Representations of Semisimple Lie Groups

Anthony W. Knapp
Peter E. Trapa

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Introduction

These lectures lead by a relatively straight path from the end of a one-semester course in Lie groups through the Langlands classification of irreducible admissible representations of linear connected reductive groups. The lectures are for the most part distillations from the first author's books [**K1**] and [**K2**], which have extensive bibliographies. A continuation of this introduction, a section called "Motivation", places the content of these lectures in a broader representation-theoretic context by relating the subject matter to the so-called unitarity problem.

The prerequisites are the elementary theory of Lie groups as in Chevalley [**C**], including elementary facts about Lie algebras over $\mathbb{R}$ and $\mathbb{C}$. In addition, it is assumed that the reader knows standard material from algebra, analysis, and point-set topology as is commonly taught in first-year graduate courses in the United States. Any advance knowledge of complex semisimple Lie algebras, universal enveloping algebras, and representation theory of finite or compact groups would be quite helpful for orientation, but no such knowledge is really assumed.

Too much mathematics is involved along the path of these lectures to allow time for many proofs. Instead these lectures work a great deal with examples. It is a wonderful feature of representation theory that examples are easily at hand and much of the general behavior can be anticipated from fairly simple examples.

Lecture 1 gives some basic definitions but is otherwise exclusively about examples. For the most part, the group under discussion is the group $G = SL(2, \mathbb{R})$ of real 2-by-2 matrices of determinant 1 under multiplication. A number of concepts from later lectures are introduced at this stage in the context of this G. Lecture 2 defines semisimple groups in general, gives a host of examples, and examines the structure theory of these groups and their Lie algebras.

[1]Department of Mathematics, State University of New York, Stony Brook, NY 11794–3651
E-mail address: `aknapp@ccmail.sunysb.edu`
[2]School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540
E-mail address: `ptrapa@math.ias.edu`

Lecture 3 develops the abstract representation theory of compact groups and the theory of induced representations. Both these notions are tools for what is to come. The theory for compact groups is also a prototype for the more general theory that will follow, and a certain class of induced representations provides easy examples of infinite-dimensional representations of semisimple groups that are almost irreducible. Lecture 4 specializes to the representation theory of compact connected Lie groups, where the Theorem of the Highest Weight parametrizes the irreducible representations of such a group. Universal enveloping algebras figure into the proof of this theorem.

Lecture 5 begins to develop tools for handling infinite-dimensional representations. The center of the universal enveloping algebra of a semisimple Lie algebra turns out to be large. The Harish-Chandra isomorphism identifies the center and makes it available for defining a nontrivial invariant, known as the infinitesimal character, of an irreducible infinite-dimensional representation. When using the Lie algebra to work with an infinite-dimensional representation, one works only with nice vectors in the representation space, and these are the next objects of study.

Lecture 6 discusses global characters, which are tools for characterizing more complicated irreducible representations. The first part of Lecture 7 discusses the most important of these representations, the discrete series. The remainder of Lecture 7 and the first part of Lecture 8 discuss some preliminary notions for the Langlands classification, and the end of Lecture 8 actually states the Langlands classification and gives its meaning for $SL(2,\mathbb{R})$.

At the end of each lecture is a brief section "Notes" that tells where one may do further reading about the material of the lecture. The Notes refer to expository sources, not repeating historical information that may be found in those sources. For most lectures, a few gentle exercises appear in a section at the end of the lecture. Readers who seek more exercises may consult [**K1**] and [**K2**]. References are collected at the very end of this article.

We use the following notation beyond what one might expect. The dual of a vector space V is V', the transpose of a matrix or linear transformation L is L^t, and the conjugate transpose of a matrix L is L^*. For a topological group G, G_0 denotes the identity component; this is a closed normal subgroup. For a Lie group G with Lie algebra $\mathfrak{g}$, Ad and ad denote the natural adjoint actions of G and $\mathfrak{g}$, respectively, on $\mathfrak{g}$. If V is a complex vector space, $S(V)$ denotes the symmetric algebra of V. Generally we use the expressions $Z_A(B)$ and $N_A(B)$ to denote the centralizer and normalizer of B in A; these notions need to be interpreted according to the nature of the objects A and B.

Motivation

Let us place the content of the lectures in a broader representation-theoretic context. This discussion will help show the importance of the Langlands classification and indicate the nature of the path we shall take to get there. Necessarily we shall have to make some definitions quickly here that will later be made with more deliberation and with more use of examples. The context for this section is an overview of one of the fundamental problems in the representation theory of Lie groups: the problem of determining the "irreducible unitary representations" of a

Lie group G. We shall be largely interested only in such representations in the case where G is "semisimple" or, more generally, "reductive". This section consists only of motivation, and the reader who wants to do so may postpone looking at it until any later stage of the lectures.

A *unitary representation* of G is a continuous norm-preserving group action of G by linear transformations on a Hilbert space. It is *irreducible* if it is nonzero and the only G invariant closed subspaces are 0 and the whole space. The interest is in classifying the irreducible unitary representations of G up to the obvious kind of equivalence. The set of these equivalence classes is called the *unitary dual*, and we use the term *unitarity problem* to refer to the effort to find the unitary dual. A substantial fraction of Lie theory since 1950 was developed to tackle the unitarity problem, and the point of this section is to provide motivation for this lecture series from the point of view of this problem.

We begin with some motivation for the study of unitary representations. The classical theory of Fourier series decomposes an arbitrary function in $L^2(S^1)$, S^1 being the circle group, into a discrete sum of imaginary exponentials $e^{in\theta}$. We may regard the Hilbert space $L^2(S^1)$ with the translation group action as a unitary representation of S^1. When the exponentials $e^{in\theta}$ are viewed as homomorphisms of S^1 into the multiplicative group of nonsingular 1-by-1 complex matrices, the exponentials are precisely the irreducible unitary representations of S^1. Thus one aspect of the classical theory of Fourier series is that the unitary representation of S^1 on $L^2(S^1)$ gets decomposed into a discrete "sum", with limits allowed, of irreducible unitary representations.

From this point of view, the theory of the Fourier transform on $\mathbb{R}$ is more interesting and indicative. The noncompactness of the real line forces the decomposition of an element of $L^2(\mathbb{R})$ via the Fourier transform to be no longer discrete: the sum is replaced by an integral. Only the purely imaginary exponentials appear in the definition of the Fourier transform, and again these are all the irreducible unitary representations of $\mathbb{R}$. Once one knows about invariant measures on Lie groups, it is natural to ask for the analogous decomposition of $L^2(G)$ for any unimodular Lie group G, i.e., one having a nonzero two-sided invariant Borel measure.

It is natural also to expect that the representation of G on $L^2(G)$ by left translation, say, will decompose discretely if G is compact and continuously if G is noncompact. In 1947 Bargmann made the remarkable discovery that the analysis of $L^2(G)$ for G equal to the group $SL(2, \mathbb{R})$ of 2-by-2 real matrices of determinant one involves both a discrete part and a continuous part. The representations appearing in the discrete part are called "discrete series" and play a decisive role in the theory. Of course, since $L^2(G)$ is unitary, the discrete series are necessarily unitary.

The name Harish-Chandra figures prominently in the decomposition of $L^2(G)$ in the case that G is semisimple or reductive (terms that are defined in Lecture 2). In 1966 Harish-Chandra succeeded in parametrizing the discrete series for such groups. The description involves a great deal of structure, and we shall not get to it until Lecture 7.

Ten years later Harish-Chandra published the full decomposition of $L^2(G)$ for this class of groups. The irreducible unitary representations that appear in $L^2(G)$ do not nearly exhaust the unitary dual; the trivial representation is absent, for instance, if G is noncompact. Roughly, but not exactly, the members of the unitary dual that appear in $L^2(G)$ can be obtained by a process called "induction" that starts from discrete series of certain reductive subgroups of G. Induction is discussed in an example in Lecture 1 and in general in Lecture 3.

