THE LOCAL CHARACTER EXPANSION NEAR A TAME, SEMISIMPLE ELEMENT

JEFFREY D. ADLER AND JONATHAN KORMAN

ABSTRACT. Consider the character of an irreducible admissible representation of a p-adic reductive group. The Harish-Chandra-Howe local expansion expresses this character near a semisimple element as a linear combination of Fourier transforms of nilpotent orbital integrals. Under mild hypotheses, we describe an explicit region on which the local character expansion is valid. We assume neither that the group is connected, nor that the underlying field has characteristic zero.

0. INTRODUCTION

Let G denote the group of k-points of a reductive k-group $\mathbf{G}$, where k is a nonarchimedean local field. To simplify the present discussion, assume for now that $\mathbf{G}$ is connected and that k has characteristic zero. Let (π, V) denote an irreducible admissible representation of G. Let dg denote a fixed Haar measure on G, and let $C_c^\infty(G)$ denote the space of complex-valued, locally constant, compactly supported functions on G. The *distribution character* Θ_π of π is the map $C_c^\infty(G) \to \mathbb{C}$ given by $\Theta_\pi(f) := \operatorname{tr} \pi(f)$, where $\pi(f)$ is the (finite-rank) operator on V given by $\pi(f)v := \int_G f(g)\pi(g)v\, dg$. From Howe [12] and Harish-Chandra [9], the distribution Θ_π is represented by a locally constant function on the set of regular semisimple elements in G. We will denote this function also by Θ_π.

For any semisimple $\gamma \in G$, the local character expansion about γ (see [11] and [10]) is the identity

$$\Theta_\pi(\gamma\, \mathsf{e}(Y)) = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(Y),$$

valid for all regular semisimple Y in the Lie algebra $\mathfrak{m}$ of the centralizer of γ such that Y is close enough to 0. Here, the sum is over the set of nilpotent orbits $\mathcal{O}$ in $\mathfrak{m}$; $\hat{\mu}_{\mathcal{O}}$ is the function that represents the distribution that is the Fourier transform of the orbital integral $\mu_{\mathcal{O}}$ associated to $\mathcal{O}$; $c_{\mathcal{O}} = c_{\mathcal{O},\gamma}(\pi) \in \mathbb{C}$; and e is the exponential map, or some suitable substitute.

This is a qualitative result, in the sense that it gives no indication of how close Y must be to 0 in order for the identity to be valid. Many questions in harmonic analysis on G require more quantitative versions of such qualitative results. As an example of a quantitative result, DeBacker [6] has determined (under some hypotheses on G) a neighborhood of validity for the local character expansion near the identity, thus verifying a conjecture of Hales, Moy, and Prasad (see [15]).

Date: November 13, 2005.

1991 *Mathematics Subject Classification.* 22E35, 22E50, 20G25.

Key words and phrases. p-adic group, harmonic analysis, local character expansion.

In this paper, we generalize DeBacker's result for any semisimple $\gamma \in G$ satisfying mild tameness hypotheses. (See §8 for the hypotheses, and Corollary 12.10 for a precise statement of the main result.) When γ is regular, we recover a generalization of Theorem 19 of [13].

We have taken care not to assume that k has characteristic zero (though some of our hypotheses do indeed restrict the characteristic of k). The general philosophy is that whatever is true in characteristic zero should be true in large enough positive characteristic. DeBacker's results on homogeneity [6] do not assume characteristic zero. Rather, he keeps careful track of which specific hypotheses are necessary in order to make each step of his proof valid. (Previously, the validity of the local character expansion was unknown in positive characteristic.) We have done likewise.

We have also taken care not to assume that $\mathbf{G}$ is connected. Nonconnected groups are of interest for several reasons, one of which is that their characters are equivalent to twisted characters of connected groups. In order to handle this case, we need to know that characters are represented by locally constant functions on the regular set. All proofs of this result that are known to us assume either that k has characteristic zero or that $\mathbf{G}$ is connected. But it is not difficult to show that the result is true generally. We include the details in §13.

Note that Theorem 2.1.5(3) of [6] plays a key role in our proof.

Acknowledgements. We thank Stephen DeBacker, Fiona Murnaghan, Ju-Lee Kim, and Loren Spice for helpful conversations. We also thank the referee for helpful comments.

1. Notation and conventions

Let k denote a nonarchimedean local field, and let ν denote a discrete valuation on k. For any algebraic extension field E of k, ν extends uniquely to a valuation (also denoted ν) of E. Fix a complex-valued, additive character Λ on k that is nontrivial on the ring R of integers in k and trivial on the prime ideal of R.

For a reductive k-group $\mathbf{G}$, let $\mathbf{G}^\circ$ denote its connected part, and let $\mathfrak{g}$ denote its Lie algebra. Let $\mathfrak{g}^*$ denote the dual of $\mathfrak{g}$. Let $G = \mathbf{G}(k)$, the group of k-rational points of $\mathbf{G}$; and let $\mathfrak{g} = \mathfrak{g}(k)$ and $\mathfrak{g}^* = \mathfrak{g}(k)^*$. Let $\mathbf{Z_G}$ denote the center of $\mathbf{G}$.

We use similar notation and font conventions for other groups. That is, given a group $\mathbf{M}$, we have $\mathfrak{m}$, M, etc.

Let Ad (resp. ad) denote the adjoint or coadjoint representation of $\mathbf{G}$ (resp. $\mathfrak{g}$) on $\mathfrak{g}$ or $\mathfrak{g}^*$. Let Int denote the conjugation action of G on itself. For an element or subset S in G and an element or subset L in $\mathfrak{g}$ or $\mathfrak{g}^*$ (resp. G), we will sometimes write ${}^S L$ instead of $\mathrm{Ad}(S)L$ (resp. $\mathrm{Int}(S)L$).

An element $g \in G$ is *semisimple* if $\mathrm{Ad}(g)$ is a semisimple linear transformation of $\mathfrak{g}$. When $g \in G^\circ$, this is equivalent to g belonging to a torus. For a subset S of G, let S^{ss} denote the set of semisimple elements in S (so $S^{\mathrm{ss}} = S \cap G^{\mathrm{ss}}$). An element $g \in G$ is *regular semisimple* if the coefficient of t^n in $\det(t - 1 + \mathrm{Ad}(g))$ is nonzero (where n is the rank of the component $g\mathbf{G}^\circ$ in $\mathbf{G}$; see [5]). We denote the set of regular semisimple elements in G by G^{reg}. Similarly we say that an element $X \in \mathfrak{g}$ is *regular semisimple* if the coefficient of t^n in $\det(t - \mathrm{ad}(X))$ is nonzero. We denote the set of regular semisimple elements in $\mathfrak{g}$ by $\mathfrak{g}^{\mathrm{reg}}$.

For a subset S of $\mathfrak{g}$ (resp. G) let $[S]$ denote the characteristic function of S on $\mathfrak{g}$ (resp. G).

Call an element $X \in \mathfrak{g}$ *nilpotent* if there is some one-parameter subgroup λ of $\mathbf{G}$ defined over k such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$. Let $\mathcal{N}_{\mathfrak{g}}$ denote the set of nilpotent elements in $\mathfrak{g}$, and $\mathcal{O}_{\mathfrak{g}}(0)$ the set of nilpotent orbits under the adjoint action of G on $\mathcal{N}_{\mathfrak{g}}$. We will leave out the subscript when it is understood. One can similarly define a set $\mathcal{N}^*$ of nilpotent elements in $\mathfrak{g}^*$.

For any compact group K, let $K^\wedge$ denote the set of equivalence classes of irreducible, continuous representations of K. We will not always distinguish between a representation and its equivalence class. Recall that if K is abelian, then $K^\wedge$ is a group.

Let $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{ r+ \mid r \in \mathbb{R}\}$ and extend the ordering on $\mathbb{R}$ to one on $\widetilde{\mathbb{R}}$ as follows: for all $r, s \in \mathbb{R}$,

$$\begin{aligned} r < s+ &\quad \text{if and only if} \quad r \le s; \\ r+ < s+ &\quad \text{if and only if} \quad r < s; \\ r+ < s &\quad \text{if and only if} \quad r < s. \end{aligned}$$

If $r \in \mathbb{R}$, define $(r+)+$ to be $r+$. There is a natural way to extend the additive structure on $\mathbb{R}$ to an additive structure on $\widetilde{\mathbb{R}} \cup \{\infty\}$.

For any k-group $\mathbf{G}$, let $\mathbf{X}^*(\mathbf{G})$ denote the lattice of characters of $\mathbf{G}$. For any k-torus $\mathbf{T}$ in $\mathbf{G}$, let $\Phi = \Phi(\mathbf{G}, \mathbf{T}) \subset \mathbf{X}^*(\mathbf{T})$ denote the absolute root system of $\mathbf{G}$ with respect to $\mathbf{T}$. We can also interpret Φ as the set of nontrivial eigencharacters for the adjoint action of $\mathbf{T}$ on $\mathfrak{g}$.

Suppose $\mathbf{T}$ is maximal in $\mathbf{G}$. If $\alpha \in \Phi$, then let $\mathbf{U}_\alpha$ denote the root group corresponding to α. This group need not be defined over k, but is defined over a splitting field E of $\mathbf{T}$ over k. Let $\mathfrak{g}(E)_\alpha \subset \mathfrak{g}(E)$ denote the root space corresponding to α.

Let $\Psi(\mathbf{G}, \mathbf{T})$ denote the set of affine roots of $\mathbf{G}$ with respect to $\mathbf{T}$ and ν. If $\psi \in \Psi(\mathbf{G}, \mathbf{T})$, let $\dot{\psi} \in \Phi(\mathbf{G}, \mathbf{T})$ denote the gradient of ψ. We denote the root lattice in $\mathfrak{g}(E)_{\dot{\psi}}$ corresponding to ψ by $\mathfrak{g}(E)_\psi$ [15, 3.2].

2. Apartments and buildings.

For any extension E/k of finite ramification degree, let $\mathcal{B}(\mathbf{G}, E)$ denote the extended Bruhat-Tits building of $\mathbf{G}$ over E. Note that if E/k is Galois, then $\mathcal{B}(\mathbf{G}, k)$ embeds naturally in the set of $\mathrm{Gal}(E/k)$-fixed points of $\mathcal{B}(\mathbf{G}, E)$, with equality when E/k is tame (see [17, (5.11)]).

