ON THE LOCAL CONSTANCY OF CHARACTERS

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Abstract. The character of an irreducible admissible representation of a p-adic reductive group is known to be a constant function in some neighborhood of any regular semisimple element γ in the group. Under certain mild restrictions on γ, we give an explicit description of a neighborhood of γ on which the character is constant.

Introduction

Let k be a p-adic field of characteristic zero, and let G be a connected reductive algebraic group defined over k. We denote by G the group of k-rational points of G, and by g the Lie algebra of G. Let π be an irreducible admissible representation of G, and let Θπ be the (distribution) character of π. In [7] Harish-Chandra showed that Θπ can be represented by a function (also denoted by Θπ) which is locally integrable on G and locally constant on the set Greg of regular semisimple elements in G. Thus for any γ ∈ Greg there exists some neighborhood of γ on which the character is constant. In [8, Theorem 2, p. 483], R. Howe gave an elementary proof of Harish-Chandra’s result for general linear groups. In this paper we give a precise version of local constancy (near compact regular semisimple tame elements) for all reductive groups. The outline of the approach given here follows the elementary argument of Howe.

Let g_{x,r} (resp. G_{x,r}) be the Moy-Prasad lattices [10] in g (resp. open compact subgroups of G), normalized as in [9, §1.2]. Let G_cpt denote the set of compact elements in G. For a maximal k-torus T, let T_r denote its filtration subgroups (section 0.3). Let ρ(π) denote the depth of π [10, §5].

Fix a regular semisimple element γ and let T := C_G(γ)^0 be the connected component of its centralizer; T is a maximal k-torus in G. We assume that it splits over some tamely ramified finite Galois extension E of k. Let T denote the group of k-rational points of T. When γ ∈ T ∩ G_cpt we attach to it the nonnegative rational number s(γ). Using the filtration subgroups T_r and the parameter s(γ), we characterize a neighborhood of γ on which the character Θπ is constant. Whether or not this neighborhood of constancy is maximal is not addressed here.

The main result of this paper is the following (Theorem 20).

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Theorem. Let \( r = \max\{s(\gamma), \rho(\pi)\} + s(\gamma) \). The character \( \Theta_{\pi} \) is constant on the set \( G(\gamma T_r+) \).

We now give a brief sketch of the proof. Let \( K \) be any open compact subgroup of \( G \). Decompose \( \Theta_{\pi} \) into a countable sum of ‘partial trace’ operators \( \Theta_d \), according to the irreducible representations \( d \) of \( K \) (see section 3). For \( G = \text{GL}_n \), Howe proved [8, p. 499] the following key fact. If \( X \) is a compact subset of \( G_{\text{reg}} \), then \( \Theta_d \) vanishes on \( X \) for all \( d \) not in a certain finite set \( F \) (which depends only on \( X \)). It follows (see proof of Theorem 20), that \( \Theta_{\pi}(f) = \int_X \left( \sum_{d \in F} \Theta_d \right)(x)f(x)dx \) for all \( f \in C_0^\infty(X) \). Hence \( \Theta_{\pi} \) is represented on \( X \) by the locally constant function \( \sum_{d \in F} \Theta_d \).

The main part of this paper is concerned with formulating an analogue of the above key fact for reductive groups (see Corollary 18). The rational number \( s(\gamma) \), defined in section 1, is used (Corollary 18) to make a precise choice of a set \( X \) and a subgroup \( K \). Corollary 18 characterizes a finite set \( F \) of representations, such that for all \( d \) not in \( F \), \( \Theta_d \) vanishes on \( X \) (see Remark 14 for the significance of this fact). Thus the representations \( d \in F \) are those which play a role in understanding the character \( \Theta_{\pi} \) near \( \gamma \). The proof of this corollary relies on a special case (Corollary 13), in which we only consider 1-dimensional \( d \). Such representations have an explicit description in terms of cosets in the lie algebra \( g \). In section 2 we develop the technical tools, using Moy-Prasad lattices, to handle these cosets. Once we have a characterization of the set \( F \), we can make precise statements about the neighbourhood of constancy of the character near \( \gamma \) (Theorem 20).

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**Notation and Conventions**

Let \( k \) be a \( p \)-adic field (a finite extension of some \( \mathbb{Q}_p \)) with residue field \( \mathbb{F}_p \). Let \( \nu \) be a valuation on \( k \) normalized such that \( \nu(k^\times) = \mathbb{Z} \).

For any algebraic extension field \( E \) of \( k \), \( \nu \) extends uniquely to a valuation (also denoted \( \nu \)) of \( E \).

We denote the ring of integers in \( E \) by \( R_E \) (write \( R \) for \( R_k \)), and the prime ideal in \( R_E \) by \( \wp_E \) (write \( \wp \) for \( \wp_k \)).

Let \( G \) be a connected reductive group defined over \( k \), and \( G(E) \) the group of \( E \)-rational points of \( G \). We denote by \( G \) the group of \( k \)-rational points of \( G \). Denote the Lie algebras of \( G \) and \( G(E) \) by \( g \) and \( g(E) \), respectively. Write \( g \) for the Lie algebra of \( k \)-rational points of \( g \).

Let \( \mathcal{N} \) be the set of nilpotent elements in \( g \). There are different notions of nilpotency, but since we assume that \( \text{char}(k) = 0 \), these notions all coincide.

Let \( \text{Ad} \) (resp. \( \text{ad} \)) denote the adjoint representation of \( G \) (resp. \( g \)) on its Lie algebra \( g \). For elements \( g \in G \) and \( X \in g \) (resp. \( x \in G \)) we will
sometimes write $gX$ (resp. $g_x$) instead of $\text{Ad}(g)X$ (resp. $gxg^{-1}$). For a subset $S$ of $g$ (resp. $G$) let $^G S$ denote the set $\{^g s \mid g \in G \text{ and } s \in S\}$.

