all z in the center $Z(G)$ of G and that $s(g\gamma g^{-1}) = s(\gamma)$ for all $g \in G$.

We will need the following basic properties of $s(\gamma)$.

Lemma 5. *Suppose $\gamma \in T_{cpt}$ and $\gamma' \in T_{s(\gamma)+}$.*

(1) *$s(\gamma\gamma') = s(\gamma)$ and for $\alpha \in \Phi(\mathbf{T}, E)$, we have $|\alpha(\gamma\gamma') - 1| = |\alpha(\gamma) - 1|$.*
(2) *$\gamma\gamma' \in T_{cpt}$.*

Proof. (1) Fix $r > s(\gamma) \geq 0$ such that $T_r = T_{s(\gamma)+}$. With this notation $\gamma' \in T_r$. By the alternative description of T_r, for any $\chi \in \mathbf{X}^*(\mathbf{T}, E)$, $\chi(\gamma') = 1 + \mu'$ where $\nu(\mu') \geq r$. Thus for any $\alpha \in \Phi(\mathbf{T}, E)$, $\alpha(\gamma') = 1 + \lambda'$ where $\nu(\lambda') \geq r$.

Note that since each $\alpha \in \Phi(\mathbf{T}, E)$ is continuous, $\alpha(T(E)_{cpt}) \subset R_E^\times$. Since $\gamma \in T_{cpt} \subset T(E)_{cpt}$ we get that $\alpha(\gamma)$ is a unit.

Now $\alpha(\gamma\gamma') - 1 = \alpha(\gamma)\alpha(\gamma') - 1 = \alpha(\gamma)(1 + \lambda') - 1 = (\alpha(\gamma) - 1) + \alpha(\gamma)\lambda'$. Using $\nu(\alpha(\gamma) - 1) =: s_\alpha(\gamma)$, $\alpha(\gamma)$ is a unit, and $\nu(\lambda') \geq r > s(\gamma) \geq s_\alpha(\gamma)$, we have $\nu(\alpha(\gamma\gamma') - 1) = \nu(\alpha(\gamma) - 1))$ (or equivalently $|\alpha(\gamma\gamma') - 1| = |\alpha(\gamma) - 1)|$) for all $\alpha \in \Phi(\mathbf{T}, E)$. Thus $s(\gamma\gamma') := \max_\alpha\{\nu(\alpha(\gamma\gamma') - 1)\} = \max_\alpha\{\nu(\alpha(\gamma) - 1)\} =: s(\gamma)$.

(2) Since γ and γ' are in T_{cpt}, so is their product. □

Corollary 6. *Let $\gamma \in T$ be a compact regular semisimple element. Then $\gamma T_{s(\gamma)+} \subset G^{reg}$.*

Proof. For $t \in T \cap G^{reg}$, following [7, §18], define

$$D_{G/T}(t) := \det\left(\mathrm{Ad}(t) - 1)\right)|_{\mathfrak{g}/\mathfrak{t}} = \prod_{\alpha \in \Phi(\mathbf{T}, E)} (\alpha(t) - 1).$$

Then $t \in T \cap G^{reg} \Leftrightarrow D_{G/T}(t) \neq 0 \Leftrightarrow |D_{G/T}(t)| \neq 0$. Using Lemma 5 with $\gamma \in T \cap G_{cpt}$ and $\gamma' \in T_{s(\gamma)+}$, we get $|D_{G/T}(\gamma\gamma')| = \prod_\alpha |\alpha(\gamma\gamma') - 1| = \prod_\alpha |\alpha(\gamma) - 1| = |D_{G/T}(\gamma)| \neq 0$. □

2. Some Technical Lemmas

The next lemma will generalize the following example.