Every maximal k-split torus $\mathbf{S}$ in $\mathbf{G}$ has an associated apartment $\mathcal{A}(\mathbf{S}, k)$ in $\mathcal{B}(\mathbf{G}, k)$. Let $\mathbf{T}$ be a maximal k-torus in $\mathbf{G}$ containing $\mathbf{S}$. Then $\mathbf{T}$ splits over some Galois extension E, so $\mathbf{T}$ has an apartment $\mathcal{A}(\mathbf{T}, E)$ in $\mathcal{B}(\mathbf{G}, E)$. The Galois fixed point set of the apartment of $\mathbf{T}$ in $\mathcal{B}(\mathbf{G}, E)$ is the apartment of $\mathbf{S}$ in $\mathcal{B}(\mathbf{G}, k)$ [23, §2.6].

Suppose $\mathbf{M} \subset \mathbf{G}$ is an E-Levi k-subgroup (that is, $\mathbf{M}(E)$ is a Levi subgroup of $\mathbf{G}(E)$) for some finite Galois extension E/k. There is a natural family of $\mathrm{Gal}(E/k)$-equivariant embeddings of $\mathcal{B}(\mathbf{M}, E)$ into $\mathcal{B}(\mathbf{G}, E)$. When E/k is tame, this in turn induces a natural family of embeddings of $\mathcal{B}(\mathbf{M}, k)$ into $\mathcal{B}(\mathbf{G}, k)$. (In general, there is no canonical way to pick a distinguished member of this family. However, all such embeddings share the same image, and no statement we make will depend on the choice of embedding.)

We shall require a generalization of the previous paragraph.

Lemma 2.1. *Let $\gamma \in G^{ss}$ and $\mathbf{M} = C_{\mathbf{G}}(\gamma)$. Then for every tame extension E/k of finite ramification degree, we have an $\mathbf{M}(E)$-equivariant, toral embedding $i_E \colon \mathcal{B}(\mathbf{M}, E) \longrightarrow \mathcal{B}(\mathbf{G}, E)$. Moreover, these maps can be chosen to be compatible in the following sense: if E and E' are two such fields and $E \supseteq E'$, then the restriction of i_E to $\mathcal{B}(\mathbf{M}, E')$ is $i_{E'}$.*

Proof. For each Galois extension E/k of finite ramification degree, Theorem 2.2.1 and Lemma 2.3.4 of [14] together provide us with a family I_E of toral, $\mathbf{M}^\circ(E)$-equivariant embeddings of $\mathcal{B}(\mathbf{M}^\circ, E) = \mathcal{B}(\mathbf{M}, E)$ into $\mathcal{B}(\mathbf{G}^\circ, E) = \mathcal{B}(\mathbf{G}, E)$. Moreover, $\mathrm{Gal}(E/k)$ acts on I_E. For an embedding $f \in I_E$ and an element $m \in \mathbf{M}(E)$, define ${}^m f \colon \mathcal{B}(\mathbf{M}, E) \to \mathcal{B}(\mathbf{G}, E)$ by ${}^m f(x) = m f(m^{-1}x)$. Since ${}^m f = f$ for all $m \in \mathbf{M}^\circ(E)$, we thus have an action of the compact group $\mathbf{M}(E)/\mathbf{M}^\circ(E) \rtimes \mathrm{Gal}(E/k)$. By the convexity of I_E, this action must have fixed points, which then form a family of toral embeddings that are equivariant under both $\mathrm{Gal}(E/k)$ and $\mathbf{M}(E)$.

We now prove the final statement of the lemma. Let E/k be the maximal tame subextension of a finite extension over which both $\mathbf{M}^\circ$ and $\mathbf{G}^\circ$ split. Choose an embedding i_E as in the previous paragraph. For any tame L/E, $\mathcal{B}(\mathbf{M}, E)$ generates $\mathcal{B}(\mathbf{M}, L)$ as an $\mathbf{M}(L)$-space, so our choice of i_E determines a choice of i_L. Thus, it will be enough to show compatibility between i_L and $i_{L'}$ for all tame, Galois L/k and all $L \supseteq L' \supseteq k$. The restriction of i_L to $\mathcal{B}(\mathbf{M}, L')$ has image in the set of $\mathrm{Gal}(L/L')$-fixed points of $\mathcal{B}(\mathbf{G}, L)$. But from [21] or [17], this set is $\mathcal{B}(\mathbf{G}, L')$. $\square$

3. Moy-Prasad filtrations.

Let $\mathbf{T}$ be a maximal k-torus in $\mathbf{G}$ with splitting field E over k. Let $\mathbf{T}(E)_0$ denote the parahoric subgroup of $\mathbf{T}(E)$. For $r \in \widetilde{\mathbb{R}}$, define

$$\mathfrak{t}(E)_r := \left\{ X \in \mathfrak{t}(E) \;\middle|\; \nu(d\chi(X)) \geq r \text{ for all } \chi \in \mathbf{X}^*(\mathbf{T}) \right\}$$

and for $r > 0$,

$$\mathbf{T}(E)_r := \left\{ t \in \mathbf{T}(E)_0 \;\middle|\; \nu(\chi(t) - 1) \geq r \text{ for all } \chi \in \mathbf{X}^*(\mathbf{T}) \right\}.$$

For each $(x, r) \in \mathcal{B}(\mathbf{G}, E) \times \widetilde{\mathbb{R}}$, Moy and Prasad define lattices $\mathfrak{g}(E)_{x,r}$ in $\mathfrak{g}(E)$ and $\mathfrak{g}(E)^*_{x,r}$ in $\mathfrak{g}(E)^*$. When $r \geq 0$, they define a normal subgroup $\mathbf{G}(E)_{x,r}$ of the parahoric subgroup $\mathbf{G}(E)_x$ of $\mathbf{G}(E)$. In particular, for all $x \in \mathcal{A}(\mathbf{T}, E)$ and $r \in \widetilde{\mathbb{R}}$,

$$\mathfrak{g}(E)_{x,r} = \mathfrak{t}(E)_r \oplus \sum_{\psi \in \Psi(\mathbf{G}, \mathbf{T}), \psi(x) \geq r} \mathfrak{g}(E)_\psi \tag{3.1}$$

Similarly, $\mathbf{G}(E)_{x,r}$ is defined in terms of the filtrations on $\mathbf{T}(E)$ and on root groups. These definitions depend on the normalization of the valuation ν; our normalization agrees with that of Yu [24]. Thus, for example, for any $\alpha \in k^\times$, $\alpha \cdot \mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+\nu(\alpha)}$.

However, the definitions do not depend on the choice of $\mathbf{T}$ containing x in its apartment. Note that for all $r \in \widetilde{\mathbb{R}}$, $\mathfrak{g}(E)_{x,r+} = \cup_{s>r} \mathfrak{g}(E)_{x,s}$ and $\mathbf{G}(E)_{x,|r|+} = \cup_{s>|r|} \mathbf{G}(E)_{x,s}$.

Moy and Prasad also define $\mathfrak{g}_{x,r}$ and $G_{x,r}$ (irrespective of whether or not $\mathbf{G}$ is k-split). The above normalization was chosen to have the following property [1, 1.4.1]: when E/k is tame and $x \in \mathcal{B}(\mathbf{G}, k)$, we have

$$\mathfrak{g}_{x,r} = \mathfrak{g}(E)_{x,r} \cap \mathfrak{g}, \qquad \text{and (for } r > 0), \quad G_{x,r} = \mathbf{G}(E)_{x,r} \cap G.$$

We will also use the following notation. For $r \in \widetilde{\mathbb{R}}$, let

$$\mathfrak{g}_r = \cup_{x \in \mathcal{B}(\mathbf{G},k)} \mathfrak{g}_{x,r} \quad \text{and (for } r \geq 0) \quad G_r = \cup_{x \in \mathcal{B}(\mathbf{G},k)} G_{x,r}.$$

It is proven in [2] that $\mathfrak{g}_r$ (resp. G_r) is a G-domain: a G-invariant, open and closed subset of $\mathfrak{g}$ (resp. G).

For any $x \in \mathcal{B}(\mathbf{G}, k)$ and any $0 < r \le t \le 2r$, the group $(G_{x,r}/G_{x,t})$ is abelian. Under many conditions (for example, if $\mathbf{G}$ contains a tamely ramified maximal torus, or if $\mathbf{G}$ is simply connected), there exists a (G_x-equivariant) isomorphism (see [15] or [24])

$$G_{x,r}/G_{x,t} \longleftrightarrow \mathfrak{g}_{x,r}/\mathfrak{g}_{x,t}\ , \tag{3.2}$$

and thus an isomorphism

$$(G_{x,r}/G_{x,t})^{\wedge} \longleftrightarrow \mathfrak{g}^*_{x,(-t)+}/\mathfrak{g}^*_{x,(-r)+}\ . \tag{3.3}$$

Yu [25] has defined a more complicated filtration on $\mathbf{T}$ than the one above. Using this filtration to define $G_{x,r}$, he shows that (3.3) is valid for all $\mathbf{G}$. However, for the groups that we will consider, Yu's filtration is equivalent to the one above.

4. Singular depth

From now on, $\mathbf{G}$ is a reductive k-group, $\gamma \in G^{\mathrm{ss}}$, and $\mathbf{M}$ is the centralizer of γ in $\mathbf{G}$.

Definition 4.1. For $m \in M$, let $s(m) = s^{\mathbf{G}}_{\mathbf{M}}(m) := \max_{\alpha \in A_m}\{0,\ \nu(\alpha - 1)\}$, where A_m is the set of generalized eigenvalues of the action of $\mathrm{Ad}(m)$ on $\mathfrak{g}/\mathfrak{m}$.

Remark 4.2. If $\mathbf{G}$ is connected and m is regular, then the definition of $s(m)$ given above agrees with the definition in [13, §1].

Remark 4.3. Note that $s(mz) = s(m)$ for all $z \in Z_G$ and that $s(hmh^{-1}) = s(m)$ for all $h \in M$.

Remark 4.4. Suppose $m \in M$, and E/k is an extension that contains all of the generalized eigenvalues of both $\mathrm{Ad}(\gamma)$ and $\mathrm{Ad}(m)$ acting on $\mathfrak{g}/\mathfrak{m}$. Since $\mathrm{Ad}(\gamma)$ and $\mathrm{Ad}(m)$ commute, we can write $\mathfrak{g}(E)/\mathfrak{m}(E)$ as a direct sum of subspaces V_i, where V_i is simultaneously an α_i-eigenspace for $\mathrm{Ad}(\gamma)$ and a generalized β_i-eigenspace for $\mathrm{Ad}(m)$.