Let $n$ denote the (absolute) rank of $G$. We say that an element $g \in G$ is regular semisimple if the coefficient of $t^n$ in $\det(t-1+\text{Ad}(g))$ is nonzero. We denote the set of regular semisimple elements in $G$ by $G^{\text{reg}}$. Similarly we say that an element $X \in g$ is regular semisimple if the coefficient of $t^n$ in $\det(t-\text{ad}(X))$ is nonzero. We denote the set of regular semisimple elements in $g$ by $g^{\text{reg}}$. Let $G_{\text{cpt}}$ denote the set of compact elements in $G$. For a subset $S$ of $G$ we will sometimes write $S_{\text{cpt}}$ for $S \cap G_{\text{cpt}}$.

For a subset $S$ of $g$ (resp. $G$) let $[S]$ denote the characteristic function of $S$ on $g$ (resp. $G$).

For any compact group $K$, let $K^h$ denote the set of equivalence classes of irreducible, continuous representations of $K$.

Let $\pi$ be an irreducible admissible representation of $G$. We denote by $\Theta_\pi$ the character of $\pi$ thought of as a locally constant function on the set $G^{\text{reg}}$. Let $\rho(\pi)$ denote the depth of $\pi$ [10, §5].

0. Preliminaries

0.1. Apartments and buildings. For a finite extension $E$ of $k$, let $\mathcal{B}(G, E)$ denote the extended Bruhat-Tits building of $G$ over $E$; write $\mathcal{B}(G)$ for $\mathcal{B}(G, k)$. It is known (e.g. [13]) that if $E$ is a tamely ramified finite Galois extension of $k$ then $\mathcal{B}(G, k)$ can be embedded into $\mathcal{B}(G, E)$ and its image is equal to the set of Galois fixed points in $\mathcal{B}(G, E)$. If $T$ is a maximal $k$-torus in $G$ that splits over $E$, let $\mathcal{A}(T, E)$ be the corresponding apartment over $E$. Let $X^*(T, E)$ (resp. $X_*(T, E)$) denote the group of $E$-rational characters (resp. cocharacters) of $T$.

It is known in the tame case [1, §1.9] that there is a Galois equivariant embedding of $\mathcal{B}(T, E)$ into $\mathcal{B}(G, E)$, which in turn induces an embedding of $\mathcal{B}(T, k)$ into $\mathcal{B}(G, k)$. Such embeddings are only unique modulo translations by elements of $X_*(T, k) \otimes \mathbb{R}$, however their images are all the same and are equal to the set $\mathcal{A}(T, E) \cap \mathcal{B}(G, k)$. From now on we fix a $T$-equivariant embedding $i : \mathcal{B}(T, k) \rightarrow \mathcal{B}(G, k)$, and use it to regard $\mathcal{B}(T, k)$ as a subset of $\mathcal{B}(G, k)$; write $x$ for $i(x)$.

Notation. We write $\mathcal{A}(T, k)$ for $\mathcal{A}(T, E) \cap \mathcal{B}(G, k)$. This is well defined independent of the choice of $E$ [15]. Moreover, $\mathcal{A}(T, k)$ is the set of Galois fixed points in $\mathcal{A}(T, E)$.

We remark that the image of $\mathcal{B}(T, E)$ in $\mathcal{B}(G, E)$ is the apartment $\mathcal{A}(T, E)$, while the image of $\mathcal{B}(T, k)$ in $\mathcal{B}(G, k)$ is the set $\mathcal{A}(T, k)$.

0.2. Moy-Prasad filtrations. Regarding $G$ as a group over $E$, Moy and Prasad (see [10] and [11]) define lattices in $g(E)$ and subgroups of $G(E)$.

We can and will normalize (with respect to the normalized valuation $v$) the indexing $(x, r) \in \mathcal{B}(G, E) \times \mathbb{R}$ of these lattices and subgroups as in [9, §1.2]. We will denote the (normalized) lattices by $g(E)_{x, r}$, and the (normalized) subgroups by $G(E)_{x, [r]}$. 
If $w_E$ is a uniformizing element of $E$, and $e = e(E/k)$ is the ramification index of $E$ over $k$, then these normalized lattices (resp. subgroups) satisfy $w_E \mathfrak{g}(E)_{x,r} = \mathfrak{g}(E)_{x,r+\frac{e}{2}}$. Write $\mathfrak{g}_{x,r}$ (resp. $G_{x,[r]}$) for $\mathfrak{g}(k)_{x,r}$ (resp. $G(k)_{x,[r]}$).

The above normalization was chosen to have the following property \cite{1, 1.4.1]: when $E$ is a tamely ramified Galois extension of $k$ and $x \in B(G, k) \subset B(G, E)$, we have

\begin{equation}
\text{(1)} \quad \mathfrak{g}_{x,r} = \mathfrak{g}(E)_{x,r} \cap \mathfrak{g}, \quad \text{and (for } r > 0) \quad G_{x,r} = \mathfrak{g}(E)_{x,r} \cap G.
\end{equation}

We will also use the following notation. Let $r \in \mathbb{R}$ and $x \in B(G)$.

- $\mathfrak{g}_{x,r+} = \cup_{s > r} \mathfrak{g}_{x,s}$ and $G_{x,[r]+} = \cup_{s > [r]} G_{x,s}$.

- $G_r = \cup_{s \in B(G)} G_{x,s}$ and $G_{r+} = \cup_{s > r} G_s$ for $r \geq 0$.

The lattices $\mathfrak{g}_{x,r+}$ (resp. groups $G_{x,[r]+}$) have analogous properties to those of $\mathfrak{g}_{x,r}$ (resp. $G_{x,[r]}$). The set $G_0$ is the set of compact elements $G_{cpt}$. We remark that $G_{cpt} \subset G(E)_{cpt} \cap G$, and in general they need not be equal \cite{3, §2.2.3].