Example 7. $\mathbf{G} = \mathbf{GL_2}$, $\mathbf{T}$ a k-split maximal torus. Choose $x_0 \in \mathcal{B}(\mathbf{G}, k)$ so that $G_{x_0,0} = GL_2(R)$. Any $X \in \mathcal{N} \cap (\mathfrak{g}_{x_0,r} \smallsetminus \mathfrak{g}_{x_0,r+})$ is of the form ${}^k\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, for some $k \in G_{x_0,0}$ (see [5, 9.2.1]). Thus

$$\begin{aligned} X &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \\ &= \frac{x}{ad-bc} \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix}. \end{aligned}$$

Write $X = Y + Z$ as in (2) and note that the depth of X with respect to the filtration $\{\mathfrak{g}_{x_0,r}\}_{r\in\mathbb{R}}$ of $\mathfrak{g}$ is controlled by Z. This is the case since $\max\{\nu(a^2), \nu(-c^2)\} \geq \nu(ac)$ and $ad - bc \in R^\times$.

Lemma 8. *Fix $x \in \mathcal{B}(\mathbf{T}, k)$ and $r \in \mathbb{R}$. For $X \in \mathcal{N} \cap (\mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+})$, write $X = Y + Z$ as in (2). Then $Z \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+}$.*

Proof. We first prove the case when the maximal k-torus $\mathbf{T}$ is k-split and then reduce the general case to this case.

Split case. Assume $\mathbf{T}$ is k-split. Note that $\mathfrak{t}^\perp = \oplus_{\alpha\in\Phi(\mathbf{T})}\mathfrak{g}_\alpha$. Fix a system of simple roots Δ in $\Phi(\mathbf{T})$ and choose a Chevalley basis for $\mathfrak{g}$ as in [1, §1.2]. Such a basis contains elements H_b and E_b in $\mathfrak{g}$ for each $b \in \Phi(\mathbf{T})$. If $\mathbf{G}$ is semisimple, then the set $\{H_b |\ b \in \Delta\} \cup \{E_b |\ b \in \Phi(\mathbf{T})\}$ is a basis for $\mathfrak{g}$. These elements also satisfy the commutation relations listed in [1, 1.2.1]. With respect to this choice of Chevalley basis, the adjoint representation is determined by the following formulas [1, 1.2.5]:

$$(5)\quad \left\{ \begin{array}{rcl} \mathrm{Ad}(e_b(\lambda))E_c &=& \left\{ \begin{array}{lll} E_b & if & c = b \\ E_c + \lambda H_b - \lambda^2 E_b & if & c = -b \\ \sum_{i\geq 0} M_{b,c;i}\, \lambda^i\, E_{ib+c} & if & c \neq \pm b \end{array} \right. \\ \mathrm{Ad}(t)E_c &=& c(t)E_c \\ \mathrm{Ad}(e_b(\lambda))H &=& H - db(H)\lambda E_b \\ \mathrm{Ad}(t)H &=& H \end{array} \right.$$

for all $H \in \mathrm{Lie}(T)$, all $t \in T$ and all $\lambda \in k$. Here e_b is the unique map $e_b : \mathbf{Add} \longrightarrow \mathbf{G}$ such that $\mathrm{d}e_b(1) = E_b$ (de_b is the derivative of e_b); and $M_{b,c;i}$ are constants with $M_{b,c;0} = 1$.

Let B be the Borel subgroup associated to Δ (with Levi decomposition $B = TN$ and opposite Borel $\overline{B} = T\overline{N}$). We have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$, where $\mathfrak{n} := \mathrm{Lie}(N)$ and $\overline{\mathfrak{n}} := \mathrm{Lie}(\overline{N})$. Note that $\mathfrak{n} \oplus \overline{\mathfrak{n}} = \oplus_{\alpha\in\Phi(\mathbf{T})}\mathfrak{g}_\alpha = \mathfrak{t}^\perp$. Recall that $G_{x,0}$ acts on $\mathfrak{g}_{x,r}$ (and on $\mathfrak{g}_{x,r+}$).