Lemma 4.5. *If $m \in M_r^{ss}$, then $s(m) \ge r$.*

Proof. Pick a maximal k-torus $\mathbf{T}$ in $\mathbf{M}$ with $m \in T$, and a splitting field E for $\mathbf{T}$. Then

$$m \in M_r \cap T \subset \mathbf{M}(E)_r \cap \mathbf{T}(E) = \mathbf{T}(E)_r,$$

where the last equality follows from Theorem 4.1.5(1) of [7]. Thus, $\nu(\chi(m)-1) \ge r$ for all $\chi \in \mathbf{X}^*(\mathbf{T})$. In particular, this is true for all $\chi \in \Phi(\mathbf{G}, \mathbf{T}) \subset \mathbf{X}^*(\mathbf{T})$. □

Lemma 4.6. *If $m \in M_{s(\gamma)+}$, then $s(\gamma m) = s(\gamma)$. If γ is compact mod Z_G, then so is γm, and conversely. If γ is semisimple, then so is γm.*

Proof. Let V_i, α_i, and β_i be as in Remark 4.4. For all i, Lemma 4.5 implies that $\nu(\beta_i - 1) > s(\gamma) \ge \nu(\alpha_i - 1)$, and thus

$$\nu(\alpha_i\beta_i - 1) = \nu\big((\alpha_i - 1)(\beta_i - 1) - (\alpha_i - 1) + (\beta_i - 1)\big) = \nu(\alpha_i - 1).$$

Thus, $s(\gamma m) = s(\gamma)$. The second statement of the lemma follows from the fact that α_i is a unit if and only if so is $\alpha_i\beta_i$. If $\mathrm{Ad}(m)$ is diagonalizable over some field extension, then $\mathrm{Ad}(m)$ and $\mathrm{Ad}(\gamma)$ are simultaneously diagonalizable, so the last statement follows. □

For $m \in M$, following [10, §18], define

$$D_{G/M}(m) := \det\Big((\mathrm{Ad}(m) - 1)\big|_{\mathfrak{g}/\mathfrak{m}}\Big).$$

(When $M^\circ = G^\circ$, define $D_{G/M} \equiv 1$.)

For $r \geq 0$ let

$$M'_r := \big\{ m \in M_r \bigm| D_{G/M}(\gamma m) \neq 0 \big\},$$
$$M''_r := \big\{ m \in M_r \bigm| \gamma m \in G^{\mathrm{reg}} \big\}.$$

Note that $M''_r \subset M'_r$, and these are open, dense subsets of M_r.

Corollary 4.7. $M'_{s(\gamma)+} = M_{s(\gamma)+}$.

Proof. Let $m \in M_{s(\gamma)+}$. Let V_i, α_i, and β_i be as in Remark 4.4. As in the proof of Lemma 4.6, $\nu(\alpha_i\beta_i - 1) = \nu(\alpha_i - 1)$ for all i. Thus,

$$|D_{G/M}(\gamma m)| = \prod_i |\alpha_i\beta_i - 1|^{\dim V_i} = \prod_i |\alpha_i - 1|^{\dim V_i} = |D_{G/M}(\gamma)| \neq 0.$$

Therefore, $m \in M'_{s(\gamma)+}$. □

5. Intertwining

Definition 5.1. Let K be a compact open subgroup of G and let $d \in K^\wedge$. For $g \in G$, recall that ${}^g d$ is the representation of gKg^{-1} given as ${}^g d(gkg^{-1}) := d(k)$.

Definition 5.2. If d and d' are continuous representations of compact subgroups K and K' (respectively) of G, then let $[d : d'] = \dim_{\mathbb{C}} \mathrm{Hom}_{K \cap K'}(d, d')$.

Lemma 5.3. *Let K and L be compact subgroups of G, and let N be a compact subgroup of K. Let $d \in K^\wedge$ and let χ denote a one-dimensional representation of L. Let $d_1 \oplus \cdots \oplus d_n$ be a decomposition of d into representations of N. If $0 \neq [\chi : d]$ then for some d_i, $0 \neq [\chi : d_i]$.*

Proof. We have $0 < [\chi : d] \leq [\chi : d|_N] = \sum_i [\chi : d_i]$. Therefore, $[\chi : d_i] > 0$ for some i. □

6. Partial traces

From now on, let (π, V) denote an irreducible admissible representation of G. Let Θ_π denote the distribution character of π. This distribution is represented by a locally constant function (also denoted Θ_π) on G^{reg} (see §13). Let $\rho(\pi)$ denote the depth of π [15, §5].

For any irreducible representation d of a compact open subgroup K, let V_d denote the (K, d)-isotypic subspace of V. Let E_d denote the K-equivariant projection from V to V_d. Define the distribution Θ_d by $\Theta_d(F) := \mathrm{tr}(E_d\, \pi(F)\, E_d)$ for all $F \in C_c^\infty(G)$. Then Θ_d, which can be thought of as the 'partial trace of π with respect to d', is represented by the locally constant function $\Theta_d(x) := \mathrm{tr}(E_d\, \pi(x)\, E_d)$ on G. It follows from the definitions that

$$\Theta_\pi(F) = \sum_{d \in K^\wedge} \Theta_d(F) \quad \text{for all } F \in C_c^\infty(G). \tag{6.1}$$

Note that for each fixed F, $\pi(F)$ has finite rank, so all but finitely many terms in this sum vanish.

Lemma 6.2. *(1) If $x \in G$ and $k \in K$, then $\Theta_d(kxk^{-1}) = \Theta_d(x)$.*

(2) If $h \in \ker(d)$, then $\Theta_d(xh) = \Theta_d(x)$.

Proof. The first statement is [13, Lemma 14]. Since $\pi(h)E_d = E_d$, we have

$$\Theta_d(xh) = \mathrm{tr}(E_d\,\pi(x)\pi(h)\,E_d) = \mathrm{tr}(E_d\,\pi(x)\,E_d) = \Theta_d(x). \quad \square$$

Let N be a compact open subgroup of K, and let $d \in K^\wedge$. Considered as a representation of N, d decomposes into a finite sum of distinct irreducible representations d_i with multiplicities α_i:

$$\alpha_1 d_1 \oplus \cdots \oplus \alpha_n d_n\,.$$

For each i, let V_{d,d_i} denote the (N, d_i)-isotypic subspace of V_d. Let E_{d,d_i} denote the N-equivariant projection from V to V_{d,d_i}. For $x \in G$, define

$$\Theta_{d,d_i}(x) := \mathrm{tr}(E_{d,d_i}\pi(x)E_{d,d_i}).$$

Remark 6.3. Note that $\Theta_d(x) = \sum_i \Theta_{d,d_i}(x)$. Moreover, Θ_{d,d_i} has invariance properties analogous to those given for Θ_d in Lemma 6.2.

Proposition 6.4. *Let $g \in G$. If $\Theta_d(g) \neq 0$, then $0 \neq [d : {}^g d]$. If $\Theta_{d,d_i}(g) \neq 0$, then $0 \neq [d_i : {}^g d_i]$.*

Proof. Define a pairing $\langle\ ,\ \rangle$ on V_d with respect to which d is unitary. Let $\{v_j\}$ be a basis for V_d. For $k_1, k_2 \in K$,

$$\begin{aligned}
\Theta_d(k_1 x k_2) &= \mathrm{tr}(E_d\,\pi(k_1 x k_2)\,E_d)\\
&= \sum_j \langle E_d \pi(k_1 x k_2) v_j, v_j\rangle\\
&= \sum_j \langle \pi(k_1) E_d \pi(x)\pi(k_2) v_j, v_j\rangle.
\end{aligned}$$

Fixing k_2 and letting k_1 vary, we have a sum of matrix coefficients of d. Note that by Lemma 6.2(1) we have,

$$\begin{aligned}
\Theta_d(k_1 x k_2) &= \Theta_d(k_2 k_1 x k_2 k_2^{-1})\\
&= \Theta_d(k_2 k_1 x)\\
&= \sum_j \langle \pi(k_2) E_d \pi(k_1 x) v_j, v_j\rangle.
\end{aligned}$$

Fixing k_1 and letting k_2 vary, we again have a sum of matrix coefficients of d. Therefore, our first statement follows from Corollary 14.3 of [10].

To prove the second statement, note that since E_{d,d_i} is a projection onto a subspace of V_{d_i}, if $\Theta_{d,d_i}(g) \neq 0$, then $\Theta_{d_i}(g) \neq 0$. $\square$

Lemma 6.5. *Fix i. Let χ be a character of a closed subgroup N' of N. Let $g \in G$. Suppose*

$$0 \neq \int_{N'} \Theta_{d,d_i}(gn)\chi(n)\,dn\,.$$

Then $d_i|_{N'} = \bar{\chi}$.

Proof. For all $n \in N'$, $\pi(n)E_{d,d_i}$ acts on V_{d,d_i} via the scalar $d_i(n)$. Let $\{a_j\}$ be the multi-set of diagonal entries of a matrix that represents, with respect to some basis, the action of $E_{d,d_i}\pi(g)$ on V_{d,d_i}. Then for some j,

$$0 \neq a_j \int_{N'} d_i(n)\chi(n)\,dn.$$

The conclusion then follows from Lemma 14.2 of [10]. $\square$

7. Harmonic analysis

From distributions on G to distributions on M. For the distribution Θ_π (or any other distribution) on G, we follow Harish-Chandra [10, §18] (see also [20]) to define a distribution θ on M that captures the behavior of Θ_π near γ.

Fix $r \geq 0$. The proof of Proposition 1 of [20] is valid for general reductive groups, so we may apply it to see that the following (surjective) map is everywhere submersive:

$$\begin{array}{ccc} G \times M'_r & \xrightarrow{p} & {}^G(\gamma M'_r) \\ (g, m) & \longmapsto & g\gamma m g^{-1}. \end{array}$$

Theorem 7.1. *There exists a unique, surjective, linear map*

$$\begin{array}{ccc} C_c^\infty(G \times M'_r) & \longrightarrow & C_c^\infty({}^G(\gamma M'_r)) \\ \alpha & \longmapsto & f_\alpha\ , \end{array}$$

such that for all $F \in C_c^\infty(G)$

$$\int_G F(x) f_\alpha(x)\, dx = \int_{G\times M'_r} F(g\gamma m g^{-1})\alpha(g,m)\, dg\, dm,$$

and $\operatorname{supp}(f_\alpha) \subset p(\operatorname{supp}(\alpha))$.

Proof. This is Theorem 11, p. 49, of [8] applied to the map p above. (Note that in *loc. cit.*, it is not assumed that the characteristic of k is zero nor that $\mathbf{G}$ is connected.) $\square$

Remark 7.2. The set ${}^G(\gamma M'_r)$ is an *open* (since p is submersive), G-invariant neighborhood of γ in G, so $C_c^\infty({}^G(\gamma M'_r)) \subset C_c^\infty(G)$. Note that the set ${}^G(\gamma M_r)$ is not necessarily open.