**Lemma 1.** Let $\gamma$ be a compact regular semisimple element, and consider the maximal $k$-torus $T := C_G(\gamma)^{r}$. Suppose that $T$ splits over a tamely ramified finite Galois extension $E$ of $k$. Then $\gamma$ fixes $B(T,k)$ pointwise.

**Proof.** Recall that $\gamma$ acts on $A(T,E)$ by translations \cite{14, §1}. Since $\gamma$ belongs to a compact subgroup, it has a fixed point $x \in B(G, E)$.

If $\gamma$ acts on $A(T,E)$ by a nontrivial translation, then for any $y \in A(T,E)$ there is an $n \in \mathbb{N}$ such that $d(x,y) \neq d(x, \gamma^n y)$. This contradicts the fact that the action preserves distances. So $\gamma$ must act trivially on $A(T,E)$. In particular, $\gamma$ fixes $A(T,k)$, and hence $B(T,k)$, pointwise. \square

### 2.3. Root decomposition.

Let $T$ be a maximal $k$-torus in $G$ that splits over a tamely ramified finite Galois extension $E$ of $k$. Let $\Phi(T,E)$ denote the set of roots of $G$ with respect to $E$ and $T$, and let $\Psi(T,E)$ denote the corresponding set of affine roots of $G$ with respect to $E$, $T$ and $\nu$. When $T$ is $k$-split, we also write $\Phi(T)$ for $\Phi(T,k)$ (resp. $\Psi(T)$ for $\Psi(T,k)$). If $\psi \in \Psi(T,E)$, let $\psi \in \Phi(T,E)$ be the gradient of $\psi$, and let $\mathfrak{g}(E)_{\psi} \subset \mathfrak{g}(E)$ be the root space corresponding to $\psi$. We denote the root lattice in $\mathfrak{g}(E)_{\psi}$ corresponding to $\psi$ by $\mathfrak{g}(E)_{\psi}$ \cite{10, 3.2].

For $x \in A(T,E)$ and $r \in \mathbb{R}$, let $t(E)_r := t(E) \cap \mathfrak{g}(E)_{x,r}$ and $t(E)_{r+} := t(E) \cap \mathfrak{g}(E)_{x,r+}$. Note that $t(E)_r$ and $t(E)_{r+}$ are defined independent of the choice of $x \in A(T,E)$. Similarly one defines the subgroups $T(E)_r$ and $T(E)_{r+}$ for $r \geq 0$; they have analogous properties. Note that using our conventions we will sometimes denote $T(E)_0$ by $T(E)_{cpt}$.

An alternative description is \cite{9, §2.1}: for $r \in \mathbb{R}$,

$$t(E)_r = \{ \Gamma \in t(E) | \nu(d\chi(\Gamma)) \geq r \quad \text{for all } \chi \in X^+(T,E) \}$$

and for $r > 0$,

$$T(E)_r = \{ t \in T(E) | \nu(\chi(t) - 1) \geq r \quad \text{for all } \chi \in X^+(T,E) \}.$$
Since $G$ splits over $E$, we have

$$g(E)_{x,r} = t(E)_r \oplus \sum_{\psi \in \Phi(T,E), \psi(x) \geq r} g(E)_{\psi},$$

$$g(E)_{x,r+} = t(E)_{r+} \oplus \sum_{\psi \in \Phi(T,E), \psi(x) > r} g(E)_{\psi}.$$

Let $t := \text{Lie}(T)$, and define $t^{\perp} := (\text{Ad}(\gamma) - 1)g$. We have the following decomposition [7, §18]

$$g = t \oplus t^{\perp}.$$  

We write $X = Y + Z$ with respect to this decomposition; when convenient, we also write $X_t$ for $Y$.

Fix $x \in B(T,k) \subset B(G,k)$ and $r \in \mathbb{R}$. Write $t_r$ for $t \cap g_{x,r}$ (resp. $t_{r+}$ for $t \cap g_{x,r+}$); as mentioned earlier, these definitions are independent of $x$. Define $t^{\perp}_{x,r} := t^{\perp} \cap g_{x,r}$ (resp. $t^{\perp}_{x,r+} := t^{\perp} \cap g_{x,r+}$). We have [1, 1.9.3],

$$g_{x,r} = t_r \oplus t^{\perp}_{x,r},$$

$$g_{x,r+} = t_{r+} \oplus t^{\perp}_{x,r+}.$$  

0.4. Hypotheses. (HB) There is a nondegenerate $G$-invariant symmetric bilinear form $B$ on $g$ such that we can identify $g^*_{x,r}$ with $g_{x,r}$ via the map $\Omega : g \rightarrow g^*$ defined by $\Omega(X)(Y) = B(X,Y)$.

Groups satisfying the above hypothesis are discussed in [4, §4].

Fix $r \in \mathbb{R}_{>0}$ and $x \in B(G,k)$. For any $r \leq t \leq 2r$ the group $(G_{x,r}/G_{x,t})$ is abelian. By hypothesis (HB), there exists a $(G_{x,0}$-equivariant) isomorphism (see [1, §1.7] or [12, p.16])

$$(G_{x,r}/G_{x,t})^\wedge \cong g_{x,(-t)+}/g_{x,(-r)+}.$$  

1. Regular depth

From now on let $\gamma \in G^{reg}$, and assume that the $k$-torus $T := C_G(\gamma)^0$ splits over a tamely ramified finite Galois extension $E$ of $k$. We attach to $\gamma$ the following rational number $s(\gamma)$.

Definition 2. For each $\alpha \in \Phi(T,E)$ let $s_{\alpha}(\gamma) := \nu(\alpha(\gamma) - 1)$ and define $s(\gamma) := \max \{ s_{\alpha}(\gamma) \mid \alpha \in \Phi(T,E) \}$.