As we said, these lectures deal with representations largely in the case that G is semisimple or reductive. Before proceeding with the motivation, let us comment on how the unitarity problem for these special groups fits into the theory for general Lie groups. The Levi decomposition for Lie algebras says that a real Lie algebra is the semidirect product of a solvable Lie algebra and a semisimple Lie algebra, and it follows that a connected Lie group is, up to a covering group, the semidirect product of a connected solvable Lie group and a semisimple group. The unitary dual of solvable Lie groups is by now fairly well understood, and, at least for "Type I" groups, the effect on the unitarity problem of the semidirect product construction is understood, too. Thus the unitarity problem for semisimple groups is the main thing standing in the way of solving the unitarity problem for all Lie groups of Type I. As was noted in the previous paragraaph, a portion of the unitary dual of a semisimple group G is obtained by induction starting from discrete series of certain reductive subgroups of G. In this construction, it is not enough to use semisimple subgroups, and thus we are led to study reductive groups instead. Fortunately the analogous subgroups of a reductive G are still reductive, so that we do not need to enlarge our class of groups a second time. This matter is discussed more in Lectures 2 and 3.

For the remainder of this section, we shall assume that G is reductive in the sense of Lecture 2. Such a group has a maximal compact subgroup, which we denote by K, and G is topologically the product of K and a Euclidean space. Early on, Harish-Chandra realized that the Hilbert space structure of an irreducible unitary representation is so rigid that it is essentially superfluous. This matter is treated in detail at the end of Lecture 5, but we indicate the basic picture here. In the first place, the restriction of any unitary representation from G to K is a discrete "sum" (allowing limits) of irreducible unitary representations of K (Lecture 3), and, in the case of an irreducible representation of G, each equivalence class of irreducible unitary representations of K is represented only finitely many times in this sum. This finite-multiplicity property plays such a prominent role that we make it into a definition: we say that a representation of G with this property is *admissible*. Irreducible unitary representations are admissible.

Let us now assume only that the given unitary representation of G on V is admissible. Let V_K be the algebraic direct sum within V of the spaces of the irreducible representations of K. This is a countable-dimensional dense subspace of V, and we disregard its topology. The idea is that for many purposes we may replace V by V_K and work with the representation algebraically. The action of G on V can be differentiated on V_K to give V_K the structure of a module for the universal enveloping algebra $U(\mathfrak{g})$ of the complexification $\mathfrak{g}$ of the Lie algebra of G. In the passage from G to $U(\mathfrak{g})$, any information that helps distinguish covering groups of G from G itself is lost. On the other hand, K acts on V_K by construction. Since K captures the fundamental group of G, the action of K keeps track of the information that helps distinguish covering groups of G from G itself. The space V_K, equipped with the actions of $U(\mathfrak{g})$ and K, is called the *underlying* $(\mathfrak{g}, K)$ *module* of V. The restriction of the G invariant Hermitian inner product on the Hilbert space V is invariant in the natural senses under the actions of K and the Lie algebra of G, and the underlying $(\mathfrak{g}, K)$ module is said to be *infinitesimally unitary.*

One is led to define a $(\mathfrak{g}, K)$ module to be a vector space together with a left $U(\mathfrak{g})$ module structure and a locally finite linear action of K satisfying some expected compatibility conditions. When the underlying $(\mathfrak{g}, K)$ modules of two admissible unitary representations satisfy the expected notion of isomorphism, the original representations are said to be *infinitesimally equivalent*. It is not immediately obvious that $(\mathfrak{g}, K)$ modules form the appropriate setting for studying unitary representations, but they do: Every "irreducible" infinitesimally unitary $(\mathfrak{g}, K)$ module is the underlying $(\mathfrak{g}, K)$ module of an essentially unique irreducible unitary representation, two irreducible unitary representations are equivalent if and only if their underlying $(\mathfrak{g}, K)$ modules are infinitesimally equivalent, and all questions of reducibility of an admissible unitary representation can in principle be addressed on the level of underlying $(\mathfrak{g}, K)$ modules. If we write $\widehat{G}_u$ for the set of equivalence classes of irreducible unitary representations of G, what we are saying is that the original problem of identifying $\widehat{G}_u$ amounts to the same thing as finding all classes of infinitesimally unitary $(\mathfrak{g}, K)$ modules, the classes being defined by infinitesimal equivalence. In this formalism the Hilbert spaces have disappeared from the picture.

Because of this passage from G to $U(\mathfrak{g})$ and K, we may expect that the representation theory of compact groups will be an important tool in studying unitary representations of noncompact groups. We study the abstract theory of representations of compact groups K in Lecture 3, and we identify the unitary dual of K in Lecture 4. In fact, it is important to remember that K itself is an example of a reductive group. Thus we cannot expect to say more about the representation theory of reductive groups than we can about the representation theory of K.

A natural starting point in the classification of irreducible $U(\mathfrak{g})$ modules is restriction to a large abelian subalgebra of $U(\mathfrak{g})$. In Lecture 5 we shall see that the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is large and that $Z(\mathfrak{g})$ acts by scalars in any irreducible $U(\mathfrak{g})$ module. Thus we may attach an algebra homomorphism from $Z(\mathfrak{g})$ into $\mathbb{C}$ to any irreducible $(\mathfrak{g}, K)$ module X. In Lecture 5 we give Harish-Chandra's result establishing that the set of all nonzero such homomorphisms is canonically isomorphic to $\mathfrak{h}'/W$. Here $\mathfrak{h}'$ is the dual of a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. If λ is the Weyl group group orbit in $\mathfrak{h}'$ attached to X, we say that X has *infinitesimal character* λ. All these matters are in Lecture 5.

Unlike the case of irreducible unitary representations, two irreducible admissible representations can have infinitesimally equivalent underlying $(\mathfrak{g}, K)$ modules without being equivalent by bounded operators. In defining what set of equivalence classes of irreducible admissible representations to consider, we have to decide what kinds of equivalence to use. We choose infinitesimal equivalence. Let $\widehat{G}_a$ be the set of classes of irreducible admissible representations. Then $\widehat{G}_u \subseteq \widehat{G}_a$ in a natural way.

The set $\widehat{G}_a$ is large enough to contain the classes of unitary representations, but small enough to have a well-developed character theory. Characters are discussed in detail in Lecture 6, but we mention a few highlights here. In the theory of finite groups, the character of a finite-dimensional representation is the numerical-valued function on G whose value at $g \in G$ is the trace of the action by g. Characters are constant on conjugacy classes of G, and one knows that characters provide a powerful means to pass between representations of a group and the structure of the set of conjugacy classes in G. When G is compact, a similar theory is applicable (Lecture 3). If G is also connected, each conjugacy class meets any particular maximal toral subgroup of G, according to Lecture 6, and the analysis of representations of G can thereby be reduced to the theory for the toral subgroup. In particular, the Weyl Character Formula of Lecture 6 gives a relatively simple expression for all irreducible characters of a compact connected Lie group.

Because elements in $\widehat{G}_a$ may be infinite dimensional if G is noncompact, some serious care is required in generalizing character theory to the noncompact case. It turns out that we can attach to each element of $\widehat{G}_a$, not a conjuagation-invariant function on G, but rather a conjugation-invariant distribution on G called a global character. The modifier "global" is meant to indicate that the distribution is attached initially, not to a $(\mathfrak{g}, K)$ module itself, but instead to a global representation of G with a given underlying $(\mathfrak{g}, K)$ module. Such globalizations of a $(\mathfrak{g}, K)$ module always exist—typically there are many—but the global character does not depend on the choice of globalization. Because of this fact, we need not be careful to specify whether we are dealing with an irreducible admissible $(\mathfrak{g}, K)$ module or one of its globalizations. As in the case of finite groups, global characters are complete invariants in the sense that two elements of $\widehat{G}_a$ are infinitesimally equivalent if and only if their global characters coincide. In fact, the global characters corresponding to the members of $\widehat{G}_a$ are linearly independent.