Fix an open compact subgroup K of G; let 1_K denote its characteristic function. We have the following diagram:

$$\begin{array}{ccccccc} C_c^\infty(M_r) & \xrightarrow{\text{restr.}} & C_c^\infty(M'_r) & \longrightarrow & C_c^\infty(G \times M'_r) & \longrightarrow & C_c^\infty({}^G(\gamma M'_r)) \\ \\ f & \longmapsto & f & \longmapsto & \alpha & \longmapsto & f_\alpha \end{array}$$

where the first arrow is the restriction map; the second arrow is given by $\alpha(g,m) := 1_K(g) f(m)$; and the third arrow is the map of Theorem 7.1. Note that the support of f_α is contained in ${}^K(\gamma M'_r)$, an open, K-invariant neighborhood of γ in G.

Let $\Theta = \Theta_\pi$ denote the distribution character of (π, V).

Definition 7.3. Define the distribution θ on M_r by $\theta(f) := \Theta_\pi(f_\alpha)$.

Lemma 7.4. *Normalize the measure on G so that K has total measure 1. Then for each $f \in C_c^\infty(M_r)$*

$$\theta(f) = \sum_{d \in K^\wedge} \int_{M_r} \Theta_d(\gamma m) f(m)\, dm,$$

where the sum is over a finite *set (which depends on f) of representations.*

Note that a similar statement appears on p. 78 of [10].

Proof. Combine equation (6.1) with

$$\begin{aligned}
\Theta_d(f_\alpha) &:= \int_G \Theta_d(x) f_\alpha(x)\,dx \\
&= \int_{G\times M'_r} \Theta_d(g\gamma m g^{-1}) 1_K(g) f(m)\,dg\,dm \\
&= \int_{K\times M'_r} \Theta_d(k\gamma m k^{-1}) f(m)\,dk\,dm \\
&= \int_{M'_r} \Theta_d(\gamma m) f(m)\,dm \qquad \text{(by Lemma 6.2)} \\
&= \int_{M_r} \Theta_d(\gamma m) f(m)\,dm. \quad \square
\end{aligned}$$

For the following lemma, note that ${}^K(\gamma M''_r)$ is a K-invariant neighborhood of γ in G^{reg}, and that ${}^K(\gamma M''_r)$ is dense in ${}^K(\gamma M'_r) \supset \mathrm{supp}(f_\alpha)$. From §13, Θ_π is represented by a locally constant function on G^{reg}.

Lemma 7.5. *The distribution θ is represented on M''_r by the function $\theta(m) := \Theta_\pi(\gamma m)$.*

Proof. For all $f \in C_c^\infty(M_r)$, the support of the function f_α is a subset of G^{reg}, so the first integral below makes sense:

$$\begin{aligned}
\theta(f) = \Theta_\pi(f_\alpha) &:= \int_{G^{\mathrm{reg}}} \Theta_\pi(x) f_\alpha(x)\,dx \\
&= \int_{{}^K(\gamma M''_r)} \Theta_\pi(x) f_\alpha(x)\,dx \\
&= \int_{K\times M''_r} \Theta_\pi(k\gamma m k^{-1}) f(m)\,dk\,dm \\
&= \int_{M''_r} \Theta_\pi(\gamma m) f(m)\,dm. \quad \square
\end{aligned}$$

8. Hypotheses

From now on, we will make certain assumptions: Let $\mathbf{G}$, γ, and $\mathbf{M}$ be as in §4. That is, $\mathbf{G}$ is a reductive linear algebraic k-group, $\gamma \in G^{\mathrm{ss}}$, and $\mathbf{M}$ is the centralizer of γ in $\mathbf{G}$.

Hypothesis 8.1. The eigenvalues of $\mathrm{Ad}(\gamma)$ belong to a tamely ramified extension of k.

Hypothesis 8.2. The order of $\gamma \mathbf{Z_G G}^\circ \in \mathbf{G}/\mathbf{Z_G G}^\circ$ is prime to the residue characteristic of k.

We will use this hypothesis to prove Lemma 9.1. However, there are other conditions that would also suffice.

Hypothesis 8.3. Using some system of embeddings as in Lemma 2.1 to identify $\mathcal{B}(\mathbf{M}, E)$ with a subset of $\mathcal{B}(\mathbf{G}, E)$ for each tame E/k, we have that for each such E, and for $r \in \widetilde{\mathbb{R}}$ and $x \in \mathcal{B}(\mathbf{M}, k)$,

$$\begin{aligned}
&\mathfrak{m}(E)^*_r = \mathfrak{m}(E)^* \cap \mathfrak{g}(E)^*_r && \mathfrak{m}(E)^*_{x,r} = \mathfrak{m}(E)^* \cap \mathfrak{g}(E)^*_{x,r} \\
&\mathbf{M}(E)_r = \mathbf{M}(E) \cap \mathbf{G}(E)_r && \mathbf{M}(E)_{x,r} = \mathbf{M}(E) \cap \mathbf{G}(E)_{x,r} \qquad (r > 0).
\end{aligned}$$

We will not pursue here the question of when this hypothesis holds. For now, we note that it holds when one of the following holds for some tame E/k: $\mathbf{M}^\circ$ is an E-Levi subgroup of $\mathbf{G}^\circ$; $\mathbf{M}^\circ$ and $\mathbf{G}^\circ$ are both E-split, and have the same E-split rank; or $\mathbf{G} = R_{E/k}\mathbf{H} \rtimes \mathrm{Gal}(E/k)$ for some E-group $\mathbf{H}$ (where $R_{E/k}$ denotes restriction of scalars), and $\gamma \in \mathrm{Gal}(E/k)$. (See Lemmata 2.2.3 and 2.2.9 of [3].)

Hypothesis 8.4. There is a nondegenerate G-invariant symmetric bilinear form B on $\mathfrak{g}$ such that we can identify $\mathfrak{g}^*_{x,r}$ with $\mathfrak{g}_{x,r}$ via the map $\mathfrak{g} \to \mathfrak{g}^*$ defined by $X \longmapsto (Y \mapsto B(X,Y))$.

(Groups satisfying this hypothesis are discussed in [4, §4].) Thus, we can (and eventually will) identify $\mathfrak{g}_{x,r}$ with $\mathfrak{g}^*_{x,r}$ and $\mathfrak{m}_{x,r}$ with $\mathfrak{m}^*_{x,r}$

The following hypothesis concerns the existence of a "mock exponential" map.

Hypothesis 8.5. Let $e = \max\{\rho(\pi), s(\gamma)\}$. There exists a bijection $\mathsf{e} : \mathfrak{g}_{e+} \longrightarrow G_{e+}$ such that for all $x \in \mathcal{B}(\mathbf{G},k)$ and all $e < r \le s \le t \le 2s$, we have that e induces the group isomorphism $\mathfrak{g}_{x,s}/\mathfrak{g}_{x,t} \to G_{x,s}/G_{x,t}$ of (3.2). Moreover, for all $s > e$, $\mathsf{e}(\mathfrak{m}_s) = M_s$, and

(1) for $r, s > e$, all $X \in \mathfrak{m}_{x,r}$, and all $Y \in \mathfrak{m}_{x,s}$, we have $\mathsf{e}(X)\cdot\mathsf{e}(Y) \equiv \mathsf{e}(X+Y)$ modulo $M_{x,(r+s)}$;

(2) for all $m \in M$ we have $\mathrm{Int}(m) \circ (\mathsf{e}\,|_{\mathfrak{m}_{e+}}) = (\mathsf{e}\,|_{\mathfrak{m}_{e+}}) \circ \mathrm{Ad}(m)$;

(3) for all $x \in \mathcal{B}(\mathbf{M},k)$, all $s,t \in \widetilde{\mathbb{R}}$ with $s > e$, all $Y \in \mathfrak{g}_{x,s}$, and all $X \in \mathfrak{g}_{x,t}$, we have ${}^{\mathsf{e}(Y)}X - X \in \mathfrak{g}_{x,s+t}$.

Note that item (3) in the hypothesis asserts, for all $x \in \mathcal{B}(\mathbf{M},k)$, a weaker version of Proposition 1.6.3 of [1], and the remainder of the hypothesis is a weaker version of Hypothesis 3.2.1 in [6]. Item (1) implies that e carries a Haar measure on $\mathfrak{m}$ into a Haar measure on M.

Mock exponential maps are known to exist in several situations. For example, for $\mathbf{GL}_n(k)$, the map $X \mapsto (1+X)$ works. For a classical group that splits over a tame extension of k, with odd residue characteristic, the Cayley transform works. If k has characteristic zero and e is sufficiently large, then the exponential map works.

We will need the next two hypotheses in order to apply Theorem 11.9.

Hypothesis 8.6. Assume that $\mathbf{M}$ satisfies Hypothesis 3.4.3 of [6], concerning the convergence of nilpotent orbital integrals.

This is automatically satisfied when k has characteristic zero.

Hypothesis 8.7. Assume that k and $\mathbf{M}$ satisfy the hypotheses of Theorem 2.1.5(3) of [6].

9. Lie algebra decompositions

Lemma 9.1. *Let E/k be a tame extension containing the eigenvalues of* $\mathrm{Ad}(\gamma)$. *For all $x \in \mathcal{B}(\mathbf{M},k)$, and $r \in \widetilde{\mathbb{R}}$, we have*

$$\mathfrak{g}(E)_{x,r} = \bigoplus_\alpha (\mathfrak{g}(E)_{x,r} \cap \mathfrak{g}(E)_{\gamma,\alpha}),$$

where the sum is over the set of eigenvalues of the action of $\mathrm{Ad}(\gamma)$ *on* $\mathfrak{g}(E)$, *and each* $\mathfrak{g}(E)_{\gamma,\alpha}$ *is the associated eigenspace.*

Proof. Let n be the order of $\gamma \mathbf{Z_G G}^\circ \in \mathbf{G}/\mathbf{Z_G G}^\circ$. Then $\gamma^n \in (\mathbf{Z_G})(E)\mathbf{T}(E)$ for some maximal torus $\mathbf{T}$ such that $\mathbf{T}/\mathbf{Z}_{\mathbf{G}}^\circ$ is E-split. Since γ^n acts trivially on $\mathfrak{t}(E)$ and acts on each $\mathbf{T}$-root space in $\mathfrak{g}(E)$, we have from (3.1) that

$$\mathfrak{g}(E)_{x,r} = \bigoplus_\beta (\mathfrak{g}(E)_{x,r} \cap \mathfrak{g}(E)_{\gamma^n,\beta}),$$

where the sum runs over the eigenvalues β of $\mathrm{Ad}(\gamma^n)$, and each $\mathfrak{g}(E)_{\gamma^n,\beta}$ denotes the corresponding eigenspace. Thus it is enough to show that for each β,

$$\mathfrak{g}(E)_{x,r} \cap \mathfrak{g}(E)_{\gamma^n,\beta} = \bigoplus_{\alpha^n=\beta} (\mathfrak{g}(E)_{x,r} \cap \mathfrak{g}(E)_{\gamma,\alpha}).$$

This will follow from Lemma 9.3 and Hypothesis 8.2. □

We may identify $\mathfrak{m}$ with the 1-eigenspace of $\mathrm{Ad}(\gamma)$ in $\mathfrak{g}$. Define $\mathfrak{m}^\perp$ to be the sum of all of the other eigenspaces of $\mathrm{Ad}(\gamma)$. Identify $\mathfrak{m}^*$ with $\{X \in \mathfrak{g}^* \mid X(\mathfrak{m}^\perp) = 0\}$ and define $\mathfrak{m}^{*\perp} := \{X \in \mathfrak{g}^* \mid X(\mathfrak{m}) = 0\}$. These objects are all defined over k, and

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{m}^{*\perp}$$

The following result is well known when $\mathbf{G}$ is connected. See [1, Proposition 1.9.2].