Remark 3. Note that $s(\gamma)$ is not the same as the depth of $\gamma$ (as defined in [2]). But for good elements [1, §2.2], these two notions agree.
Remark 4. A priori $s(\gamma) \in \mathbb{Q} \cup \{+\infty\}$, but since $\gamma$ is regular, $\alpha(\gamma) \neq 1$ for all $\alpha \in \Phi(T, E)$ and so $s(\gamma) \in \mathbb{Q}$. If $\gamma$ is compact then $s(\gamma) \geq 0$. Also note that $s(\gamma z) = s(\gamma)$ for all $z$ in the center $Z(G)$ of $G$ and that $s(g\gamma g^{-1}) = s(\gamma)$ for all $g \in G$.

We will need the following basic properties of $s(\gamma)$.

Lemma 5. Suppose $\gamma \in T_{\text{cpt}}$ and $\gamma' \in T_{s(\gamma)+}$.

1. $s(\gamma\gamma') = s(\gamma)$ and for $\alpha \in \Phi(T, E)$, we have $|\alpha(\gamma\gamma') - 1| = |\alpha(\gamma) - 1|$. 
2. $\gamma\gamma' \in T_{\text{cpt}}$.

Proof. (1) Fix $r > s(\gamma) \geq 0$ such that $T_r = T_{s(\gamma)+}$. With this notation $\gamma' \in T_r$. By the alternative description of $T_r$, for any $\chi \in X^*(T, E)$, $\chi(\gamma') = 1 + \mu' \text{ where } \nu(\mu') \geq r$. Thus for any $\alpha \in \Phi(T, E)$, $\alpha(\gamma') = 1 + \lambda'$ where $\nu(\lambda') \geq r$.

Note that since each $\alpha \in \Phi(T, E)$ is continuous, $\alpha(T(E)_{\text{cpt}}) \subset R_E^\times$. Since $\gamma \in T_{\text{cpt}} \subset T(E)_{\text{cpt}}$ we get that $\alpha(\gamma)$ is a unit.

Now $\alpha(\gamma\gamma') - 1 = \alpha(\gamma)\alpha(\gamma') - 1 = \alpha(\gamma)(1 + \lambda') - 1 = (\alpha(\gamma) - 1) + \alpha(\gamma)\lambda'$. Using $\nu(\alpha(\gamma) - 1) = s_\alpha(\gamma)$, $\alpha(\gamma)$ is a unit, and $\nu(\lambda') \geq r > s(\gamma) \geq s_\alpha(\gamma)$, we have $\nu(\alpha(\gamma\gamma') - 1) = \nu(\alpha(\gamma) - 1)$ (or equivalently $|\alpha(\gamma\gamma') - 1| = |\alpha(\gamma) - 1|$) for all $\alpha \in \Phi(T, E)$. Thus $s(\gamma\gamma') := \max\{\nu(\alpha(\gamma\gamma') - 1)\} = \max\{\nu(\alpha(\gamma) - 1)\} =: s(\gamma)$.

(2) Since $\gamma$ and $\gamma'$ are in $T_{\text{cpt}}$, so is their product. 

Corollary 6. Let $\gamma \in T$ be a compact regular semisimple element. Then $\gamma T_{s(\gamma)+} \subset G^{\text{reg}}$.

Proof. For $t \in T \cap G^{\text{reg}}$, following [7, §18], define

$$D_{G/T}(t) := \det(\text{Ad}(t) - 1)|_{G/t} = \prod_{\alpha \in \Phi(T, E)} (\alpha(t) - 1).$$

Then $t \in T \cap G^{\text{reg}} \iff D_{G/T}(t) \neq 0 \iff |D_{G/T}(t)| \neq 0$. Using Lemma 5 with $\gamma \in T \cap G_{\text{cpt}}$ and $\gamma' \in T_{s(\gamma)+}$, we get $|D_{G/T}(\gamma\gamma')| = \prod_{\alpha} |\alpha(\gamma\gamma') - 1| = \prod_{\alpha} |\alpha(\gamma) - 1| = |D_{G/T}(\gamma)| \neq 0$. 

2. Some Technical Lemmas

The next lemma will generalize the following example.

Example 7. $G = GL_2$, $T$ a $k$-split maximal torus. Choose $x_0 \in B(G, k)$ so that $G_{x_0, 0} = GL_2(R)$. Any $X \in \mathcal{N} \cap (g_{x_0, r} \cdot g_{x_0, r} +)$ is of the form $k \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. 
Lemma 8. Fix \( \mathbf{T} \in \mathcal{T} \) playing the role of \( T \). For some \( k \in \mathbb{N} \) and \( r \in \mathbb{R} \), write \( X = Y + Z \) as in (2) and note that the depth of \( X \) with respect to the filtration \( \{ \mathfrak{g}_{x,r} \}_{r \in \mathbb{R}} \) of \( \mathfrak{g} \) is controlled by \( Z \). This is the case since \( \max \{ \nu(a^2), \nu(-c^2) \} \geq \nu(ac) \) and \( ad - bc \in R^\times \).

Lemma 8. Fix \( x \in B(\mathbf{T}, k) \) and \( r \in \mathbb{R} \). For \( X \in \mathcal{N} \cap (\mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r+}) \), write \( X = Y + Z \) as in (2). Then \( Z \in \mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r+} \).

Proof. We first prove the case when the maximal \( k \)-torus \( \mathbf{T} \) is \( k \)-split and then reduce the general case to this case.