Given $X \in \mathcal{N} \cap (\mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+})$, we can use [2, Proposition 3.5.1] (with T playing the role of M) to conclude that there exists a group element $n \in N \cap G_{x,0}$ such that $({}^nX)_{\mathfrak{t}} \in \mathfrak{t}_{r+}$ (where ${}^nX$ denotes $\mathrm{Ad}(n)X$).

Write $X = Y + Z$ as in (2) and assume for a contradiction that $Z \in \mathfrak{g}_{x,r+}$. Since $X \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+}$, the assumption implies that $Y \in \mathfrak{t} \cap (\mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+}) = \mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}$.

Using the properties (5) of the Chevalley basis, one can easily check that the set $(\mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}) \oplus \mathfrak{n}$ is preserved under the action of $\mathrm{Ad}(e_b(\lambda))$ for all $b \in \Phi^+(\mathbf{T})$, where $\Phi^+(\mathbf{T})$ are the positive roots with respect to Δ. Since

$\{e_b(\lambda) \mid b \in \Phi^+(\mathbf{T})\}$ generates N, we conclude that ${}^nY \in (\mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}) \oplus \mathfrak{n}$, and hence that $({}^nY)_\mathfrak{t} \in \mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}$.

On the other hand we have ${}^nX = {}^nY + {}^nZ$, where ${}^nZ \in \mathfrak{g}_{x,r+}$. Taking the $\mathfrak{t}$ components, we get, $({}^nX)_\mathfrak{t} = ({}^nY)_\mathfrak{t} + ({}^nZ)_\mathfrak{t}$, with $({}^nZ)_\mathfrak{t} \in \mathfrak{t}_{r+}$. Since $({}^nX)_\mathfrak{t} \in \mathfrak{t}_{r+}$, we conclude that $({}^nY)_\mathfrak{t} \in \mathfrak{t}_{r+}$. This contradicts $({}^nY)_\mathfrak{t} \in \mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}$.

Hence $Z \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+}$ (note that from the decomposition (3) it is clear that $Z \in \mathfrak{g}_{x,r}$).

General case. We now assume $\mathbf{T}$ is an E-split maximal k-torus. Define $\mathfrak{t}(E)^\perp := (\mathrm{Ad}(\gamma) - 1)\mathfrak{g}(E)$. We have the following analogue of (2)

$$\boldsymbol{\mathfrak{g}}(E) = \boldsymbol{\mathfrak{t}}(E) \oplus \boldsymbol{\mathfrak{t}}(E)^\perp. \tag{6}$$

Note that $\mathfrak{t} \subset \boldsymbol{\mathfrak{t}}(E)$ and $\mathfrak{t}^\perp \subset \boldsymbol{\mathfrak{t}}(E)^\perp$. So the decomposition $X = Y + Z$ (as in (2)) for $X \in \mathfrak{g}$ is the same whether viewed in $\mathfrak{g}$ or in $\boldsymbol{\mathfrak{g}}(E)$.

Since $X \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+}$, equations (1) imply that $X \in (\boldsymbol{\mathfrak{g}}(E)_{x,r} \smallsetminus \boldsymbol{\mathfrak{g}}(E)_{x,r+}) \cap \mathfrak{g}$. Since $X \in \mathcal{N} \subset \boldsymbol{\mathcal{N}}(E)$ (where $\boldsymbol{\mathcal{N}}(E)$ is the set of nilpotent elements in $\boldsymbol{\mathfrak{g}}(E)$), we have that $X \in \boldsymbol{\mathcal{N}}(E) \cap (\boldsymbol{\mathfrak{g}}(E)_{x,r} \smallsetminus \boldsymbol{\mathfrak{g}}(E)_{x,r+})$. Now since $\mathbf{T}$ splits over E we can regard $\mathbf{G}$ over E as a split group and hence apply all the constructions of the split case above. So by the considerations of the split case above we conclude that $Z \in \boldsymbol{\mathfrak{g}}(E)_{x,r} \smallsetminus \boldsymbol{\mathfrak{g}}(E)_{x,r+}$. Intersecting with $\mathfrak{g}$ gives $Z \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+}$. $\square$

From now on we assume that γ is also compact. Recall that this implies that $s(\gamma) \geq 0$ (see Remark 4).