Corollary 9.2. *Suppose $x \in \mathcal{B}(\mathbf{M}, k)$ and $r \in \widetilde{\mathbb{R}}$. Then*

$$\mathfrak{g}_{x,r} = (\mathfrak{m} \cap \mathfrak{g}_{x,r}) \oplus (\mathfrak{m}^\perp \cap \mathfrak{g}_{x,r}). \quad \square$$

Lemma 9.3. *Let R denote the ring of integers in k, and let $\varpi \in R$ be a uniformizer. Let V be a finite-dimensional vector space over k, and L a lattice in V. Let $T\colon V \to V$ denote a diagonalizable linear map such that $T(L) = \varpi^r L$ for some $r \in \mathbb{Z}$. Let $\alpha_1, \dots, \alpha_\ell$ be the eigenvalues of T, and $V_1, \dots, V_\ell$ the corresponding eigenspaces. Suppose that for $i \neq j$, we have that $\alpha_i \not\equiv \alpha_j \bmod \varpi^{r+1}R$. Then*

$$L = \bigoplus_{i=1}^{\ell} (L \cap V_i).$$

Proof. Without loss of generality, we may replace T by $\varpi^{-r}T$, and thus assume that $T(L) = L$, so that $\alpha_i \in R^\times$ for all i. Let $x \in L$, and write $x = \sum x_i$, where $x_i \in V_i$. We must show that $x_i \in L$ for all i. Let $S = \{i \mid x_i \notin L\}$, and suppose that S is nonempty. Pick a minimal nonempty subset $I \subset S$ such that $\sum_{i\in I} \beta_i x_i \in L$ for some coefficients $\beta_i \in R^\times$. Pick such coefficients, and let y be the resulting sum. We must have $|I| > 1$. Let $i_0 \in I$. Then $\alpha_{i_0}^{-1} T(y) - y \in L$. But we may rewrite this element as

$$\sum_{i \in I \smallsetminus \{i_0\}} \Big(\frac{\alpha_i}{\alpha_{i_0}} - 1\Big) x_i,$$

contradicting the minimality of I. □

10. Some lemmata

Some results in this section are stated in terms of the Lie algebras $\mathfrak{m}$ and $\mathfrak{g}$. However, by Hypothesis 8.4, the analogous results for $\mathfrak{m}^*$ and $\mathfrak{g}^*$ are also valid (with the same proofs).

Lemma 10.1. *Let $t \in \mathbb{R}$, $x \in \mathcal{B}(\mathbf{M},k)$, and $\gamma' \in \gamma M_{x,s(\gamma)+}$. If $Z \in \mathfrak{m}^\perp \cap (\mathfrak{g}_{x,-t} \smallsetminus \mathfrak{g}_{x,(-t)+})$ then ${}^{\gamma'}Z - Z \notin \mathfrak{g}_{x,(-t+s(\gamma'))+}$.*

Proof. Write $\gamma' = \gamma m$, with $m \in M_{x,s(\gamma)+}$. Let $Z' = {}^m Z - Z$.

From Hypothesis 8.1, there is a tame extension E of k containing the eigenvalues of $\mathrm{Ad}(\gamma)$. Write $Z = \sum_\alpha Z_\alpha$, and $Z' = \sum_\alpha Z'_\alpha$, where each sum is over the set of eigenvalues for the action of $\mathrm{Ad}(\gamma)$ on $\mathfrak{g}(E)$, and each Z_α and Z'_α belongs to the corresponding eigenspace. Then ${}^{\gamma'}Z - Z = \sum_\alpha Y_\alpha$, where $Y_\alpha = \alpha Z'_\alpha + (\alpha - 1)Z_\alpha$. From our hypothesis on Z, there is some α so that $Z_\alpha \notin \mathfrak{g}(E)_{x,(-t)+}$. From Lemma 4.6, $s(\gamma') = s(\gamma)$. Therefore, from Lemma 9.1 it will be enough to show that $Y_\alpha \notin \mathfrak{g}(E)_{x,(-t+s(\gamma))+}$. Note that from Hypothesis 8.5(3), $Z' \in \mathfrak{g}_{x,(-t+s(\gamma))+}$. Therefore, Lemma 9.1 implies that $Z'_\alpha \in \mathfrak{g}(E)_{x,(-t+s(\gamma))+}$.

Suppose $\nu(\alpha) \geq 0$. Then $\alpha Z'_\alpha \in \mathfrak{g}(E)_{x,(-t+s(\gamma))+}$. By the definition of $s(\gamma)$, $(\alpha - 1)Z_\alpha \notin \mathfrak{g}(E)_{x,(-t+s(\gamma))+}$, and our conclusion follows.

Now suppose that $\nu(\alpha) < 0$. Let $q = \nu(\alpha) = \nu(\alpha - 1)$. Then $\alpha Z'_\alpha \in \mathfrak{g}(E)_{x,(q-t+s(\gamma))+}$ and $(\alpha - 1)Z_\alpha \notin \mathfrak{g}(E)_{x,(q-t)+} \supset \mathfrak{g}(E)_{x,(q-t+s(\gamma))+}$, so $Y_\alpha \notin \mathfrak{g}(E)_{x,(q-t+s(\gamma))+} \supset \mathfrak{g}(E)_{x,(-t+s(\gamma))+}$. □

Proposition 10.2. *Let $r \in \mathbb{R}$, $x \in \mathcal{B}(\mathbf{M},k)$, $X \in \mathfrak{g}$, and $\gamma' \in \gamma M_{x,s(\gamma)+}$. Write $X = Y + Z$ according to the decomposition in Corollary 9.2. If ${}^{\gamma'}X - X \in \mathfrak{g}_{x,(-r)+}$, then $Z \in \mathfrak{g}_{x,(-r-s(\gamma))+}$.*

Proof. For some $t \in \mathbb{R}$, $Z \in \mathfrak{m}^\perp \cap (\mathfrak{g}_{x,t} \smallsetminus \mathfrak{g}_{x,t+})$. By Lemma 10.1, ${}^{\gamma'}Z - Z \notin \mathfrak{g}_{x,(t+s(\gamma'))+}$. On the other hand, since ${}^{\gamma'}X - X \in \mathfrak{g}_{x,(-r)+}$ decomposes as $({}^{\gamma'}Y - Y) + ({}^{\gamma'}Z - Z) \in \mathfrak{m}_{x,(-r)+} \oplus (\mathfrak{m}^\perp \cap \mathfrak{g}_{x,(-r)+})$, we have ${}^{\gamma'}Z - Z \in \mathfrak{g}_{x,(-r)+}$. Thus $t + s(\gamma') > -r$, which implies that $Z \in \mathfrak{g}_{x,t} \subseteq \mathfrak{g}_{x,(-r-s(\gamma'))+}$. Finally, recall from Lemma 4.6 that $s(\gamma') = s(\gamma)$. □

Proposition 10.3. *If $x \in \mathcal{B}(\mathbf{M},k)$, $r \in \mathbb{R}$, and $Y \in \mathfrak{m}_{x,r} \smallsetminus \mathfrak{m}_{x,r+}$, then*

$$(Y + \mathfrak{g}_{x,r+}) \cap \mathcal{N} \neq \emptyset \quad \text{if and only if} \quad (Y + \mathfrak{m}_{x,r+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset.$$

Proof. This follows from Corollary 3.2.6 of [2] and Hypothesis 8.3. □

Definition 10.4. Let $0 < r \leq t \leq 2r$. A character $d \in (G_{x,r}/G_{x,t})^\wedge$ is called *degenerate* if the coset that corresponds to d under the isomorphism (3.3) contains nilpotent elements. One can similarly define what it means for a character of $(M_{x,r}/M_{x,t})$ to be degenerate.

Lemma 10.5. *Let $x \in \mathcal{B}(\mathbf{M},k)$, $q,t \in \widetilde{\mathbb{R}}$, and suppose $0 < q \leq t \leq 2q$. Let $d \in (G_{x,q}/G_{x,t})^\wedge$ be a character such that $[d : {}^{\gamma'}d] \neq 0$ for some $\gamma' \in \gamma M_{x,s(\gamma)+}$. Suppose that for some $v > q + s(\gamma)$, d is trivial on $M_{x,v}$. Then d is trivial on $G_{x,v}$.*

Proof. By (3.3), d corresponds to some coset $\Upsilon \in \mathfrak{g}^*_{x,(-t+)}/\mathfrak{g}^*_{x,(-q)+}$. By Lemma 1.8.1 of [1], $\Upsilon \cap {}^{\gamma'}\Upsilon \neq \emptyset$. Pick $X \in \Upsilon$ such that ${}^{\gamma'}X \in \Upsilon \cap {}^{\gamma'}\Upsilon$. Then ${}^{\gamma'}X - X \in \mathfrak{g}^*_{x,(-q)+}$. Write $X = Y + Z$ with respect to the decomposition in Corollary 9.2. By Proposition 10.2, $Z \in \mathfrak{g}^*_{x,(-q-s(\gamma))+} \subseteq \mathfrak{g}^*_{x,(-v)+}$. Hence, $X + \mathfrak{g}^*_{x,(-v)+} = Y + \mathfrak{g}^*_{x,(-v)+}$. Since d is trivial on $M_{x,v}$, we have that $Y \in \mathfrak{m}^*_{x,(-v)+}$. Therefore $X \in \mathfrak{g}^*_{x,(-v)+}$, implying that d is trivial on $G_{x,v}$. □

Remark 10.6. In fact, the proof shows that d is trivial on a slightly larger subgroup that corresponds to the lattice $\mathfrak{m}_{x,s^+} + (\mathfrak{m}^\perp \cap \mathfrak{g}_{x,s})$ (for some s such that $\mathfrak{m}_{x,s^+} = \mathfrak{m}_{x,v}$) via (3.2).