\textbf{Split case.} Assume \( \mathbf{T} \) is \( k \)-split. Note that \( t^+ = \oplus_{\alpha \in \Phi(\mathbf{T})} \mathfrak{g}_{\alpha} \). Fix a system of simple roots \( \Delta \) in \( \Phi(\mathbf{T}) \) and choose a Chevalley basis for \( \mathfrak{g} \) as in [1, §1.2]. Such a basis contains elements \( H_b \) and \( E_b \) in \( \mathfrak{g} \) for each \( b \in \Phi(\mathbf{T}) \). If \( G \) is semisimple, then the set \( \{ H_b \mid b \in \Delta \} \cup \{ E_b \mid b \in \Phi(\mathbf{T}) \} \) is a basis for \( \mathfrak{g} \). These elements also satisfy the commutation relations listed in [1, 1.2.1].

With respect to this choice of Chevalley basis, the adjoint representation is determined by the following formulas [1, 1.2.5]:

\[
\begin{align*}
\text{Ad}(e_b(\lambda))E_c &= \begin{cases} E_b & \text{if } c = b \\ E_c + \lambda H_b - \lambda^2 E_b & \text{if } c = -b \\ \sum_{i \geq 0} M_{b,c;i} \lambda^i E_{ib+c} & \text{if } c \neq \pm b \end{cases} \\
\text{Ad}(t)E_c &= e(t)E_c \\
\text{Ad}(e_b(\lambda))H &= H - db(H)\lambda E_b \\
\text{Ad}(t)H &= H
\end{align*}
\]

for all \( H \in \text{Lie}(T) \), all \( t \in T \) and all \( \lambda \in k \). Here \( e_b : \text{Add} \rightarrow G \) such that \( de_b(1) = E_b \) (\( de_b \) is the derivative of \( e_b \)); and \( M_{b,c;i} \) are constants with \( M_{b,c;0} = 1 \).

Let \( B \) be the Borel subgroup associated to \( \Delta \) (with Levi decomposition \( B = TN \) and opposite Borel \( \bar{B} = T\bar{N} \)). We have \( \mathfrak{g} = t \oplus n \oplus \bar{n} \), where \( n := \text{Lie}(N) \) and \( \bar{n} := \text{Lie}(\bar{N}) \). Note that \( n \oplus \bar{n} = \oplus_{\alpha \in \Phi(\mathbf{T})} \mathfrak{g}_{\alpha} = t^+ \). Recall that \( G_{x,0} \) acts on \( \mathfrak{g}_{x,r} \) (and on \( \mathfrak{g}_{x,r+} \)).

Given \( X \in \mathcal{N} \cap (\mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r+}) \), we can use [2, Proposition 3.5.1] (with \( T \) playing the role of \( M \)) to conclude that there exists a group element \( n \in N \cap G_{x,0} \) such that \( "X"_t \in t^+ \) (where \( "X" \) denotes \( \text{Ad}(n)X \)).

Write \( X = Y + Z \) as in (2) and assume for a contradiction that \( Z \in \mathfrak{g}_{x,r+} \).

Since \( X \in \mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r+} \), the assumption implies that \( Y \in t \cap (\mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r+}) = t_r \setminus t_{r+} \).

Using the properties (5) of the Chevalley basis, one can easily check that the set \((t_r \setminus t_{r+}) \oplus n\) is preserved under the action of \( \text{Ad}(e_b(\lambda)) \) for all \( b \in \Phi^+(\mathbf{T}) \), where \( \Phi^+(\mathbf{T}) \) are the positive roots with respect to \( \Delta \). Since
\{e_\gamma(\lambda) \mid b \in \Phi^+(T)\} \) generates \( N \), we conclude that \( nY \in (t_r \setminus t_{r+}) \oplus n \), and hence that \( \langle nY \rangle t \subseteq t_r \setminus t_{r+} \).

On the other hand we have \( nX = nY + nZ \), where \( nZ \in g_{x,r+}. \) Taking the \( t \) components, we get, \( \langle nX \rangle t = \langle nY \rangle t + \langle nZ \rangle t \), with \( \langle nZ \rangle t \subseteq t_{r+}. \) Since \( \langle nX \rangle t \subseteq t_{r+} \), we conclude that \( \langle nY \rangle t \subseteq t_{r+}. \) This contradicts \( \langle nY \rangle t \subseteq t_r \setminus t_{r+}. \)

Hence \( Z \in g_{x,r} \setminus g_{x,r+} \) (note that from the decomposition \( 3 \) it is clear that \( Z \in g_{x,r} \)).

**General case.** We now assume \( T \) is an \( E \)-split maximal \( k \)-torus. Define \( t(E)^\perp := (\text{Ad}(\gamma) - 1)g(E) \). We have the following analogue of \( 2 \)

\[
\text{(6)} \quad \gamma Z - Z \in g_{x,t} \setminus g_{x,(t^\perp)}^+. 
\]

From now on we assume that \( \gamma \) is also compact. Recall that this implies that \( s(\gamma) \geq 0 \) (see Remark 4).

**Lemma 9.** Let \( t \in \mathbb{R} \) and \( x \in B(T, k) \). If \( Z \in t^\perp \cap (g_{x,-t} \setminus g_{x,(-t)^+}) \) then \( \gamma Z - Z \notin g_{x,(-t+s(\gamma))^+}. \)

**Proof.** Using the root decomposition \( t(E)^\perp = \bigoplus_{\alpha \in \Phi^+(T, E)} \gamma g(E)_\alpha \), for \( Z \in t^\perp \subseteq t(E)^\perp \) we write \( Z = \sum Z_\alpha. \) Then \( \gamma Z - Z = \sum(Z_\alpha - Z_\alpha) = \sum(\alpha(\gamma) - 1)Z_\alpha. \)