Lemma 9. *Let $t \in \mathbb{R}$ and $x \in \mathcal{B}(\mathbf{T}, k)$. If $Z \in \mathfrak{t}^\perp \cap (\mathfrak{g}_{x,-t} \smallsetminus \mathfrak{g}_{x,(-t)+})$ then ${}^\gamma Z - Z \notin \mathfrak{g}_{x,(-t+s(\gamma))+}$.*

Proof. Using the root decomposition $\boldsymbol{\mathfrak{t}}(E)^\perp = \oplus_{\alpha \in \Phi(\mathbf{T},E)} \boldsymbol{\mathfrak{g}}(E)_\alpha$, for $Z \in \mathfrak{t}^\perp \subset \boldsymbol{\mathfrak{t}}(E)^\perp$ we write $Z = \sum Z_\alpha$. Then ${}^\gamma Z - Z = \sum({}^\gamma Z_\alpha - Z_\alpha) = \sum(\alpha(\gamma) - 1)Z_\alpha$.

By assumption $Z \notin \mathfrak{g}_{x,(-t)+}$, hence (see equations (1)) $Z \notin \boldsymbol{\mathfrak{g}}(E)_{x,(-t)+}$. Thus for some $\alpha \in \Phi(\mathbf{T}, E)$, $Z_\alpha \notin \boldsymbol{\mathfrak{g}}(E)_{x,(-t)+}$, and so by definition of $s_\alpha(\gamma)$, $(\alpha(\gamma) - 1)Z_\alpha \notin \boldsymbol{\mathfrak{g}}(E)_{x,(-t+s_\alpha(\gamma))+}$. It follows by definition of $s(\gamma)$, that $(\alpha(\gamma) - 1)Z_\alpha \notin \boldsymbol{\mathfrak{g}}(E)_{x,(-t+s(\gamma))+}$. Hence ${}^\gamma Z - Z = \sum(\alpha(\gamma) - 1)Z_\alpha \notin \boldsymbol{\mathfrak{g}}(E)_{x,(-t+s(\gamma))+}$. Intersecting with $\mathfrak{g}$ we get that ${}^\gamma Z - Z \notin \mathfrak{g}_{x,(-t+s(\gamma))+}$. $\square$

Proposition 10. *Let $r \in \mathbb{R}$ and $x \in \mathcal{B}(\mathbf{T}, k)$. If $X \in \mathcal{N} \cap \mathfrak{g}_{x,(-2r)+}$ satisfies ${}^\gamma X - X \in \mathfrak{g}_{x,(-r)+}$, then $X \in \mathfrak{g}_{x,(-r-s(\gamma))+}$.*

Proof. Fix $t < 2r$ such that $X \in \mathcal{N} \cap (\mathfrak{g}_{x,-t} \smallsetminus \mathfrak{g}_{x,(-t)+})$.

Write $X = Y + Z$ as in (2). By Lemma 8, $Z \in \mathfrak{t}^\perp \cap (\mathfrak{g}_{x,-t} \smallsetminus \mathfrak{g}_{x,(-t)+})$, and so by Lemma 9, ${}^\gamma Z - Z \notin \mathfrak{g}_{x,(-t+s(\gamma))+}$.

On the other hand, since γ acts trivially on Y (because $Y \in \mathfrak{t} = C_\mathfrak{g}(\gamma)$), ${}^\gamma Z - Z = {}^\gamma X - X \in \mathfrak{g}_{x,(-r)+}$.