11. A result of DeBacker

We recall the following from, for example, §4.1 of [2]. Let dX be a Haar measure on $\mathfrak{m}$. For any $C_c^\infty(\mathfrak{m})$, we define the *Fourier transform* $\hat{f} \in C_c^\infty(\mathfrak{m}^*)$ of f by

$$\hat{f}(\chi) = \int_{\mathfrak{m}} f(X) \cdot \Lambda(\chi(X))\, dX$$

for $\chi \in \mathfrak{m}^*$. Let $d\chi$ be a Haar measure on $\mathfrak{m}^*$. For $f \in C_c^\infty(\mathfrak{m}^*)$ we use the natural identification of $\mathfrak{m}^{**}$ with $\mathfrak{m}$ and define the *Fourier transform* $\hat{f} \in C_c^\infty(\mathfrak{m})$ by

$$\hat{f}(X) = \int_{\mathfrak{m}^*} f(\chi) \cdot \Lambda(\chi(X))\, d\chi$$

for $X \in \mathfrak{m}$. We normalize the measures dX and $d\chi$ so that for $X \in \mathfrak{g}$ and $f \in C_c^\infty(\mathfrak{m})$

$$\hat{\hat{f}}(X) = f(-X). \tag{11.1}$$

Recall that, from Hypothesis 8.4, we can (and eventually will) identify $\mathfrak{m}$ with its linear dual $\mathfrak{m}^*$. With this identification, the Fourier transform becomes a map from $C_c^\infty(\mathfrak{m})$ to itself. Given our normalization of measures, we have that for $x \in \mathcal{B}(\mathbf{M}, k)$ and $r \in \widetilde{\mathbb{R}}$, $[\mathfrak{m}_{x,r}]^\wedge = [\mathfrak{m}^*_{x,(-r)+}]$.

Using r and e as in Hypothesis 8.5, and the distribution θ from Definition 7.3, we now define a distribution $\widehat{\theta}$ on $\mathfrak{m}^*$ (compare [6, §2.1]).

Definition 11.2. For $f \in C_c^\infty(\mathfrak{m}^*)$, define $\widehat{\theta}(f) := \theta(\widehat{f}\,|_{\mathfrak{m}_r} \circ \mathsf{e}^{-1})$.

Remark 11.3. For $x \in \mathcal{B}(\mathbf{M}, k)$, $f \in C(\mathfrak{m}^*_{x,-s}/\mathfrak{m}^*_{x,(-r)+})$ if and only if $\widehat{f} \in C(\mathfrak{m}_{x,r}/\mathfrak{m}_{x,s+}) \subset C_c^\infty(\mathfrak{m}_r)$. Hence in this case $\widehat{\theta}(f) = \theta(\widehat{f} \circ \mathsf{e}^{-1})$.

The following were defined in [6, §2.1].

Definition 11.4. For any M-invariant subset S of $\mathfrak{m}$ or $\mathfrak{m}^*$, let $J(S)$ denote the space of M-invariant distributions supported on S.

Definition 11.5. Suppose $-s \le -r$. Define the spaces of distributions

$$\widetilde{J}(\mathfrak{m})_{x,-s,(-r)+} = \left\{ T \in J(\mathfrak{m}) \;\middle|\; \begin{array}{l} \text{for } f \in C(\mathfrak{m}_{x,-s}/\mathfrak{m}_{x,(-r)+}), \text{ if } T(f) \neq 0 \\ \text{then } \operatorname{supp}(f) \cap (\mathcal{N}_{\mathfrak{m}} + \mathfrak{m}_{x,(-s)+}) \neq \emptyset \end{array} \right\},$$

and

$$\widetilde{J}(\mathfrak{m})_{(-r)+} = \bigcap_{x \in \mathcal{B}(\mathbf{M},k)} \bigcap_{-s \le -r} \widetilde{J}(\mathfrak{m})_{x,-s,(-r)+}.$$

Definition 11.6. Define the space of functions

$$\mathcal{D}(\mathfrak{m})_{(-r)+} := \sum_{x \in \mathcal{B}(\mathbf{M},k)} C_c(\mathfrak{m}/\mathfrak{m}_{x,(-r)+}).$$

Remark 11.7. One can define $\widetilde{J}(\mathfrak{m}^*)_{x,-s,(-r)^+}$, $\widetilde{J}(\mathfrak{m}^*)_{(-r)+}$, and $\mathcal{D}(\mathfrak{m}^*)_{(-r)+}$ similarly.

Remark 11.8. The function $f \in C_c^\infty(\mathfrak{m}^*)$ lies in $C_c^\infty(\mathfrak{m}^*_r)$ if and only if $\widehat{f} \in \mathcal{D}(\mathfrak{m})_{(-r)+}$ (see Definition 4.2.1 and Lemma 4.2.3 of [2]).

If T is a distribution on $\mathfrak{m}$, then we let $\operatorname{res}_{\mathcal{D}(\mathfrak{m})_{(-r)+}} T$ denote the restriction of T to $\mathcal{D}(\mathfrak{m})_{(-r)+}$. The following homogeneity result was proved by DeBacker [6, Theorem 2.1.5]. From a remark in the introduction to *loc. cit.*, the result does not require that $\mathbf{M}$ be connected.

Theorem 11.9. $\operatorname{res}_{\mathcal{D}(\mathfrak{m})_{(-r)+}} \widetilde{J}(\mathfrak{m})_{(-r)+} = \operatorname{res}_{\mathcal{D}(\mathfrak{m})_{(-r)+}} J(\mathcal{N}_{\mathfrak{m}})$.

12. The Main Theorem

Recall that we are assuming the hypotheses in §8. Let $\widehat{\theta}$ be as in definition 11.2.

Theorem 12.1. *Let $r > \max\{\rho(\pi), 2s(\gamma)\}$. Then $\widehat{\theta} \in \widetilde{J}(\mathfrak{m}^*)_{(-r)+}$.*

Proof. It is enough to show that $\widehat{\theta} \in \widetilde{J}(\mathfrak{m}^*)_{x,-s,(-r)+}$ for all $x \in \mathcal{B}(\mathbf{M}, k)$ and all $s, r \in \mathbb{R}$ such that $s \geq r$. Fix $x \in \mathcal{B}(\mathbf{M}, k)$ and $s \geq r$, and take $f \in C(\mathfrak{m}^*_{x,-s}/\mathfrak{m}^*_{x,(-r)+})$. Suppose $0 \neq \widehat{\theta}(f)$. We need to show that $\operatorname{supp}(f) \cap (\mathcal{N}^*_{\mathfrak{m}} + \mathfrak{m}^*_{x,(-s)+}) \neq \emptyset$. By the linearity of $\widehat{\theta}$, it suffices to show this for $f = [Y + \mathfrak{m}^*_{x,(-r)+}]$, where $Y \in \mathfrak{m}^*_{x,-s}$. In other words, it suffices to show that the character $\chi := \mathsf{e}^{-1} \circ \Lambda_Y$ of $M_{x,s}/M_{x,s+}$ is degenerate, where $\Lambda_Y := \Lambda \circ Y$. We have

$$\widehat{f}(W) = \Lambda_Y(W)[\mathfrak{m}_{x,r}](W) = \Lambda_Y(W) \sum_{\overline{Z} \in \mathfrak{m}_{x,r}/\mathfrak{m}_{x,s}} [\overline{Z}](W).$$

Thus $0 \neq \widehat{\theta}(f) = \theta(\widehat{f} \circ \mathsf{e}^{-1})$ implies that for some $Z \in \mathfrak{m}_{x,r}$, $0 \neq \theta(h)$, where $h \in C_c^\infty(M_r)$ is defined by $h(\mathsf{e}(W)) = \Lambda_Y(W)[Z + \mathfrak{m}_{x,s}](W)$. Using Lemma 7.4 with $r' := r - s(\gamma)$, we get

$$0 \neq \theta(h) = \sum_{d \in G^\wedge_{x,r'}} \int_{M_r} \Theta_d(\gamma m) h(m)\, dm.$$

Thus for some $d \in G^\wedge_{x,r'}$,

$$0 \neq \int_{M_r} \Theta_d(\gamma m) h(m)\, dm = \int_{M_{x,r}} \Theta_d(\gamma m) h(m)\, dm, \tag{12.2}$$

where the last equality holds because h is supported on $M_{x,r}$.

Pick minimal $t \in \mathbb{R}$ such that d is trivial on $G_{x,t+}$. Let $q = \max\{r', (\frac{t}{2})+\}$. Since $q \geq r'$, one can restrict d to $G_{x,q}$. Since the group $G_{x,q}/G_{x,t+}$ is abelian, this restriction decomposes into a finite sum of irreducible, one-dimensional representations $d_1 \oplus \cdots \oplus d_n$. From equation (12.2) and Remark 6.3, we see that for some i, and for all $v > s$,

$$\begin{aligned} 0 &\neq \int_{M_{x,r}} \Theta_{d,d_i}(\gamma m) h(m)\, dm \\ &= \sum_{\dot{n} \in M_{x,r}/M_{x,v}} \int_{M_{x,v}} \Theta_{d,d_i}(\gamma \dot{n} m) h(\dot{n} m)\, dm \\ &= \sum_{\dot{n} \in M_{x,r}/M_{x,v}} h(\dot{n}) \int_{M_{x,v}} \Theta_{d,d_i}(\gamma \dot{n} m)\, dm, \end{aligned}$$

where we think of each sum as running over a set of coset representatives for $M_{x,r}/M_{x,v}$, and in the last line above we use the fact that h is constant on $M_{x,s+}$. Therefore, for some $\dot{n}$,

$$0 \neq \int_{M_{x,v}} \Theta_{d,d_i}(\gamma \dot{n} m)\, dm. \tag{12.3}$$

If we assume that $v \geq q$, then we may apply Lemma 6.5 to see that d_i is trivial on $M_{x,v}$. Moreover, (12.3) implies that $\Theta_{d,d_i}(\gamma') \neq 0$ for some $\gamma' \in \gamma M_{x,s(\gamma)+}$, which by Proposition 6.4 means that $0 \neq [d_i : {}^{\gamma'}d_i]$. If we further assume that $v > q + s(\gamma)$, then we may apply Lemma 10.5 to see that $d_i|_{G_{x,v}} \equiv 1$. Since d is an irreducible representation of $G_{x,r'}$, it follows that $G_{x,r'}$ permutes the d_i's transitively. Therefore, d is trivial on $G_{x,v}$, so $v > t$. Since this is true for all v satisfying $v > s$ and $v > q + s(\gamma)$, we see that

$$0 < t \leq \max\{s, q + s(\gamma)\}. \tag{12.4}$$

We now use this inequality to prove four others:

$$s > (\tfrac{t}{2})+ \tag{12.5}$$

$$r + s > t \tag{12.6}$$

$$s > q \tag{12.7}$$

$$s \geq t. \tag{12.8}$$

The first two of these follow trivially from the last. However, we prove them independently because we want to isolate the one part of this paper, the proof of (12.8), that relies on the hypothesis that $r > 2s(\gamma)$.