By assumption \( Z \notin g_{x,(-t)^+} \), hence (see equations \( 1 \)) \( Z \notin g(E)_{x,(-t)^+}. \) Thus for some \( \alpha \in \Phi^+(T, E) \), \( Z_\alpha \notin g(E)_{x,(-t)^+} \), and so by definition of \( s_\alpha(\gamma) \), \( (\alpha(\gamma) - 1)Z_\alpha \notin g(E)_{x,(-t+s_\alpha(\gamma))^+}. \) It follows by definition of \( s(\gamma) \), that \( (\alpha(\gamma) - 1)Z_\alpha \notin g(E)_{x,(-t+s(\gamma))^+}. \) Hence \( \gamma Z - Z = \sum(\alpha(\gamma) - 1)Z_\alpha \notin g(E)_{x,(-t+s(\gamma))^+}. \) Intersecting with \( g \) we get that \( \gamma Z - Z \notin g_{x,(-t+s(\gamma))^+}. \)

**Proposition 10.** Let \( r \in \mathbb{R} \) and \( x \in B(T, k) \). If \( X \in N \cap (g_{x,-2r}^+) \) satisfies \( \gamma X - X \in g_{x,(-r)^+} \) then \( X \in g_{x,(-r-s(\gamma))^+}. \)

**Proof.** Fix \( t < 2r \) such that \( X \in N \cap (g_{x,-t} \setminus g_{x,(-t)^+}) \).

Write \( X = Y + Z \) as in \( 2 \). By Lemma 8, \( Z \in t^\perp \cap (g_{x,-t} \setminus g_{x,(-t)^+}), \) and so by Lemma 9, \( \gamma Z - Z \notin g_{x,(-t+s(\gamma))^+}. \)

On the other hand, since \( \gamma \) acts trivially on \( Y \) (because \( Y \in t = C_{g}(\gamma) \)), we have \( \gamma Z - Z = \gamma X - X \in g_{x,(-r)^+}. \)

Thus \( -t + s(\gamma) > -r \), or equivalently \( -t > -r - s(\gamma) \), which implies that \( X \in g_{x,-t} \subseteq g_{x,(-r-s(\gamma))^+}. \) \( \square \)
Definition 11. A character $d \in (G_{x,r}/G_{x,2r})^\wedge$ is called degenerate if under
the isomorphism (4), the corresponding coset $X + g_{x,(-r)+}$ contains nilpotent
elements.

Definition 12. Let $K$ be a compact subgroup of $G$ and $d \in K^\wedge$. For $g \in G$, let $gd$ denote the representation of $gKg^{-1}$ defined as $gd(gkg^{-1}) := d(k)$. We
say that $g$ intertwines $d$ with itself if upon restriction to $gKg^{-1} \cap K$, $d$ and
$gd$ contain a common representation (up to isomorphism) of $gKg^{-1} \cap K$.

Corollary 13. Let $x \in B(T,k)$, $r \in \mathbb{R}_{>0}$, and assume $d \in (G_{x,r}/G_{x,2r})^\wedge$ is
degenerate. If $\gamma$ intertwines $d$ with itself then $d \in (G_{x,r}/G_{x,2r+s(\gamma)})^\wedge$.

Proof. Let $X + g_{x,(-r)+}$ be the coset in $g_{x,(-2r)+}/g_{x,(-r)+}$ corresponding to
$d$ under the isomorphism (4). Since this coset is degenerate, we can assume that
$X \in N$.

Since $\gamma$ fixes $x$ (Lemma 1), $\gamma$ stabilizes $G_{x,r}$. Thus having $\gamma$ intertwine
$d$ with itself amounts to having $d \cong \gamma d$; or furthermore, since $d$ is one-
dimensional, $d = \gamma d$. Under the isomorphism (4), we get $X + g_{x,(-r)+} = 
\gamma(X + g_{x,(-r)+})$, or equivalently that $\gamma X - X \in g_{x,(-r)+}$. Now apply Proposition 10 to conclude that $X \in g_{x,(-r-s(\gamma))+}$, which under the isomorphism
(4) gives that $d \in (G_{x,r}/G_{x,2r+s(\gamma)})^\wedge$. \hfill \Box

3. Partial Traces

Let $(\pi,V)$ be an irreducible admissible representation of $G$. Let $K$ be
an open compact subgroup of $G$. Let $V = \bigoplus_{d \in K^\wedge} V_d$ be the decomposition
of $V$ into $K$-isotypic components. Let $E_d$ denote the $K$-equivariant
projection from $V$ to $V_d$. For $f \in C_c^\infty(G)$ define the distribution $\Theta_d(f) :=
\text{trace}(E_d \pi(f) E_d)$, the ‘partial trace of $\pi$ with respect to $d$’. The distribution
$\Theta_d$ is represented by the locally constant function $\Theta_d(x) := \text{trace}(E_d \pi(x) E_d)$
on $G$. Recall that it is known that the distribution $\Theta_d(f) := \text{trace}(\pi(f))$ is
also represented by a locally constant function, $\Theta_\pi$, on $G^{reg}$; we will not use
this fact here. It follows from the definitions that as distributions

$$\Theta_\pi(f) = \sum_{d \in K^\wedge} \Theta_d(f) \quad \text{for all} \quad f \in C_c^\infty(G).$$

Remark 14. For (some) $\omega \subset G^{reg}$ compact, Corollary 19 and the proof of
Theorem 20 will imply that, for all $f \in C_c^\infty(\omega)$, this sum is finite.

Lemma 15. $\Theta_d(kxk^{-1}) = \Theta_d(x)$ for all $x \in G$ and all $k \in K$.

Proof. Since $E_d$ is $K$-equivariant, it commutes with $\pi(k)$ for all $k \in K$.

$$\Theta_d(kxk^{-1}) = \text{trace}(E_d \pi(kxk^{-1}) E_d)$$
$$= \text{trace}(E_d \pi(k) \pi(x) \pi(k^{-1}) E_d)$$
$$= \text{trace}(\pi(k) E_d \pi(x) E_d \pi(k^{-1}))$$
$$= \text{trace}(E_d \pi(x) E_d) = \Theta_d(x).$$ \hfill \Box
Let $N$ be an open compact subgroup of $G$ which is normal in $K$. Suppose $g \in G$ normalizes $K$ and $N$. Let $d \in K^\wedge$. Considered as a representation of $N$, $d$ decomposes into a finite sum of irreducible representations

$$d_1 \oplus \cdots \oplus d_n.$$  

**Proposition 16.** Suppose $\Theta_d(g) \neq 0$. Then $d \cong g^*d$ as representations of $K$ and also for some $i \in \{1, \cdots, n\}$, $d_i \cong g^*d_i$ as representations of $N$.