Thus $-t + s(\gamma) > -r$, or equivalently $-t > -r - s(\gamma)$, which implies that $X \in \mathfrak{g}_{x,-t} \subseteq \mathfrak{g}_{x,(-r-s(\gamma))+}$. $\square$

Definition 11. A character $d \in (G_{x,r}/G_{x,2r})^\wedge$ is called degenerate if under the isomorphism (4), the corresponding coset $X+\mathfrak{g}_{x,(-r)+}$ contains nilpotent elements.

Definition 12. Let K be a compact subgroup of G and $d \in K^\wedge$. For $g \in G$, let ${}^g d$ denote the representation of gKg^{-1} defined as ${}^g d(gkg^{-1}) := d(k)$. We say that g intertwines d with itself if upon restriction to $gKg^{-1} \cap K$, d and ${}^g d$ contain a common representation (up to isomorphism) of $gKg^{-1} \cap K$.

Corollary 13. *Let $x \in \mathcal{B}(\mathbf{T}, k)$, $r \in \mathbb{R}_{>0}$, and assume $d \in (G_{x,r}/G_{x,2r})^\wedge$ is degenerate. If γ intertwines d with itself then $d \in (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$.*

Proof. Let $X + \mathfrak{g}_{x,(-r)+}$ be the coset in $\mathfrak{g}_{x,(-2r)+}/\mathfrak{g}_{x,(-r)+}$ corresponding to d under the isomorphism (4). Since this coset is degenerate, we can assume that $X \in \mathcal{N}$.

Since γ fixes x (Lemma 1), γ stabilizes $G_{x,r}$. Thus having γ intertwine d with itself amounts to having $d \cong {}^\gamma d$; or furthermore, since d is one-dimensional, $d = {}^\gamma d$. Under the isomorphism (4), we get $X + \mathfrak{g}_{x,(-r)+} = {}^\gamma(X + \mathfrak{g}_{x,(-r)+})$, or equivalently that ${}^\gamma X - X \in \mathfrak{g}_{x,(-r)+}$. Now apply Proposition 10 to conclude that $X \in \mathfrak{g}_{x,(-r-s(\gamma))+}$, which under the isomorphism (4) gives that $d \in (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$. □

3. Partial Traces

Let (π, V) be an irreducible admissible representation of G. Let K be an open compact subgroup of G. Let $V = \bigoplus_{d\in K^\wedge} V_d$ be the decomposition of V into K-isotypic components. Let E_d denote the K-equivariant projection from V to V_d. For $f \in C_c^\infty(G)$ define the distribution $\Theta_d(f) := \operatorname{trace}(E_d\,\pi(f)\,E_d)$, the 'partial trace of π with respect to d'. The distribution Θ_d is represented by the locally constant function $\Theta_d(x) := \operatorname{trace}(E_d\,\pi(x)\,E_d)$ on G. Recall that it is known that the distribution $\Theta_\pi(f) := \operatorname{trace}\pi(f)$ is also represented by a locally constant function, Θ_π, on G^{reg}; we will not use this fact here. It follows from the definitions that as distributions

$$\Theta_\pi(f) = \sum_{d\in K^\wedge} \Theta_d(f) \quad \text{for all } f \in C_c^\infty(G).$$

Remark 14. For (some) $\omega \subset G^{reg}$ compact, Corollary 19 and the proof of Theorem 20 will imply that, for all $f \in C_c^\infty(\omega)$, this sum is *finite.*

Lemma 15. *$\Theta_d(kxk^{-1}) = \Theta_d(x)$ for all $x \in G$ and all $k \in K$.*

Proof. Since E_d is K-equivariant, it commutes with $\pi(k)$ for all $k \in K$.

$$\begin{aligned}\Theta_d(kxk^{-1}) &= \operatorname{trace}(E_d\pi(kxk^{-1})E_d)\\ &= \operatorname{trace}(E_d\pi(k)\pi(x)\pi(k^{-1})E_d)\\ &= \operatorname{trace}(\pi(k)E_d\pi(x)E_d\pi(k^{-1}))\\ &= \operatorname{trace}(E_d\pi(x)E_d) = \Theta_d(x).\end{aligned}$$