Unlike what happens in the compact case, the conjugacy classes of a noncompact G need not meet a particular abelian subgroup. What does happen, however, is that each conjugacy class in an open dense set meets exactly one of a given particular finite set of abelian subgroups. The fact that each element of $\widehat{G}_a$ has an infinitesimal character implies that the global character satisfies a large system of differential equations coming from the center of $Z(\mathfrak{g})$, and the simple structure of most conjugacy classes essentially reduces the system to a system of differential equations on Euclidean space involving the infinitesimal character. This is discussed in Lecture 6. These equations can be solved, and the result is that the global character takes a particularly simple form on an open dense set; in fact, the formula for the global character bears a striking resemblance to the Weyl Character Formula in the compact case. A deep theorem of Harish-Chandra asserts that the complement of the open dense set of conjugacy classes cannot contribute anything interesting to the global character, and as a consequence one deduces that, for each infinitesimal character, the space of solutions satisfying the system of differential equations mentioned above is finite dimensional. Since the global characters of members of $\widehat{G}_a$ are linearly independent, each of the sets of members of $\widehat{G}_a$ of infinitesimal character λ is finite.

In any event, the unitarity problem can be stated as follows: parametrize the set $\widehat{G}_a$, and determine the subset of parameters corresponding to unitary representations. The rephrased problem is potentially useful only because it is possible to parametrize $\widehat{G}_a$. This is the content of the Langlands classification as treated in Lecture 8, and a subsequent classification of "irreducible tempered" representations by Knapp and Zuckerman. We now briefly outline the shape of the Langlands classification.

The Langlands classification (Lecture 8) first builds a family of irreducible admissible representations of G from a special kind of irreducible admissible representations of certain reductive subgroups of G. This construction proceeds by parabolic induction (Lecture 3). The hard step is to show that this family exhausts the irreducible admissible representations of G. The technique here is to analyze the asymptotics of the matrix coefficients. Lecture 7 deals with asymptotics; the core of the idea is as follows. It turns out that we can write $G = KAK$, where A is a certain Euclidean group. Given an admissible representation of G, we can think of the elements of G acting by some infinite-dimensional matrix whose coefficients are functions on G. If we arrange our basis to be compatible with the restriction of the representation to K, then the KAK decomposition implies that a block of matrix coefficients is determined by its restriction to the Euclidean space A, and it makes sense to study its asymptotics there.

The idea of the classification is then to take an irreducible admissible representation and study the asymptotics of its matrix coefficients. If these coefficients do not have the best possible growth characteristics at infinity, then it is possible to write the original representation as parabolically induced by means of an irreducible admissible representation of a smaller reductive group H with best possible decay characteristics at infinity. "Best possible decay characteristics at infinity" means, for this purpose, that the block of matrix coefficients is in $L^{2+\varepsilon}(H)$ for every $\varepsilon > 0$, hence that the representation is almost in the discrete series of H; in this case we call the representation *tempered*. Qualitatively the theorem is that the irreducible admissible representations of G are obtained by parabolic induction by means of irreducible tempered admissible representations of certain subgroups H of G. When the subgroup H is G itself, we recover the irreducible tempered admissible representations of G, in a kind of tautology. These tempered representations of G require a separate analysis, which is carried out earlier in Lecture 8, and they are obtained as constituents of representations essentially induced from discrete series of the subgroups H of G.

Thus we are left with problem of locating $\widehat{G}_u$ in terms of the above parametrization. We have an easy chance of succeeding only if we can relate the property of being unitary to the asymptotic growth of matrix coefficients. Already this seems like an unattainable goal: the discrete series, whose matrix coefficients are in L^2, are unitary; and so is the trivial representation, whose unique matrix coefficient is a constant function. This example is a little misleading—the trivial representation is anomalous in some sense—but even the more precise statements that are available are not very useful. The unitary representations simply do not fit nicely into the Langlands classification. This behavior helps account for the fact that the unitarity problem, if considered group by group, remains open except for $GL(n, \mathbb{R})$, $GL(n, \mathbb{H})$, complex classical groups, and most groups for which the dimension of the Euclidean group A above is ≤ 2.

LECTURE 1
Some Representations of $SL(n, \mathbb{R})$

Group Representations for the Case $n = 2$

We are going to be studying group representations, and we begin with some examples. The group in question will be $SL(n, \mathbb{R})$, the group of real n-by-n matrices of determinant 1. Mostly we consider $SL(2, \mathbb{R})$ for the time being.

A *representation* of G on a *complex* vector space V is defined to be a homomorphism Φ of G into the group of invertible linear transformations from V to itself. If G and V are topological, the map $G \times V \to V$ is assumed continuous.

Equivalently, a representation is a group action $G \times V \to V$ with $(g, v) \mapsto v$ linear for all $g \in G$.

The continuity condition for a representation Φ of G on a Hilbert space V is equivalent with the condition that $\|\Phi(x)\|$ be bounded in a neighborhood of the identity and that $\Phi(x)v \to v$ as $x \to 1$ for each v in a dense subset of V. (See the exercises.)

Here are three beginning examples. We do not need to specialize G to $SL(n, \mathbb{R})$ yet.

Example 1. Let G be any subgroup of $GL(n, \mathbb{C})$, i.e., the group of all n-by-n real or complex matrices, and let $V = \mathbb{C}^n$. Define $\Phi(g)v = gv$, matrix product. This Φ is called the *standard representation* of G.

Example 2. Suppose that a group G acts on a set X. Let V be the vector space of all functions $f : X \to \mathbb{C}$, and define $(\Phi(g)f)(x) = f(g^{-1}x)$. This Φ is called the *left regular representation* of G on the space of all functions on X. The use of g^{-1} rather than g in the formula for Φ makes Φ a homomorphism instead of an antihomomorphism.

Example 3. Let Φ be a representation of G on V. An *invariant subspace* U of V, i.e., a vector subspace such that $\Phi(G)U \subseteq U$, defines by restriction a representation of G on U. This is called a *subrepresentation*.

A *unitary representation* Φ of G on V is a representation in which V is a Hilbert space, finite or infinite dimensional, and each $\Phi(g)$ is *unitary*, i.e., invertible linear and also norm-preserving. The condition "norm-preserving" means that $\|\Phi(g)v\| = \|v\|$ for all v and g, and it is equivalent with the condition that the inner product satisfy $(\Phi(g)v_1, \Phi(g)v_2) = (v_1, v_2)$ for all v_1, v_2, and g.

Example. The space $V = \mathbb{C}^n$ may be made into a Hilbert space in the usual way. Define the *unitary group* $U(n)$ to be the subgroup of all matrices $g \in GL(n, \mathbb{C})$ with $g^*g = 1$, where $(\,\cdot\,)^*$ denotes adjoint. In Example 1 if G is a subgroup of the unitary group $U(n)$, then Φ is unitary. If G is any subgroup of $GL(n, \mathbb{C})$ that is not contained in $U(n)$, then the Φ in Example 1 is not unitary.

The obvious notion of isomorphism for two representations of the same group G is known as "equivalence": The representations (Φ_1, V_1) and (Φ_2, V_2) are *equivalent* if there is an invertible linear map $E : V_1 \to V_2$ such that the action of G on V_1 by Φ_1 matches the action of G on V_2 by Φ_2. In symbols, $E\Phi_1(g) = \Phi_2(g)E$ for all $g \in G$. If G, V_1, and V_2 are topological, then E and E^{-1} are assumed continuous.

Now let us specialize G to be $SL(2, \mathbb{R})$, the group of all 2-by-2 real matrices of determinant 1.

Here are some examples of finite-dimensional representations of $G = SL(2, \mathbb{R})$. We start from Example 1, the standard representation of G on $\mathbb{C}^2$. Since a representation is in particular a group action, we can use Example 2 to form the corresponding left regular representation of G on the space of functions from $\mathbb{C}^2$ to $\mathbb{C}$. Restrict in the sense of Example 3 to the subrepresentation on the space of polynomial functions in two variables homogeneous of degree N. This means that $P\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$ is a linear combination of monomials $z_1^{N-k} z_2$ as k varies, and the representation is given by

$$\left(\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right).$$

The dimension of the space is $N + 1$ because the monomials form a basis.