**Proof.** We refer to the appendix. Since $g$ permutes the $V_d$'s (Theorem 22.1), $0 \neq \Theta_d(g) = \text{trace}(E_d \pi(g)E_d)$ implies that $g$ must stabilize $V_d$. Fix a decomposition (1) as in Theorem 22.2, and let $E_{W_i}$ denote the $K$-equivariant projections onto $W_i$. Since $E_d = \sum E_{W_i}$, trace($E_d \pi(g)E_d$) $\neq 0$ implies that for some $i$, $g$ must stabilize $W_i$, and that trace($E_{W_i} \pi(g)E_{W_i}$) $\neq 0$. By Theorem 22.3, $d \cong g^*d$, which proves the first part of the theorem.

Now as a representation of $N$,

$$W_i \cong \bigoplus_j W_{i,d_j},$$

where $W_{i,d_j}$ are the $d_j$-isotypic components of $W_i$. Since $g$ stabilizes $N$, it must permute the $W_{i,d_j}$'s (Theorem 22.1). Since $E_{W_i} = \sum E_{W_{i,d_j}}$, having trace($E_{W_i} \pi(g)E_{W_i}$) $\neq 0$ implies that for some $j$, $g$ must stabilize $W_{i,d_j}$, and that trace($E_{W_{i,d_j}} \pi(g)E_{W_{i,d_j}}$) $\neq 0$. Fix a decomposition (1) as in Theorem 22.2 for $W_{i,d_j}$:

$$W_{i,d_j} \cong \bigoplus_N d_j.$$

Since $E_{W_{i,d_j}} = \sum E_{d_j}$, trace($E_{W_{i,d_j}} \pi(g)E_{W_{i,d_j}}$) $\neq 0$ implies that $g$ must stabilize one of the $d_j$'s. By Theorem 22.3, $d_j \cong g^*d_j$, which proves the second part of the theorem. $\Box$

The following theorem and corollaries are used in the proof of Theorem 20 to show that for $f$ with compact support, the sum $\sum_{d \in K^\wedge} \Theta_d(f)$ is finite (see also Remark 14).

**Theorem 17.** Fix $x \in \mathcal{B}(T, k)$ and let $r > \text{max}\{s(\gamma), \rho(\pi)\}$. If $d \in (G_{x,r})^\wedge$ satisfies $\Theta_d(g) \neq 0$, then $d \subset (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$.

**Proof.** If $d$ is trivial we are done, so assume it is not. Let $t$ be the smallest number such that $d|_{G_{x,t}}$ is trivial (so in particular $d|_{G_{x,t}}$ is nontrivial).

**Case** $t < 2r$: Pick $s \leq 2r$ such that $G_{x,s} = G_{x,t+s}$. Consider $d$ as an element of $(G_{x,r}/G_{x,2r})^\wedge$. By Proposition 16, $\Theta_d(g) \neq 0$ implies that $d \cong g^*d$. Also, $\Theta_d(g) \neq 0$ implies that $d \subset \pi|_{G_{x,r}}$; since $r > \rho(\pi)$ this means that $d$ is degenerate (see [5, §7.6]). Now apply Corollary 13.

**Case** $t \geq 2r$: Note that $\frac{r}{2} \geq r > s(\gamma)$. For $\epsilon > 0$ such that $\frac{r}{2} > \frac{s(\gamma)}{2} + s(\gamma)$, let $s = t + \epsilon$. By making $\epsilon$ smaller if necessary, we can make sure that $G_{x,s} = G_{x,t+s}$. Note that $t > \frac{r}{2} + \frac{s(\gamma)}{2} + s(\gamma) = \frac{s(\gamma)}{2} + s(\gamma)$. 

Since $\frac{r}{2} > \frac{r}{2} \geq r$ it makes sense to restrict $d$ to $G_{x,\frac{r}{2}}$ and think of it as an element of $(G_{x,\frac{r}{2}}/G_{x,s})^\wedge$. As a representation of $G_{x,\frac{r}{2}}/G_{x,s}$, $d$ decomposes into a finite sum of irreducible (one-dimensional) representations

$$d_1 \oplus \cdots \oplus d_n.$$

Let $X_i + g_{x,(-\frac{s}{2})+}$ be the coset in $g_{x,(-s)+}/g_{x,(-\frac{s}{2})+}$ corresponding to $d_i$ under the isomorphism (4).

By Proposition 16, $0 \neq \Theta_d(\gamma)$ implies that for some $j$, $d_j \cong \gamma d_j$.

Now $d \subset \pi|_{G_{x,r}}$, implies that $d_j \subset \pi|_{G_{x,\frac{r}{2}}}$ and since $\frac{s}{2} > r > \rho(\pi)$ we have that $d_j$ is degenerate. Apply Corollary 13 to $d_j$ to conclude that $d_j \in (G_{x,\frac{r}{2}}/G_{x,s+s(\gamma)}\wedge)$. In particular $d_j$ is trivial on $G_{x,\frac{r}{2}+s(\gamma)}$, and hence on $G_{x,t}$.