□

Let N be an open compact subgroup of G which is normal in K. Suppose $g \in G$ normalizes K and N. Let $d \in K^\wedge$. Considered as a representation of N, d decomposes into a finite sum of irreducible representations

$$d_1 \oplus \cdots \oplus d_n \,.$$

Proposition 16. *Suppose $\Theta_d(g) \neq 0$. Then $d \cong {}^g d$ as representations of K and also for some $i \in \{1, \cdots, n\}$, $d_i \cong {}^g d_i$ as representations of N.*

Proof. We refer to the appendix. Since g permutes the $V_{d'}$'s (Theorem 22.1), $0 \neq \Theta_d(g) = \mathrm{trace}(E_d \pi(g) E_d)$ implies that g must stabilize V_d. Fix a decomposition (‡) as in Theorem 22.2, and let E_{W_i} denote the K-equivariant projections onto W_i. Since $E_d = \sum E_{W_i}$, $\mathrm{trace}(E_d \pi(g) E_d) \neq 0$ implies that for some i, g must stabilize W_i, and that $\mathrm{trace}(E_{W_i} \pi(g) E_{W_i}) \neq 0$. By Theorem 22.3, $d \cong {}^g d$, which proves the first part of the theorem.

Now as a representation of N,

$$W_i \underset{N}{=} \bigoplus_j W_{i,d_j},$$

where W_{i,d_j} are the d_j-isotypic components of W_i. Since g stabilizes N, it must permute the W_{i,d_j}'s (Theorem 22.1). Since $E_{W_i} = \sum E_{W_{i,d_j}}$, having $\mathrm{trace}(E_{W_i} \pi(g) E_{W_i}) \neq 0$ implies that for some j, g must stabilize W_{i,d_j}, and that $\mathrm{trace}(E_{W_{i,d_j}} \pi(g) E_{W_{i,d_j}}) \neq 0$. Fix a decomposition (‡) as in Theorem 22.2 for W_{i,d_j}:

$$W_{i,d_j} \underset{N}{\cong} \bigoplus d_j .$$

Since $E_{W_{i,d_j}} = \sum E_{d_j}$, $\mathrm{trace}(E_{W_{i,d_j}} \pi(g) E_{W_{i,d_j}}) \neq 0$ implies that g must stabilize one of the d_j's. By Theorem 22.3, $d_j \cong {}^g d_j$, which proves the second part of the theorem. □

The following theorem and corollaries are used in the proof of Theorem 20 to show that for f with compact support, the sum $\sum_{d \in K^\wedge} \Theta_d(f)$ is finite (see also Remark 14).

Theorem 17. *Fix $x \in \mathcal{B}(\mathbf{T}, k)$ and let $r > \max\{s(\gamma), \rho(\pi)\}$. If $d \in (G_{x,r})^\wedge$ satisfies $\Theta_d(\gamma) \neq 0$, then $d \in (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$.*

Proof. If d is trivial we are done, so assume it is not. Let t be the smallest number such that $d|_{G_{x,t+}}$ is trivial (so in particular $d|_{G_{x,t}}$ is nontrivial).

Case $t < 2r$: Pick $s \leq 2r$ such that $G_{x,s} = G_{x,t+}$. Consider d as an element of $(G_{x,r}/G_{x,2r})^\wedge$. By Proposition 16, $\Theta_d(\gamma) \neq 0$ implies that $d \cong {}^\gamma d$. Also, $\Theta_d(\gamma) \neq 0$ implies that $d \subset \pi|_{G_{x,r}}$; since $r > \rho(\pi)$ this means that d is degenerate (see [5, §7.6]). Now apply Corollary 13.