Note that (12.5) is obvious in the case where $t \leq s$. So assume that $t \leq q + s(\gamma)$. Recall that q is either r' or $(\frac{t}{2})+$. If $q = r'$, then $s \geq r = q + s(\gamma) \geq t > t/2+$. If $q = (\frac{t}{2})+$, then $t \leq (\frac{t}{2}+) + s(\gamma)$. Since $t \in \mathbb{R}$, $t \leq t/2 + s(\gamma)$, so $t/2 \leq s(\gamma) < r \leq s$. Since $s \in \mathbb{R}$, we have (12.5).

To prove (12.6), note that (12.5) implies that $s(\gamma) + s > (\frac{t}{2}+) + s(\gamma)$, so $r + s > (\frac{t}{2}+) + s(\gamma)$. Since $r + s > r$, we have $r + s > \max\{r, \frac{t}{2} + s(\gamma)+\} = q + s(\gamma)$. Since $r + s > s$, we have $r + s > \max\{s, q + s(\gamma)\} \geq t$.

To prove (12.7), note that $s \geq r > r'$. From (12.5), we conclude that $s > \max\{r', \frac{t}{2}+\} = q$.

To prove (12.8), we use (12.4) to reduce to the case where $q + s(\gamma) \geq t$. If $q = r'$, then $s \geq r = r' + s(\gamma) = q + s(\gamma) \geq t$. If $q = \frac{t}{2}+$, then since $q + s(\gamma) \geq t$, we have $s(\gamma)+ \geq \frac{t}{2}$. Since $s(\gamma) \in \mathbb{R}$, $s(\gamma) \geq \frac{t}{2}+$. Therefore (using the hypothesis that $r > 2s(\gamma)$), we have

$$s \geq r \geq 2s(\gamma)+ = (s(\gamma)+) + s(\gamma) \geq (\tfrac{t}{2}+) + s(\gamma) = q + s(\gamma) \geq t.$$

We have

$$\begin{aligned}
0 &\neq \int_{M_{x,r}} \Theta_{d,d_i}(\gamma m) h(m)\, dm \\
&= \int_{\mathfrak{m}_{x,r}} \Theta_{d,d_i}(\gamma\, \mathsf{e}(W)) \Lambda_Y(W) [Z + \mathfrak{m}_{x,s}](W)\, dW \\
&= \Lambda_Y(Z) \int_{\mathfrak{m}_{x,s}} \Theta_{d,d_i}(\gamma\, \mathsf{e}(Z + W)) \Lambda_Y(W)\, dW.
\end{aligned}$$

Now let $z := \mathsf{e}(Z)$. From (12.6), d_i is trivial on $M_{x,r+s}$. Apply Hypothesis 8.5(1) together with Remark 6.3 to obtain:

$$0 \neq \int_{\mathfrak{m}_{x,s}} \Theta_{d,d_i}(\gamma z\, \mathsf{e}(W)) \Lambda_Y(W)\, dW = \int_{M_{x,s}} \Theta_{d,d_i}(\gamma z m) \chi(m)\, dm.$$

From (12.7) and Lemma 6.5, $0 \neq [\bar{\chi} : d_i]$. From (12.8), d_i is trivial on $G_{x,s+}$. By Remark 10.6, the restriction $\tilde{d}$ of d_i to $G_{x,s}$ is represented by a coset $Y + \mathfrak{g}^*_{x,(-s)+}$ where $Y \in \mathfrak{m}^*_{x,-s}$. Since $s > \rho(\pi)$, Theorem 3.5 of [16] implies that $\tilde{d}$ is degenerate. Thus, $(Y + \mathfrak{g}^*_{x,(-s)+}) \cap \mathcal{N}^* \neq \emptyset$. Use Proposition 10.3 to conclude that

$(Y+\mathfrak{m}^*_{x,(-s)+})\cap\mathcal{N}^*_{\mathfrak{m}}\neq\emptyset$, and hence that $\bar{\chi}=d_i|_{M_{x,s}}=\tilde{d}|_{M_{x,s}}$ is degenerate. Thus, χ, is degenerate. □

From now on, use Hypothesis 8.4 to identify $\mathfrak{m}^*_{x,r}$ with $\mathfrak{m}_{x,r}$ for all $x\in\mathcal{B}(\mathbf{M},k)$ and all $r\in\widetilde{\mathbb{R}}$. For $\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)$, let $\mu_{\mathcal{O}}$ denote the corresponding nilpotent orbital integral and let $\widehat{\mu}_{\mathcal{O}}$ denote its Fourier transform (both are distributions). Note that in general the set $\mathcal{O}_{\mathfrak{m}}(0)$ can have infinite cardinality. However, Hypothesis 8.7 puts restrictions on k and $\mathbf{G}$, which guarantee that the cardinality is finite.

Corollary 12.9. *Let $r>\max\{\rho(\pi),2s(\gamma)\}$. Then*

$$\theta(f\circ\mathsf{e}^{-1})=\sum_{\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)}c_{\mathcal{O}}\widehat{\mu}_{\mathcal{O}}(f)\quad\text{for all } f\in C_c^\infty(\mathfrak{m}_r^{reg}),$$

where $c_{\mathcal{O}}=c_{\mathcal{O},\gamma}(\pi)$ are complex constants that depend on $\mathcal{O}$, γ and π.

Proof. Let $f\in C_c^\infty(M_r)$. Then by Remark 11.8, $\widehat{f}\in\mathcal{D}(\mathfrak{m})_{(-r)+}$. Let $\check{\widehat{f}}(X)=\widehat{f}(-X)$. For all $\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)$, let $-\mathcal{O}$ denote $\{X\in\mathfrak{m}\mid -X\in\mathcal{O}\}$, and note that $-\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)$. Then for some coefficients $\{c_{\mathcal{O}}\mid\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)\}$, we have

$$\begin{aligned}\theta(f\circ\mathsf{e}^{-1})&=\widehat{\theta}(\check{\widehat{f}}) && \text{by (11.1)}\\ &=\sum_{\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)}c_{-\mathcal{O}}\mu_{-\mathcal{O}}(\check{\widehat{f}}) && \text{by Theorems 11.9 and 12.1}\\ &=\sum_{\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)}c_{\mathcal{O}}\mu_{\mathcal{O}}(\widehat{f})\\ &=\sum_{\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)}c_{\mathcal{O}}\widehat{\mu}_{\mathcal{O}}(f).\end{aligned}$$ □

From Hypothesis 8.6 and work of Huntsinger (see Theorem A.1.2 of [3]), it is known that $\widehat{\mu}_{\mathcal{O}}$ is represented by a locally constant function (which we will also denote $\widehat{\mu}_{\mathcal{O}}$) on $\mathfrak{m}^{\mathrm{reg}}$. (When k has characteristic zero, this is a result of Harish-Chandra [10, Theorem 4.4].) From §13, Θ_π is also represented by a locally constant function.

Corollary 12.10. *Let $r>\max\{\rho(\pi),2s(\gamma)\}$. Then*

$$\Theta_\pi(\gamma\,\mathsf{e}(Y))=\sum_{\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)}c_{\mathcal{O}}\widehat{\mu}_{\mathcal{O}}(Y)$$

for all $Y\in\mathfrak{m}''_r:=\mathsf{e}^{-1}(M''_r)$.

Proof. For $m\in M''_r$, we have $\gamma m\in G^{\mathrm{reg}}$. Let $\mathbf{T}=C_{\mathbf{G}}(\gamma m)^\circ$. From Lemmata 1 and 2 of [5], $0\neq\det((\mathrm{Ad}(\gamma m)-1)_{\mathfrak{g}/\mathfrak{t}})$. Since $\mathrm{Ad}(\gamma m)=\mathrm{Ad}(\gamma)\,\mathrm{Ad}(m)=\mathrm{Ad}(m)$ on $\mathfrak{m}/\mathfrak{t}$, $0\neq\det((\mathrm{Ad}(m)-1)|_{\mathfrak{m}/\mathfrak{t}})$, and so $m\in M^{\mathrm{reg}}$. Thus, $M''_r\subset M^{\mathrm{reg}}$, and thus $\mathfrak{m}''_r\subset\mathfrak{m}^{\mathrm{reg}}$, and so the right-hand side of the equation makes sense. Since $\gamma M''_r\subset G^{\mathrm{reg}}$, the left-hand side makes sense from Proposition 13.1. The result now follows from Corollary 12.9 and Lemma 7.5. □

Corollary 12.11. *Let $r>\max\{\rho(\pi),2s(\gamma)\}$. Then*

$$\Theta_\pi(g\gamma\,\mathsf{e}(Y)g^{-1})=\sum_{\mathcal{O}\in\mathcal{O}_{\mathfrak{m}}(0)}c_{\mathcal{O}}\widehat{\mu}_{\mathcal{O}}(Y)\quad\text{for all}\quad Y\in\mathfrak{m}''_r,\ g\in G.$$

Proof. This follows from Corollary 12.10 and the G-invariance of Θ_π. □

Remark 12.12. When γ is regular, we have that $\mathbf{M}^\circ$ is a torus, and so the only nilpotent orbit in $\mathfrak{m}$ is the 0 orbit. Thus in this case there is only one orbital integral in the character expansion and its Fourier transform is a constant function. This means that the domain of validity of the local character expansion near a regular semisimple element is a domain on which Θ_π is constant. Thus we recover a generalization of the main result of [13].

Remark 12.13. It would be desirable, for applications of motivic integration to character theory, to have a version of Theorem 12.1 (and thus of its corollaries) that is valid under the weaker hypothesis that $r > \max\{\rho(\pi), s(\gamma)\}$. In order to obtain such a theorem, one would have to replace the one part of the proof of Theorem 12.1 that assumes $r > 2s(\gamma)$: the proof of inequality (12.8). This inequality allows us to apply Remark 10.6, a slight strengthening of Lemma 10.5, to the character d_i. However, if we had a version of Lemma 10.5 strong enough to apply directly to the representation d (which is not necessarily one dimensional), then (12.8) would be unnecessary. We will pursue this matter elsewhere.

13. Appendix: Characters are locally constant

In this section only, k denotes a nonarchimedean local field, $\mathbf{G}$ denotes an arbitrary reductive k-group, and Θ_π denotes the distribution character of an admissible irreducible representation π of G. We place no other hypotheses on k or G. In particular, we do not assume any of the hypotheses in §8.