The above representations turn out to be *irreducible*, having no (closed) invariant subspaces. Up to equivalence, these are all the irreducible finite-dimensional representations. We omit the proof.

Theorem 1. *Every finite-dimensional* unitary *representation Φ of $SL(2, \mathbb{R})$ is trivial, i.e., has $\Phi(g) = 1$ for all g.*

Proof. Say Φ maps $SL(2, \mathbb{R})$ into $U(n)$. Since

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & r^2x \\ 0 & 1 \end{pmatrix},$$

all $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ with $x > 0$ are conjugate. So all $\Phi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ with $x > 0$ are conjugate. Since $U(n)$ is compact, its conjugacy classes are closed. Thus the limit $\Phi\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1$ is in the conjugacy class. Hence $\Phi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = 1$ for $x > 0$. Similarly $\Phi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = 1$ for $x < 0$, and $\Phi\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right) = 1$. The elements $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$ together generate $SL(2, \mathbb{R})$. So $\Phi(g) = 1$ for all g.

Yet there are nontrivial infinite-dimensional unitary representations. The group $SL(2, \mathbb{R})$ has a nonzero left-invariant Borel measure dx ("left Haar measure"). We obtain the *left regular representation* of $SL(2, \mathbb{R})$ by using the action of G on G and then taking the action on functions. The representation space is $L^2(SL(2, \mathbb{R}), dx)$, and the action is

$$(\Phi(g)f)(x) = f(g^{-1}x).$$

This representation Φ is unitary by the invariance of dx:

$$\|\Phi(g)f\|^2 = \int_G |f(g^{-1}x)|^2\,dx = \int_G |f(x)|^2\,dx = \|f\|^2.$$

Proving continuity requires observing by dominated convergence that

$$\int_G |f(g^{-1}x) - f(g_0^{-1}x)|^2\,dx \longrightarrow 0$$

as $g \to g_0$ for $f \in C_{\text{com}}(G)$ and knowing that $C_{\text{com}}(G)$ is dense in $L^2(G)$.

Similarly $SL(2,\mathbb{R})$ has a "right Haar measure" (it is actually the same as left Haar measure), and the *right regular representation* of $SL(2,\mathbb{R})$ on $L^2(SL(2,\mathbb{R}), dx)$ has $(\Phi(g)f)(x) = f(xg)$. Note that the formula for this action uses g and not g^{-1}.

The above unitary representations, namely the left and right regular representations of G on $L^2(G)$ are not close to irreducible. We shall next give some examples of unitary representations that are irreducible except in one particular case. The idea for constructing a nearly irreducible family of representations is to start from a transitive group action of G on a small coset space G/H and pass to the regular representation on functions. It turns out that the group-action property remains valid when certain kinds of coefficients, called *multipliers*, are included in the formula, and we use multipliers that make the representations unitary.

The examples we have in mind are the members of the *principal series* of $SL(2,\mathbb{R})$ in the *noncompact picture*. The principal series consists of two infinite families $\mathcal{P}^{+,iv}$ and $\mathcal{P}^{-,iv}$ of representations. Here v is an arbitrary element of $\mathbb{R}$, and we get one member of each family for each v. The space is $L^2(\mathbb{R})$ for each representation, and the action is given by

$$\mathcal{P}^{+,iv}\begin{pmatrix} a & b \\ c & d\end{pmatrix} f(x) = |-bx+d|^{-1-iv} f\left(\frac{ax-c}{-bx+d}\right),$$

$$\mathcal{P}^{-,iv}\begin{pmatrix} a & b \\ c & d\end{pmatrix} f(x) = (\operatorname{sgn}(-bx+d))|-bx+d|^{-1-iv} f\left(\frac{ax-c}{-bx+d}\right).$$

The group action property needs to be checked in each case. The unitarity property is proved by an easy change of variables.

We shall sketch a proof that the principal series representations are almost irreducible. But first we isolate a fundamental property of any unitary representation:

$(*)$ If U is a closed invariant subspace, so is $U^\perp$.

The argument in obvious notation is that

$$(\mathcal{P}(g)u^\perp, u) = (u^\perp, \mathcal{P}(g)^*u) = (u^\perp, \mathcal{P}(g)^{-1}u) = (u^\perp, \mathcal{P}(g^{-1})u) \in (u^\perp, U) = 0.$$

Unitarity has been used in the formula $\mathcal{P}(g)^* = \mathcal{P}(g)^{-1}$.

Returning to the question of irreducibility of the principal series, let E be the orthogonal projection on a closed invariant subspace U. This commutes with all $\mathcal{P}(g)$ by $(*)$. The operator $\mathcal{P}\left(\begin{smallmatrix} 1 & 0 \\ y & 1\end{smallmatrix}\right)$ acts in $L^2(\mathbb{R})$ by translation by $-y$. So E commutes with translations on $L^2(\mathbb{R})$. From Fourier analysis on $\mathbb{R}$, one knows therefore that $(Ef)\widehat{\ }(\xi) = m(\xi)\widehat{f}(\xi)$ for some $m \in L^\infty(\mathbb{R})$. where $\widehat{f}(\xi)$ denotes the Fourier transform $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx$. The equality $E^2 = E$ shows that $m^2 = m$ a.e. Hence m takes values in $\{0,1\}$ a.e. Analysis of commutativity of

E with $\mathcal{P}\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$, together with an application of Fubini's Theorem, shows m is constant a.e. on each half line. Thus the only nontrivial closed invariant subspaces are the two spaces of all members of $L^2(\mathbb{R})$ whose Fourier transforms are 0 on one of the two half lines.

Actually it turns out that all of $\mathcal{P}^{+,iv}$ and $\mathcal{P}^{-,iv}$ are irreducible except for $\mathcal{P}^{-,0}$, which is reducible. This fact is quite a bit harder to prove.

Let us consider another realization of the principal series, the *induced picture.* Historically it was not so easy to realize that the noncompact picture could be transformed to the induced picture. But once this transformation has been carried out, we obtain a framework that readily generalizes to groups other than $SL(2,\mathbb{R})$. Before defining the induced picture, we give names to some subgroups of $G = SL(2,\mathbb{R})$:

$$\begin{aligned} K &= \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}, \\ M &= \{\pm 1\}, \\ A &= \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \,\Big|\, r > 0 \right\}, \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}, \\ \overline{N} &= \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \right\}. \end{aligned}$$

Starting from the data $(+, iv)$ or $(-, iv)$, define

$$\sigma\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{cases} \varepsilon & \text{if } - \\ 1 & \text{if } +, \end{cases}$$

where $\varepsilon = \pm 1$, and

$$\nu\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = ivt \quad \text{and} \quad \rho\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = t.$$

Then $man \mapsto e^{\nu \log a}\sigma(m)$ is a representation of MAN. It is one-dimensional and unitary.