Case $t \geq 2r$: Note that $\frac{t}{2} \geq r > s(\gamma)$. For $\epsilon > 0$ such that $\frac{t}{2} > \frac{\epsilon}{2} + s(\gamma)$, let $s = t + \epsilon$. By making ϵ smaller if necessary, we can make sure that $G_{x,s} = G_{x,t+}$. Note that $t > \frac{t}{2} + \frac{\epsilon}{2} + s(\gamma) = \frac{s}{2} + s(\gamma)$.

Since $\frac{s}{2} > \frac{t}{2} \geq r$ it makes sense to restrict d to $G_{x,\frac{s}{2}}$ and think of it as an element of $(G_{x,\frac{s}{2}}/G_{x,s})^\wedge$. As a representation of $G_{x,\frac{s}{2}}/G_{x,s}$, d decomposes into a finite sum of irreducible (one-dimensional) representations

$$d_1 \oplus \cdots \oplus d_n .$$

Let $X_i + \mathfrak{g}_{x,(-\frac{s}{2})+}$ be the coset in $\mathfrak{g}_{x,(-s)+}/\mathfrak{g}_{x,(-\frac{s}{2})+}$ corresponding to d_i under the isomorphism (4).

By Proposition 16, $0 \neq \Theta_d(\gamma)$ implies that for some j, $d_j \cong {}^{\gamma}d_j$.

Now $d \subset \pi|_{G_{x,r}}$, implies that $d_j \subset \pi|_{G_{x,\frac{s}{2}}}$ and since $\frac{s}{2} > r > \rho(\pi)$ we have that d_j is degenerate. Apply Corollary 13 to d_j to conclude that $d_j \in (G_{x,\frac{s}{2}}/G_{x,\frac{s}{2}+s(\gamma)})^\wedge$. In particular d_j is trivial on $G_{x,\frac{s}{2}+s(\gamma)}$, and hence on $G_{x,t}$.

Since $G_{x,r}$ normalizes $G_{x,\frac{s}{2}}$, it acts by permutations on the d_i's. Since d is irreducible, this action is transitive. Hence all the d_i's are conjugate by elements of $G_{x,r}$. By the conjugation of the d_i's and the fact that $d_j|_{G_{x,t}} = 1$ it follows that $d_i|_{G_{x,t}} = 1$ for all i, and so d itself is trivial on $G_{x,t}$. This contradicts the definition of t. Hence this case is not possible and $t < 2r$. $\square$

Corollary 18. *Fix $x \in \mathcal{B}(\mathbf{T}, k)$ and let $r > \max\{s(\gamma), \rho(\pi)\}$. Let X denote $\gamma T_{r+s(\gamma)}$, a compact subset of $T \cap G^{reg}$. If $d \in (G_{x,r})^\wedge$ satisfies $\Theta_d(\gamma') \neq 0$ for some $\gamma' \in X$, then $d \in (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$.*

Proof. Lemma 5 implies that γ' fixes x and that $s(\gamma')=s(\gamma)$. Now apply Theorem 17 to γ'. $\square$

Corollary 19. *Fix $x \in \mathcal{B}(\mathbf{T}, k)$ and let $r > \max\{s(\gamma), \rho(\pi)\}$. Let ω denote ${}^{G_{x,r}}(\gamma T_{r+s(\gamma)})$, an open compact subset of G^{reg}. Then Θ_d vanishes on ω for all $d \notin (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$. Furthermore, $\Theta_d(x) = \Theta_d(\gamma)$ for all $x \in \omega$ and all $d \in (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$.*

Proof. Follows immediately from Lemma 15 and Corollary 18. $\square$

4. Proof of the Main Theorem

Let $r > \max\{s(\gamma), \rho(\pi)\}$. Denote the finite set $(G_{x,r}/G_{x,r+s(\gamma)})^\wedge$ by F.