Proposition 13.1. *The character Θ_π is represented by a function that is locally constant on G^{reg}.*

Recall that we needed this in order to prove Cor. 12.10.

Of course, when k has characteristic zero and $\mathbf{G}$ is connected, the above is a result of Howe [12] and Harish-Chandra [10]. (Moreover, Harish-Chandra [10] showed that $|D_G|^{1/2}\Theta_\pi$ is locally integrable, a result that Clozel [5] extended to nonconnected groups. But we do not need this.) When k has arbitrary characteristic and $\mathbf{G}$ is connected, Harish-Chandra [9] gave an alternative proof of local constancy (which is independent of local integrability), except that he proves Theorem 1 of *loc. cit.* (the "submersion principle") only in characteristic zero. According to Harish-Chandra, Borel knew a characteristic-free proof of the submersion principle. However, the first such published proof is due to Gopal Prasad, and appears as Appendix B in [3].

Thus, the only novelty in our result above is that we simultaneously relax the requirements that k have characteristic zero and that $\mathbf{G}$ be connected.

The proof, on the other hand, involves no novelty at all, since the characteristic-free proofs mentioned above can be generalized without difficulty to the case of nonconnected groups. Here we outline how to do so.

We look first at Prasad's proof of the submersion principle. Describing the changes necessary to generalize the proof would take about as much space as the proof itself. So for the reader's convenience, we present the generalized proof, even though it is almost a word-for-word copy of Prasad's.

Proposition 13.2 (Harish-Chandra)**.** *Suppose k is an arbitrary field, $\mathbf{G}$ is a reductive k-group, and P is a parabolic subgroup of G°. For $x \in G$, let x^* denote the* *image of x under the projection of G onto $G^* = G/P$. Fix $\gamma \in G^{reg}$. The mapping*

$$x \to ({}^x\gamma)^*$$

from G to G^* is everywhere submersive.

Proof (G. Prasad). Let $\phi_\gamma \colon G \to G^*$ denote the map $x \to ({}^x\gamma)^*$. Since $\phi_\gamma(xy) = \phi_{{}^y\gamma}(x)$, it is enough to show that ϕ_γ is submersive at the identity.

Define

$$V := \{{}^{\gamma^{-1}}X - X \mid X \in \mathfrak{g}\}.$$

We must show that $\mathfrak{g} = V + \mathfrak{p}$ where $\mathfrak{p}$ is the Lie algebra of P.

Without loss of generality, we assume that k is algebraically closed.

The identity component of $C_G(\gamma)$ is a torus T of G. Let B be a Borel subgroup containing T and let U denote its unipotent radical. Let T' be a maximal torus lying in $B \cap P$. Let $\mathfrak{b}$ (resp., $\mathfrak{t}$, resp., $\mathfrak{t}'$, resp. $\mathfrak{u}$) denote the Lie algebra of B (resp. T, resp., T', resp. U). Note that $\mathfrak{g}$ decomposes as a direct sum of $\mathrm{Ad}(\gamma)$-eigenspaces, and V contains all of the non-trivial eigenspaces, so $\mathfrak{t} + V = \mathfrak{g}$. Thus, it is enough to show that $\mathfrak{t} \subset V + \mathfrak{p}$. Since $\mathfrak{t}'$ is contained in $\mathfrak{p}$ and $\mathfrak{u} \subset V$, we conclude that $\mathfrak{u} + \mathfrak{t}' \subset V + \mathfrak{p}$. However, $\mathfrak{b} = \mathfrak{u} + \mathfrak{t}'$, so $\mathfrak{t} \subset \mathfrak{b} \subset V + \mathfrak{p}$, and so we are finished. □

We now turn to the rest of Harish-Chandra's proof of local constancy. Now that the submersion principle has been established in the generality we require, the rest of the proof works without change, except for the places where Harish-Chandra uses the Cartan decomposition in the proof of his Theorem 2 (§4 of [9]). We now describe how to generalize this decomposition in a way that meets the requirements of Harish-Chandra's proof.

Let K denote a special maximal compact subgroup of G°, i.e., the stabilizer of a special vertex $x \in \mathcal{B}(\mathbf{G}^\circ, k)$. Pick a maximal k-split torus $\mathbf{A}$ such that x lies in the apartment of $A = \mathbf{A}(k)$. Let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{A})$. Then $\mathbf{M}$ is a minimal Levi k-subgroup of $\mathbf{G}^\circ$, and there is some minimal parabolic k-subgroup $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{MN}$.

Recall that G° has the Cartan decomposition for $G^\circ = KM^+K$, where we can define M^+ to be the set of all $m \in M$ such that $m^{-1}(P \cap G_{x,r})m \subseteq P \cap G_{x,r}$ for all $r > 0$. Harish-Chandra uses the Cartan decomposition, together with the following fact: there is a base $\{K_r\}_{r \in \mathbb{N}}$ of compact, open, normal subgroups of K such that for each r, there is a compact open subgroup $P_r \subset P$ such that $m^{-1}P_r m \subset K_r$ for all $m \in M^+$. So we only need to prove the following:

Lemma 13.3. *There is a subset $\widetilde{M}^+ \subset G$ such that $G = K\widetilde{M}^+K$, and such that for all $m \in \widetilde{M}^+$, we have that*

$$(*) \qquad m^{-1}(P \cap G_{x,r})m \subseteq G_{x,r} \quad \textit{for all } r > 0.$$

Proof. Let $\widetilde{M}$ denote the set of elements of G that normalize the pair (P, A). Then $\widetilde{M} \cap G^\circ = M$. Since G° acts transitively on the set of all pairs consisting of a maximal k-split torus and a minimal parabolic subgroup containing it, we have that every coset in G/G° contains an element of $\widetilde{M}$. That is, $\widetilde{M}/M$ is naturally isomorphic to G/G°. Thus, $G = K\widetilde{M}K$.

Define $\widetilde{M}^+$ to be the set of all elements of $\widetilde{M}$ that satisfy condition $(*)$ of the lemma.

Since M/A is compact, M has a unique Moy-Prasad filtration: for $r > 0$, $M_r = M_{x,r} = M \cap G_{x,r}$. Since $\widetilde{M}$ normalizes M, it also normalizes each M_r. Each group

$G_{x,r}$ has an Iwahori factorization with respect to $P = MN$. Thus, an element m of $\widetilde{M}$ lies in $\widetilde{M}^+$ if and only if $m^{-1}(N \cap G_{x,r})m \subseteq N \cap G_{x,r}$ for all $r > 0$.

Recall that $A \cap M^+$ contains an element m_0 with the property that $m_0^{-1}(N \cap G_{x,r})m_0 \subset N \cap G_{x,r+1}$ for all $r > 0$. Thus, for any $m \in \widetilde{M}$, there is some positive i such that $mm_0^i \in \widetilde{M}^+$. This implies that $\widetilde{M}^+$ meets every coset of $\widetilde{M}/M$, and thus that $G = K\widetilde{M}^+K$. □

References

[1] J. D. Adler, *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math **185** (1998), 1–32.

[2] J. D. Adler and S. DeBacker, *Some applications of Bruhat-Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group*, Michigan Math. J., **50** (2002), no. 2, 263–286.

[3] ______, *Murnaghan-Kirillov theory for supercuspidal representations of tame general linear groups*, J. Reine Angew. Math. **575** (2004), 1–35.

[4] J. D. Adler and A. Roche, *An intertwining result for p-adic groups*, Canad. J. Math. **52** (2000), no. 3, 449–467.

[5] L. Clozel, *Characters of non-connected, reductive p-adic groups*, Canad. J. Math. **39** (1987), no. 1, 149–167.

[6] S. DeBacker, *Homogeneity results for invariant distributions of a reductive p-adic group*, Ann. Sci. École Norm. Sup. (4) **35** (2002), 391–422.

[7] ______, *Some applications of Bruhat-Tits theory to harmonic analysis on a reductive p-adic group*, Michigan Math. J. **50** (2002), no. 2, 241–261.

[8] Harish-Chandra (notes by G. van Dijk), *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Mathematics, vol. **162**, Springer, 1970.

[9] ______, *A submersion principle and its applications*, Geometry and Analysis: Papers Dedicated to the Memory of V. K. Patodi, Indian Academy of Sciences, Bangalore, 1980, pp. 95–102.

[10] ______, *Admissible invariant distributions on reductive p-adic groups*, Notes by Stephen DeBacker and Paul J. Sally, Jr., University Lecture Series, **16**, Amer. Math. Soc., 1999.

[11] R. Howe, *The Fourier transform and germs of characters (case of Gl_n over a p-adic field)*, Math. Ann. **208** (1974), 305–322.

[12] ______, *Some qualitative results on the representation theory of GL_n over a p-adic field*, Pacific J. Math. **73** (1977), 479–538.

[13] J. Korman, *On the local constancy of characters*, J. Lie Theory **15** (2005), no. 2, 561–573.

[14] E. Landvogt, *Some functorial properties of the Bruhat-Tits building*, J. Reine Angew. Math. **518** (2000), 213–241.

[15] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), 393–408.

[16] ______, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helvetici **71** (1996), 98–121.

[17] G. Prasad, *Galois fixed points in the Bruhat-Tits buildings of a reductive group*, Bull. Soc. Math. France **129** (2001), 169–174.

[18] G. Prasad and J.-K. Yu, *On finite group actions on reductive groups and buildings*, Invent. Math. **147** (2002), 545–560.

[19] A. Roche, *Types and Hecke algebras for principal series representations of split reductive p-adic groups*, Ann. Sci. École Norm. Sup. (4) **31** (1998), 361–413.

[20] F. Rodier, *Intégrabilité locale des caractères du groupe $GL(n, k)$ où k est un corps local de caractéristique positive*, Duke Math. J. **52** (1985), no. 3, 771–792.

[21] G. Rousseau, *Immeubles des groupes réductifs sur les corps locaux*, Thèse, Université de Paris-sud, 1977.

[22] T. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (Berlin) (A. Borel and R. Carter et al, eds.), Lecture Notes in Mathematics, vol. **131**, Springer, 1970, pp. 167–266.

[23] J. Tits, *Reductive groups over p-adic fields*, Automorphic forms, representations, and L-functions (A. Borel and W. Casselman, eds.), Proc. Symp. Pure Math., vol. **33**, part 1, AMS (1979), pp. 29–69.

[24] J.-K. Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622.

[25] ______, *Smooth models associated to concave functions in Bruhat-Tits theory*, preprint, September, 2002.

E-mail address: adler@uakron.edu

The University of Akron, Akron, OH 44325-4002

E-mail address: jkorman@math.toronto.edu

The University of Toronto, Toronto, Ontario M5S 3G3