Consider the space of functions

$$\{F \in C(G) \mid F(xman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}F(x)\},$$

where $C(G)$ denotes the space of continuous functions from G to $\mathbb{C}$, and put

$$P^{\pm,iv}(g)F(x) = F(g^{-1}x).$$

The fact that left translation by g^{-1} commutes with right translation by man (the associative law for G) is what makes it so that $P^{\pm,iv}(g)F$ is still in the above space. Given F, define LF to be essentially the restriction to $\overline{N}$. Specifically

$$(LF)(y) = F\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \qquad \text{for } y \in \mathbb{R}.$$

We check that

$$LP^{\pm,iv}(g) = \mathcal{P}^{\pm,iv}(g)L.$$

In fact, the computation is based on the identity

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma/\alpha & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix},$$

valid for matrices of determinant one, and is as follows:

$$\begin{aligned}
LP^{\pm,iv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} F(y) &= P^{\pm,iv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} F\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \\
&= F\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right) \\
&= F\begin{pmatrix} -by+d & -b \\ ay-c & a \end{pmatrix} \\
&= F\left(\begin{pmatrix} 1 & 0 \\ \frac{ay-c}{-by+d} & 1 \end{pmatrix}\begin{pmatrix} -by+d & 0 \\ 0 & (-by+d)^{-1} \end{pmatrix}\begin{pmatrix} 1 & - \\ 0 & 1 \end{pmatrix}\right) \\
&= (\text{sgn or } 1)|-by+d|^{-1-iv} F\begin{pmatrix} 1 & 0 \\ \frac{ay-c}{-by+d} & 1 \end{pmatrix} \\
&= \mathcal{P}^{\pm,iv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} LF(y).
\end{aligned}$$

Here L is onto a dense subset of $L^2(\mathbb{R})$, namely onto at least the space $C_{\text{com}}(\mathbb{R})$ of continuous functions of compact support on $\mathbb{R}$. In fact, if $f \in C_{\text{com}}(\mathbb{R})$ is given, put

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} |a|^{-1-iv}\sigma\left(\begin{smallmatrix} \operatorname{sgn} a & 0 \\ 0 & \operatorname{sgn} a \end{smallmatrix}\right) f(c/a) & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Then we check easily that $LF = f$.

We can compute the value of the norm in the induced picture that makes L preserve norms, and then L will exhibit the equivalence. The answer by a change of variables is

$$\|F\|^2 = c\int_{-\pi}^{\pi} \left| F\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right|^2 d\theta.$$

The functions on the rotation subgroup $K = \left\{\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)\right\}$ that are involved are those that transform under M by

$$F\left(k\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}\right) = \sigma\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}^{-1} F(k).$$

If we identify the rotation subgroup with the circle group $\{e^{i\theta}\}$, then these functions are identified with those having only even-numbered Fourier coefficients in the case of $\mathcal{P}^{+,iv}$, odd-numbered Fourier coefficients in the case of $\mathcal{P}^{-,iv}$.

Actually restriction to K is onto the space of L^2 functions of this kind. This fact is a consequence of a fundamental structural property that will turn out later to be a special case of the "Iwasawa decomposition" of G, namely that the multiplication mapping $K \times A \times N \to G$ is an analytic diffeomorphism onto. The proof is by inspection from the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$
$$\begin{pmatrix} a/(a^2+c^2)^{1/2} & -c/(a^2+c^2)^{1/2} \\ c/(a^2+c^2)^{1/2} & a/(a^2+c^2)^{1/2} \end{pmatrix} \begin{pmatrix} (a^2+c^2)^{1/2} & 0 \\ 0 & (a^2+c^2)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 1 \end{pmatrix}$$

We write $G = KAN$ for this decomposition.

To see that restriction to K is onto the space of L^2 functions of the appropriate kind above, we can start with a function on K satisfying the appropriate transformation law under M and extend it to $SL(2,\mathbb{R})$ so as to satisfy the transformation law under MAN. Then we get unitary representations of $SL(2,\mathbb{R})$ on $L^2(K,\sigma)$; each is the *compact picture* of the corresponding induced representation.

In summary we have three equivalent ways of viewing the principal series:

- noncompact picture
- induced picture
- compact picture.

Each has its advantages, as we shall see.

Lie Algebra Representations for the Case $n = 2$

To define "representation" for a Lie algebra, let us return to the setting that G is any Lie group. Let $\mathfrak{g}$ be its Lie algebra, identified as a real vector space with the tangent space to G at the identity 1 and having bracket structure given by bracketing the corresponding left-invariant vector fields.

We motivate the definition by the finite-dimensional case. Recall that a (finite-dimensional) group representation is a continuous homomorphism $\Phi : G \to GL(V)$ with V finite-dimensional over $\mathbb{C}$. Since a continuous homomorphism between Lie groups is necessarily smooth and since the Lie algebra of $GL(V)$ may be identified with the Lie algebra $\mathfrak{gl}(V)$ of linear maps of V to itself, we can differentiate and get a homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(V)$. The homomorphism property yields

$$\varphi[X,Y] = [\varphi(X),\varphi(Y)] = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X),$$

the second equality following from the definition of bracket in $\mathfrak{gl}(V)$. The formula for φ is

$$\varphi(X)v = \frac{d}{dt}\,\Phi(c(t))v\,|_{t=0},$$

where $c(t)$ is any smooth curve in the group with $c(0) = 1$ such that the differential of $c(t)$ at $t = 0$ satisfies $dc_0\left(\frac{d}{dt}\right) = X$. An example of such a curve $c(t)$ is $c(t) = \exp tX$.

The above considerations motivate the definition in general. If $\mathfrak{g}$ is a real Lie algebra and V is a complex vector space, a *representation* of $\mathfrak{g}$ on V is a real linear map $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ such that

$$\varphi[X,Y] = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$$

for all X and Y in $\mathfrak{g}$.

We know in the finite-dimensional case that group representations lead to Lie algebra representations on the same vector space. Let us see what to expect in the infinite-dimensional case.

We use as an example the group $G = SL(2,\mathbb{R})$, and we consider the principal series of G in the compact picture. The representation space consists of certain functions on the rotation subgroup $K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$, K acts on these functions by translations, and the Lie algebra element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ wants to differentiate the function in θ. Some smoothness condition is needed. To apply $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ repeatedly requires the function to be in $C^\infty(K)$. Then the extension from K to $G = KAN$ via the formula

$$F(kan) = e^{-(\nu+\rho)\log a} F(k)$$

is in $C^\infty(G)$ since multiplication $K \times A \times N \to G$ is a diffeomorphism, and we can differentiate by any $X \in \mathfrak{g}$. The space

$$\left\{ F \in C^\infty(K) \;\middle|\; F\left(k\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}\right) = \sigma\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}^{-1} F(k) \right\}$$

is the space of C^∞ *vectors* for the representation, and we have a representation of $\mathfrak{g}$ on this vector space (no topology). The bracket property has to be checked, but we omit this computation.

As we shall see, a space $C^\infty(V)$ of C^∞ vectors can always be defined for a representation on a Hilbert space V. We return to this matter in Lecture 5.

To understand what to expect from the Lie algebra representation obtained from an infinite-dimensional group representation, let us consider the correspondence of invariant subspaces. Let Φ be a representation of G on V, and let U be a closed invariant subspace under G. Then $U \cap C^\infty(V)$ is invariant under $\mathfrak{g}$. But if $U \subseteq C^\infty(V)$ is $\mathfrak{g}$ invariant, $\overline{U}$ need not be G invariant. Here is an example.

Example. Let the group be the 2-by-2 rotation group K as above, acting on $L^2(K)$. Take

$$U = \{C^\infty \text{ functions supported for } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\}.$$

The subspace U is invariant under differentiation in θ, hence under the action of the Lie algbera of K. But its closure is not invariant under translation in θ, hence under the group K.

The difficulty in this example can arise only with infinite-dimensional representations. In the finite-dimensional case, a subspace invariant under the Lie algebra is automatically invariant under the Lie group.

To remedy this difficulty that arises in the infinite-dimensional case, one introduces the notion of "analytic vectors" for a group representation. In the case of the principal series of $SL(2,\mathbb{R})$, the analytic vectors are exactly the real analytic functions on K. We defer the definition in the general case to Lecture 5. In any event, for a general representation on a Hilbert space V, the space of analytic vectors is denoted $C^\omega(V)$. It will be the case that if $U \subseteq C^\omega(V)$ is invariant under $\mathfrak{g}$, then $\overline{U}$ is invariant under G. What is not so obvious is that the space of analytic vectors is nonzero. Once it is known that the space of analytic vectors is nonzero, however, we see that the action of the Lie algebra does give some information about the action of the Lie group.