Theorem 20. *The distribution Θ_π is represented on the set ${}^{G}(\gamma T_{r+s(\gamma)})$ by a constant function.*

Proof. Using Corollary 19, we have for all $f \in C_c^\infty(G)$ whose support is contained in ω,

$$\Theta_\pi(f) = \sum_{d \in (G_{x,r})^\wedge} \Theta_d(f) = \sum_{d \in F} \int_\omega \Theta_d(x) f(x) dx = \sum_{d \in F} \int_\omega \Theta_d(\gamma) f(x) dx$$
$$= \int_\omega \left(\sum_{d \in F} \Theta_d(\gamma) \right) f(x) dx.$$

Thus Θ_π is represented by the constant function $\sum_{d\in F}\Theta_d(\gamma)$ on ω, i.e. $\Theta_\pi(x)=\sum_{d\in F}\Theta_d(\gamma)$ for all $x\in\omega$. Since Θ_π is conjugation invariant, we get $\Theta_\pi(gxg^{-1})=\Theta_\pi(x)=\sum_{d\in F}\Theta_d(\gamma)$ for all $x\in\omega$ and all $g\in G$. $\square$

Remark 21. This gives a new proof of the local constancy (near compact regular semisimple tame elements γ) of the character of an irreducible admissible representation for an arbitrary reductive p-adic group G.

5. Appendix

We prove some variations of Clifford theory [6, §14]. Let K and N be open compact subgroups of G, such that N is a normal subgroup of K. Let (π, V) be an irreducible admissible representation of G and let

$$V \underset{K}{=} \bigoplus_{d\in K^\wedge} V_d \tag{$\dagger$}$$

be the (canonical) decomposition of V into K-isotypic components. Here V_d denotes the d-isotypic component of V, i.e. the sum of all the K-submodules of V isomorphic to $d=(d,W)$. Each isotypic component V_d decomposes (non-canonically) into a finite sum of isomorphic copies $W_i \underset{K}{\cong} W$ of (d,W)

$$V_d \underset{K}{\cong} \bigoplus_i W_i\ . \tag{$\ddagger$}$$

Theorem 22. *Suppose $g\in G$ normalizes K and N. Then*

1. *The action of g permutes the V_d's.*
2. *Suppose g stabilizes V_d. Then there exists a decomposition ($\ddagger$) such that the action of g permutes the W_i.*
3. *Suppose W' is a K-submodule of V, isomorphic to W, and stable under the action g. Then ${}^g d\cong d$.*

Proof. (1) This follows from the fact that for any two K-submodules W' and W'' of V, if $W' \underset{K}{\cong} W''$ then $gW' \underset{K}{\cong} gW''$.

(2) Let W' be an irreducible K-submodule of V_d, isomorphic to W. Since g normalizes K and stabilizes V_d, gW' is a K-submodule of V_d. Since W' is irreducible, so is gW'. As an irreducible submodule of V_d, gW' must be isomorphic to W. By irreducibility either $W'\cap gW'=\{0\}$ or $W'=gW'$. Thus the orbit of W' under g is a collection of subspaces with trivial pairwise intersection, and so g acts on their sum as a desired. By complete reducibility of V_d (being a finite-dimensional representation of the compact group K) we can now use induction on the dimension of V_d.

(3) This follows from the following commutative diagram (in which all the arrows are isomorphisms of vector spaces and $k\in K$).

$$\begin{array}{ccccccccc}
W & \longrightarrow & W' & \xrightarrow{\pi(g)} & gW' & = & W' & \longrightarrow & W \\
{\scriptstyle d(k)}\big\downarrow & & {\scriptstyle \pi(k)}\big\downarrow & & {\scriptstyle \pi(gkg^{-1})}\big\downarrow & & {\scriptstyle \pi(k^g)}\big\downarrow & & {\scriptstyle d(k^g)}\big\downarrow{\scriptstyle {}^g d(k)} \\
W & \longrightarrow & W' & \xrightarrow{\pi(g)} & gW' & = & W' & \longrightarrow & W
\end{array}$$

□

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