Since $G_{x,r}$ normalizes $G_{x,\frac{r}{2}}$, it acts by permutations on the $d_i$'s. Since $d$ is irreducible, this action is transitive. Hence all the $d_i$'s are conjugate by elements of $G_{x,r}$. By the conjugation of the $d_i$'s and the fact that $d_j|_{G_{x,t}} = 1$ it follows that $d_i|_{G_{x,t}} = 1$ for all $i$, and so $d$ itself is trivial on $G_{x,t}$. This contradicts the definition of $t$. Hence this case is not possible and $t < 2r$. □

**Corollary 18.** Fix $x \in B(T, k)$ and let $r > \max\{s(\gamma), \rho(\pi)\}$. Let $X$ denote $\gamma T_{r+s(\gamma)}$, a compact subset of $T \cap G^{reg}$. If $d \in (G_{x,r})^\wedge$ satisfies $\Theta_d(\gamma') \neq 0$ for some $\gamma' \in X$, then $d \in (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$.

**Proof.** Lemma 5 implies that $\gamma'$ fixes $x$ and that $s(\gamma') = s(\gamma)$. Now apply Theorem 17 to $\gamma'$. □

**Corollary 19.** Fix $x \in B(T, k)$ and let $r > \max\{s(\gamma), \rho(\pi)\}$. Let $\omega$ denote $G_{x} \cdot (\gamma T_{r+s(\gamma)})$, an open compact subset of $G^{reg}$. Then $\Theta_d$ vanishes on $\omega$ for all $d \notin (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$. Furthermore, $\Theta_d(x) = \Theta_d(\gamma)$ for all $x \in \omega$ and all $d \in (G_{x,r}/G_{x,r+s(\gamma)})^\wedge$.

**Proof.** Follows immediately from Lemma 15 and Corollary 18. □

4. **Proof of the Main Theorem**

Let $r > \max\{s(\gamma), \rho(\pi)\}$. Denote the finite set $(G_{x,r}/G_{x,r+s(\gamma)})^\wedge$ by $F$.

**Theorem 20.** The distribution $\Theta_{\pi}$ is represented on the set $G(\gamma T_{r+s(\gamma)})$ by a constant function.

**Proof.** Using Corollary 19, we have for all $f \in C_c^\infty(G)$ whose support is contained in $\omega$,

$$\Theta_{\pi}(f) = \sum_{d \in (G_{x,r})^\wedge} \Theta_d(f) = \sum_{d \in F} \int_{\omega} \Theta_d(x)f(x)dx = \sum_{d \in F} \int_{\omega} \Theta_d(\gamma)f(x)dx = \int_{\omega} \left(\sum_{d \in F} \Theta_d(\gamma)\right)f(x)dx.$$
Thus $\Theta_\pi$ is represented by the constant function $\sum_{d \in F} \Theta_d(\gamma)$ on $\omega$, i.e. $\Theta_\pi(x) = \sum_{d \in F} \Theta_d(\gamma)$ for all $x \in \omega$. Since $\Theta_\pi$ is conjugation invariant, we get $\Theta_\pi(gxg^{-1}) = \Theta_\pi(x) = \sum_{d \in F} \Theta_d(\gamma)$ for all $x \in \omega$ and all $g \in G$. □

**Remark 21.** This gives a new proof of the local constancy (near compact regular semisimple tame elements $\gamma$) of the character of an irreducible admissible representation for an arbitrary reductive $p$-adic group $G$.

5. Appendix

We prove some variations of Clifford theory [6, §14]. Let $K$ and $N$ be open compact subgroups of $G$, such that $N$ is a normal subgroup of $K$. Let $(\pi, V)$ be an irreducible admissible representation of $G$ and let

$$(\dagger) \quad V = \bigoplus_{d \in K^\times} V_d$$

be the (canonical) decomposition of $V$ into $K$-isotypic components. Here $V_d$ denotes the $d$-isotypic component of $V$, i.e. the sum of all the $K$-submodules of $V$ isomorphic to $d = (d, W)$. Each isotypic component $V_d$ decomposes (non-canonically) into a finite sum of isomorphic copies $W_i \cong K W$ of $(d, W)$.

$$(\ddagger) \quad V_d \cong \bigoplus_{i} W_i .$$

**Theorem 22.** Suppose $g \in G$ normalizes $K$ and $N$. Then

1. The action of $g$ permutes the $V_d$’s.
2. Suppose $g$ stabilizes $V_d$. Then there exists a decomposition $(\ddagger)$ such that the action of $g$ permutes the $W_i$.
3. Suppose $W'$ is a $K$-submodule of $V$, isomorphic to $W$, and stable under the action $g$. Then $g^d \cong d$.

**Proof.**

1. This follows from the fact that for any two $K$-submodules $W'$ and $W''$ of $V$, if $W' \cong W''$ then $gW' \cong gW''$.
2. Let $W'$ be an irreducible $K$-submodule of $V_d$, isomorphic to $W$. Since $g$ normalizes $K$ and stabilizes $V_d$, $gW'$ is a $K$-submodule of $V_d$. Since $W'$ is irreducible, so is $gW'$. As an irreducible submodule of $V_d$, $gW'$ must be isomorphic to $W$. By irreducibility either $W' \cap gW' = \{0\}$ or $W' = gW'$. Thus the orbit of $W'$ under $g$ is a collection of subspaces with trivial pairwise intersection, and so $g$ acts on their sum as desired. By complete reducibility of $V_d$ (being a finite-dimensional representation of the compact group $K$) we can now use induction on the dimension of $V_d$.
3. This follows from the following commutative diagram (in which all the arrows are isomorphisms of vector spaces and $k \in K$).
\[
\begin{array}{c}
W \longrightarrow W' \xrightarrow{\pi(g)} gW' \longrightarrow W' \longrightarrow W \\
\downarrow d(k) \quad \downarrow \pi(k) \quad \downarrow \pi(g k g^{-1}) \quad \downarrow \pi(k g) \quad \downarrow d(k^g) \quad \downarrow g(d(k)) \\
W \longrightarrow W' \xrightarrow{\pi(g)} gW' \longrightarrow W' \longrightarrow W
\end{array}
\]

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