Representations for the Case of General n

Now quickly let us mention how some of the above considerations generalize from $SL(2,\mathbb{R})$ to $G = SL(n,\mathbb{R})$ for general n. Taking our cue from the case $n = 2$, we define the following subgroups of G:

$$\begin{aligned} K &= SO(n) = \text{rotation subgroup} \\ A &= \text{positive diagonal subgroup} \\ M &= \text{diagonal subgroup, entries } |\varepsilon| = 1 \\ N &= \text{upper-triangular group, 1's on diagonal} \\ \overline{N} &= \text{lower-triangular group, 1's on diagonal.} \end{aligned}$$

Before coming to the principal series, let us consider certain decompositions of G that will allow us to relate different pictures for the principal series.

The decomposition theorem $G = KAN$, which we saw by a direct computation for $n = 2$, continues to be valid for $G = SL(n,\mathbb{R})$. The hard step in the proof is that every member of G decomposes as a product from KAN. This follows from the Gram-Schmidt orthogonalization process in linear algebra. In fact, let $u_1, \ldots, u_n$ be the standard orthonormal basis of $\mathbb{R}^n$. Given g, form $gu_1, \ldots, gu_n$. The Gram-Schmidt process yields an orthonormal basis $v_1, \ldots, v_n$ such that $gu_1, \ldots, gu_j$ always has the same span as $v_1, \ldots, v_j$ and v_j is in

$$\mathbb{R}^+(gu_j) + \operatorname{span}\{v_1, \ldots, v_{j-1}\}.$$

If k^{-1} is the matrix that carries the column vector v_j to u_j for each j, one can check that k is in $SO(n)$ and that $k^{-1}g$ is upper-triangular with positive diagonal entries.

The more precise statement of the result is that the multiplication map $K \times A \times N \to G$ is a diffeomorphism onto G. Another relevant decomposition theorem is that the multiplication mapping $\overline{N}MAN \hookrightarrow G$ is a diffeomorphism onto an open dense subset of G whose complement has lower dimension.

Now we define the *principal series*. The straightforward setting to generalize is the *induced picture*. Let

$$\begin{aligned} \sigma &= \text{one-dimensional representation of } M \\ \nu &= \text{imaginary linear functional on diagonal subalgebra} \\ \rho &= \text{a certain real linear functional that we specify in Lecture 3.} \end{aligned}$$

The members of the induced space have

$$F(xman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}F(x),$$

and the group action by G is

$$P^{\sigma,\nu}(g)F(x) = F(g^{-1}x).$$

Initially we take the functions in question to be continuous. Then a norm has to be imposed, and the whole space for the induced representation is obtained by completion. The presence of ρ makes the resulting representation unitary.

As with $SL(2,\mathbb{R})$, there are two other pictures for principal series representations: Restriction to K gives the compact picture, while restriction to $\overline{N}$ gives the noncompact picture. For these two pictures, the Hilbert space norm is easy to specify. Both K and $\overline{N}$ have two-sided invariant measures ("Haar measures"), given in the case of $\overline{N}$ simply by Lebesgue measure in the natural matrix-entry coordinates, and the Hilbert space norms are the L^2 norms with respect to these measures. Not all members F of $L^2(K)$ are involved in the compact picture of $P^{\sigma,\nu}$, only those satisfying $F(km) = \sigma(m)^{-1}F(k)$ for each $m \in M$ and almost every $k \in K$.

Notes

As is pointed out in the Introduction, these lectures are a distillation of material in [**K1**] and [**K2**]. The section of Notes at the end of each lecture largely gives references to expository sources for further reading, quite often to [**K1**] or [**K2**]. Historical information and an extensive bibliography may be found in those two books.

Elementary Lie theory is the topic of Chevalley [**C**], particularly the first four chapters. A summary of some of this material appears in [**K2**], pp. 43–55.

Three standard books on the representation theory of semisimple groups are [**K1**], Wallach [**Wal**], and Warner [**War**]. All of these have material on abstract representation theory; in [**K1**], this material is very brief and is on pp. 10–14. For some further material in this direction, see [**Bal**].

Constructions of some finite-dimensional representations of concrete groups may be found in [**K2**], pp. 181–186. The finite-dimensional irreducible complex-linear representations of $\mathfrak{sl}(2,\mathbb{C})$ are classified in [**K2**], pp. 37–43. See also [**K1**], pp. 28–32.

Infinite-dimensional representations of $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ are discussed in more detail in [**K1**], pp. 33–42. For additional information about representations of $SL(2,\mathbb{R})$, see [**Do**].

Exercises

1. Check that the action in Example 2 of representations gives rise to a homomorphism, and not an antihomomorphism.

2. Prove that the continuity condition for a representation Φ of G on a Hilbert space V, in the presence of the homomorphism property, is equivalent with the condition that $\|\Phi(x)\|$ be bounded in a neighborhood of the identity and that $\Phi(x)v \to v$ as $x \to 1$ for each v in a dense subset of V. Conclude that a homomorphism of G into unitary operators on a Hilbert space V is continuous (and hence is a representation) if $g \mapsto \Phi(g)v$ is continuous for a dense set of vectors $v \in V$.

3. Use the result of Exercise 2 to fill in the details that the left regular representation of G on $L^2(G)$ is continuous.

4. If (Φ, V) and (Ψ, W) are representations of G, make $V \otimes W$ into the representation space of a representation $\Phi \otimes \Psi$ of G.

5. If (Φ, V) is a representation of G and V' denotes the linear dual of V, define $\Phi^t(g)(v')(v) = v'(\Phi(g^{-1})v)$ for $v \in V$ and $v' \in V'$. Show that (Φ^t, V') is a representation of G. For the situation that V is topological and Φ is continuous, construct a representation Φ^c as the subrepresentation of Φ^t on the subspace of continuous members of V'. [Φ^c is called the *contragredient* of Φ.]

6. Check the group representation property and unitarity of the unitary principal series representations of $SL(2,\mathbb{R})$.

7. Verify that the norm for the induced and noncompact pictures of the unitary principal series of $SL(2,\mathbb{R})$ match, up to a scalar factor.

LECTURE 2
Semisimple Groups and Structure Theory

Semisimple Groups and Examples

A *linear connected reductive group* is a closed connected group of real or complex matrices that is stable under conjugate transpose. A *linear connected semisimple group* is a linear connected reductive group with finite center.

For a Lie group of matrices, we know that the Lie algebra can be identified with all matrices $c'(0)$, where $c(t)$ is a smooth curve in the group with $c(0) = 1$. The bracket is $[A, B] = AB - BA$. Here are some examples of various kinds of linear connected reductive groups.

Example 1. Complex groups G with Lie algebra $\mathfrak{g}$:

a) General linear group:

$$\begin{aligned} G &= GL(n, \mathbb{C}) = \{\text{nonsingular } n\text{-by-}n \text{ matrices over } \mathbb{C}\} \\ \mathfrak{g} &= \mathfrak{gl}(n, \mathbb{C}) = \{\text{all } n\text{-by-}n \text{ matrices over } \mathbb{C}\} \end{aligned}$$

b) Special linear group:

$$\begin{aligned} G &= SL(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) \mid \det g = 1\} \\ \mathfrak{g} &= \mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \operatorname{Tr} X = 0\} \end{aligned}$$

c) Complex orthogonal group:

$$\begin{aligned} G &= SO(n, \mathbb{C}) = \{g \in SL(n, \mathbb{C}) \mid gg^t = 1\} \\ \mathfrak{g} &= \mathfrak{so}(n, \mathbb{C}) = \{X \in \mathfrak{sl}(n, \mathbb{C}) \mid X + X^t = 0\} \end{aligned}$$

d) Complex symplectic group:

$$\begin{aligned} G &= Sp(n, \mathbb{C}) = \{g \in SL(2n, \mathbb{C}) \mid g^t J g = J\} \quad \text{with } J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \\ \mathfrak{g} &= \mathfrak{sp}(n, \mathbb{C}) = \{X \in \mathfrak{sl}(2n, \mathbb{C}) \mid X^t J + JX = 0\}. \end{aligned}$$

Some people write $Sp(2n, \mathbb{C})$ for the name of (d). Among the above complex reductive groups, (a) has a one-dimensional (infinite) center, while the others have finite center and are hence semisimple.

Example 2. Compact groups G with Lie algebra $\mathfrak{g}$:

a) Rotation group:

$$\begin{aligned} G &= SO(n) = \{g \in SL(n,\mathbb{C}) \mid g^t g = 1, \text{ real entries}\} \\ \mathfrak{g} &= \mathfrak{so}(n) = \{X \in \mathfrak{sl}(n,\mathbb{C}) \mid X^t + X = 0, \text{ real entries}\} \end{aligned}$$

b) Unitary group:

$$\begin{aligned} G &= U(n) = \{g \in GL(n,\mathbb{C}) \mid \bar{g}^t g = 1\} \\ \mathfrak{g} &= \mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \overline{X}^t + X = 0\} \end{aligned}$$

c) Special unitary group:

$$\begin{aligned} G &= SU(n) = \{g \in U(n) \mid \det g = 1\} \\ \mathfrak{g} &= \mathfrak{su}(n) = \{X \in \mathfrak{u}(n) \mid \operatorname{Tr} X = 0\} \end{aligned}$$

d) "Unitary group" over quaternions $\mathbb{H}$ (up to isomorphism):

$$\begin{aligned} G &= Sp(n) = \{g \in U(2n) \mid g^t J g = J\} \\ \mathfrak{g} &= \mathfrak{sp}(n) = \{X \in \mathfrak{u}(2n) \mid X^t J + JX = 0\}. \end{aligned}$$

Example 3. More groups G with Lie algebra $\mathfrak{g}$:

a) Groups of real matrices in above complex groups. The group $GL(n,\mathbb{R})$ is disconnected and therefore is not reductive in the above definition. However, its identity component is a reductive group in the above definition.

$$\begin{aligned} G &= GL_0(n,\mathbb{R}) = \{\text{nonsingular } n\text{-by-}n \text{ matrices over } \mathbb{R}, \text{ positive determinant}\} \\ \mathfrak{g} &= \mathfrak{gl}(n,\mathbb{R}) = \{\text{all } n\text{-by-}n \text{ matrices over } \mathbb{R}\} \end{aligned}$$

$$\begin{aligned} G &= SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) \mid \det g = 1\} \\ \mathfrak{g} &= \mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr} X = 0\} \end{aligned}$$

$$\begin{aligned} G &= SO(n,\mathbb{R}) = SO(n) = \{g \in SL(n,\mathbb{R}) \mid gg^t = 1\} \\ \mathfrak{g} &= \mathfrak{so}(n,\mathbb{R}) = \mathfrak{so}(n) = \{X \in \mathfrak{sl}(n,\mathbb{R}) \mid X + X^t = 0\} \end{aligned}$$

$$\begin{aligned} G &= Sp(n,\mathbb{R}) = \{g \in SL(2n,\mathbb{R}) \mid g^t J g = J\} \\ \mathfrak{g} &= \mathfrak{sp}(n,\mathbb{R}) = \{X \in \mathfrak{sl}(2n,\mathbb{R}) \mid X^t J + JX = 0\}. \end{aligned}$$

b) Isometry groups for indefinite Hermitian forms. The group $O(m,n)$, the linear isometry group for the real quadratic form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$ in $\mathbb{R}^{m+n}$, has four components if $m > 0$ and $n > 0$. Restricting to determinant one cuts the number of components down to two. We obtain a reductive (actually semisimple) group by passing to the identity component.

$$\begin{aligned} SO_0(m,n) &= \begin{cases} \text{identity component of linear isometry group for real quadratic form} \\ x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2 \text{ in } \mathbb{R}^{m+n} \end{cases} \\ U(m,n) &= \begin{cases} \text{linear isometry group for Hermitian quadratic form} \\ |z_1|^2 + \cdots + |z_m|^2 - |z_{m+1}|^2 - \cdots - |z_{m+n}|^2 \text{ in } \mathbb{C}^{m+n} \end{cases} \\ SU(m,n) &= \{g \in U(m,n) \mid \det g = 1\}. \end{aligned}$$

We discuss the corresponding Lie algebras $\mathfrak{so}(m,n)$, $\mathfrak{u}(m,n)$, and $\mathfrak{su}(m,n)$ later in this lecture.

A linear connected reductive group G is mapped to itself by conjugate transpose, and thus inverse conjugate transpose is an automorphism Θ of G with $\Theta^2 = 1$. It is called the (global) *Cartan involution* of G.

Let $K = \{g \in G \mid \Theta g = g\}$ be the subgroup of elements in G left fixed by Θ. If G is realized in n-by-n matrices, K is a closed subgroup of $G \cap U(n)$, hence is compact.

Let θ be the differential of Θ at 1, namely negative conjugate transpose. This is an automorphism of $\mathfrak{g}$ with $\theta^2 = 1$. It is called the *Cartan involution* of $\mathfrak{g}$.

Any linear transformation whose square is 1 has $+1$ and -1 eigenspaces whose direct sum is the whole space. We write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

for the corresponding eigenspace decomposition for θ. This decomposition is called the *Cartan decomposition* of $\mathfrak{g}$. It has the following properties:

(a) $\mathfrak{k} \subseteq \{\text{skew-Hermitian matrices}\}$
(b) $\mathfrak{p} \subseteq \{\text{Hermitian matrices}\}$
(c) $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$
(d) $\mathfrak{k} =$ Lie algebra of K.

These properties are all elementary. For example, to see that the middle inclusion holds in (c), let $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$. Then

$$\theta[X,Y] = [\theta X, \theta Y] = [+X, -Y] = -[X,Y],$$

and hence $[X,Y]$ is in $\mathfrak{p}$.

Examples of $\mathfrak{k}$ and $\mathfrak{p}$.

1) Let $G = GL(n,\mathbb{C})$. Then $K = U(n)$ and

$$\begin{aligned} \mathfrak{k} &= \{\text{skew-Hermitian matrices}\} \\ \mathfrak{p} &= \{\text{Hermitian matrices}\}. \end{aligned}$$

2) Let $G = SL(n,\mathbb{R})$. Then $K = SO(n)$ and

$$\begin{aligned} \mathfrak{k} &= \{\text{real skew-symmetric matrices}\} \\ \mathfrak{p} &= \{\text{real symmetric matrices}\}. \end{aligned}$$

3) Let $G = SO(n)$ or $U(n)$ or $SU(n)$ or $Sp(n)$, i.e., any of our compact examples. Then $\Theta = 1$, $\theta = 1$, $K = G$, $\mathfrak{k} = \mathfrak{g}$, and $\mathfrak{p} = 0$.
4) Let $G = SO_0(m,n)$. If we group the $m+n$ indices into a set of m indices and a set of n indices, then

$$\begin{aligned} \mathfrak{k} &= \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \text{ real skew} \right\} \\ \mathfrak{p} &= \left\{ \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix}, \text{ real} \right\}. \end{aligned}$$

The sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Lie algebra $\mathfrak{so}(m,n)$ of $SO_0(m,n)$.

5) Let $G = U(m,n)$. Then

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \text{ skew-Hermitian} \right\}$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ \overline{Z}^t & 0 \end{pmatrix} \right\}.$$

The sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Lie algebra $\mathfrak{u}(m,n)$ of $U(m,n)$.

6) Let $G = SU(m,n)$. Then

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \text{ skew-Hermitian of trace } 0 \right\}$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ \overline{Z}^t & 0 \end{pmatrix} \right\}.$$

The sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Lie algebra $\mathfrak{su}(m,n)$ of $SU(m,n)$.

Structure Theory

All Lie algebras in this section are defined over $\mathbb{R}$ and are finite-dimensional unless stated otherwise.

A *simple Lie algebra* is a Lie algebra with dimension greater than one and with no nontrivial ideals. A *semisimple Lie algebra* is a Lie algebra with no nonzero abelian ideals.

Simple clearly implies semisimple. The full relationship between "simple" and "semisimple" will be discussed shortly.

The *Killing form* on $\mathfrak{g}$ is defined by

$$B(X,Y) = \operatorname{Tr}(\operatorname{ad} X \operatorname{ad} Y) \qquad \text{for } X \text{ and } Y \text{ in } \mathfrak{g}.$$

This is a symmetric bilinear form on $\mathfrak{g}$ that is *invariant* in the sense that

$$B((\operatorname{ad} Z)X, Y) = -B(X, (\operatorname{ad} Z)Y).$$

Example. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, and let $\{h, e, f\}$ be the basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The bracket relations among the basis vectors are given by

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

We readily compute that the matrix of B is $\begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$. For example, the entry in the second row and third column is $B(e,f) = \operatorname{Tr}(\operatorname{ad} e \operatorname{ad} f)$. The linear transformation $\operatorname{ad} e \operatorname{ad} f$ carries h to $2h$, e to $2e$, and f to 0. Thus $B(e,f) = 2 + 2 + 0 = 4$.

The theorem that gets the subject started is as follows.

Theorem (Cartan's Criterion for Semisimplicity). *A Lie algebra is semisimple if and only if its Killing form is nondegenerate.*

The word "nondegenerate" means that $B(X, \mathfrak{g}) = 0$ implies $X = 0$. Equivalently the matrix of B is to be nonsingular.

It is fairly easy to derive from this theorem the full relationship between "simple" and "semisimple".

Corollary. *A Lie algebra is semisimple if and only if it is the direct sum of simple Lie algebras that are each ideals.*

A *reductive Lie algebra* is a Lie algebra that is the direct sum of two ideals, one equal to a semisimple Lie algebra and the other equal to an abelian Lie algebra.

Proposition. *A Lie algebra is reductive if and only if each ideal* $\mathfrak{a}$ *in* $\mathfrak{g}$ *has a complementary ideal, i.e., an ideal* $\mathfrak{b}$ *with* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$.

Proposition. *If* G *is linear connected semisimple, then* $\mathfrak{g}$ *is semisimple. More generally if* G *is linear connected reductive, then* $\mathfrak{g}$ *is reductive with* $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$ *as a direct sum of ideals. Here* $Z_{\mathfrak{g}}$ *denotes the center of* $\mathfrak{g}$*, and the commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple.*

Example. $\mathfrak{gl}(n, \mathbb{R}) = \{\text{scalars}\} \oplus \mathfrak{sl}(n, \mathbb{R})$.

Let us pause to comment on other definitions of "semisimple" and "reductive" for Lie groups.

Most authors define a *semisimple Lie group* to be a connected Lie group whose Lie algebra is semisimple. Such a group G is a finite or infinite cover of the group $\mathrm{Ad}(G)$, which, relative to any basis of $\mathfrak{g}$, is a group of real matrices with a semisimple Lie algebra. A hard theorem shows that in a suitable basis $\mathrm{Ad}(G)$ is linear connected semisimple. Thus the most general semisimple Lie group is the finite or infinite cover of a linear connected semisimple group.

For example, it turns out that $SL(2, \mathbb{R})$ has a double cover and that this double cover is not isomorphic to a linear connected semisimple group.

The need for reductive Lie groups will be clearer when we consider induced representations. One wants to construct as many representations as possible by induction on the dimension of the group. The prototype is the principal series of $SL(n, \mathbb{R})$, which is constructed from representations of the diagonal group. Even when the given group is semisimple, the natural candidate subgroups to use have reductive Lie algebras (not necessarily semisimple) and may even be disconnected. This is the case with the diagonal subgroup of $SL(n, \mathbb{R})$. The exact conditions in the definition of "reductive Lie group" vary from author to author, but in any definition one wants certain important subgroups of a reductive Lie group to be reductive.

Now let us return to structure theory. Every complex matrix decomposes as the product of a unitary matrix and a positive semidefinite Hermitian matrix. The positive semidefinite matrix is unique. If the given matrix is nonsingular, the positive semidefinite matrix is positive definite, and the unitary matrix is unique. In other words, the group $G = GL(n, \mathbb{C})$ has $G = K \exp \mathfrak{p}$, where $K = U(n)$ and $\mathfrak{p}$ is the vector space of Hermitian matrices. This decomposition is called the *polar decomposition* of matrices. The generalization is called the (global) *Cartan decomposition*, and the precise statement is as follows.

Theorem. *If* G *is linear connected reductive, then* K *is compact connected and is a maximal compact subgroup of* G*. Its Lie algebra is* $\mathfrak{k}$*. Moreover, the map of* $K \times \mathfrak{p}$ *into* G *given by* $(k, X) \to k \exp X$ *is a diffeomorphism onto.*

Note in particular that the interesting part of the topology of G is carried by K.

Corollary. *If G is linear connected reductive, then the center Z_G of G satisfies*

$$Z_G = (Z_G \cap K)\exp(\mathfrak{p} \cap Z_{\mathfrak{g}}).$$

In Lecture 1 we saw that $G = SL(n, \mathbb{R})$ has a decomposition $G = KAN$, the multiplication map being a diffeomorphism onto, as a consequence of the Gram-Schmidt orthogonalization process. The generalization to all linear connected reductive G is known as the "Iwasawa decomposition". In order even to state the theorem, we need definitions of A and N. These are obtained by working with the Lie algebra.

Fix a linear connected reductive group G, and let $\mathfrak{a}$ be any maximal abelian subspace of $\mathfrak{p}$. The *trace form* on $\mathfrak{g}$ is the complex-valued real-bilinear form given by $B_0(X, Y) = \operatorname{Tr}(XY)$. This is *invariant* in the same sense as the Killing form, namely

$$B_0((\operatorname{ad} Z)X, Y) = -B_0(X, (\operatorname{ad} Z)Y).$$

Also $\langle X, Y\rangle = -B_0(X, \theta Y) = \operatorname{Tr}(XY^*)$ is a real-valued inner product on the real vector space $\mathfrak{g}$.

Proposition. *Relative to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{g}$,*

$$(\operatorname{ad} X)^* = \operatorname{ad} X^* \qquad \textit{for all } X \in \mathfrak{g}.$$

Proof. For X, Y, and Z in $\mathfrak{g}$, we have $\langle Y, (ad\, X)^* Z\rangle = \langle(\operatorname{ad} X)Y, Z\rangle = B_0((\operatorname{ad} X)Y, Z^*) = -B_0(Y, (\operatorname{ad} X)Z^*) = -\langle Y, [X, Z^*]^*\rangle = \langle Y, (\operatorname{ad} X^*)Z\rangle$.

Consequently $\operatorname{ad} X$ is Hermitian for $X \in \mathfrak{a}$. We seek a simultaneous eigenspace decomposition relative to $\operatorname{ad}\mathfrak{a}$. If X and Y are in $\mathfrak{a}$, then $[\operatorname{ad} X, \operatorname{ad} Y] = \operatorname{ad}[X, Y] = \operatorname{ad} 0 = 0$ shows that $\operatorname{ad} X$ and $\operatorname{ad} Y$ commute. Thus if $H_1, \dots, H_l$ is a basis of $\mathfrak{a}$, then $\{\operatorname{ad} H_j\}$ is a commuting family of Hermitian operators on $\mathfrak{g}$ and is simultaneously diagonalizable by the finite-dimensional Spectral Theorem. Let $V_1, \dots, V_r$ be the eigenspaces in $\mathfrak{g}$ for the different systems of eigenvalue tuples. If $\operatorname{ad} H_i$ acts as λ_{ij} on V_j, define a linear functional λ_j on $\mathfrak{a}$ by $\lambda_j(H_i) = \lambda_{ij}$. If $H = \sum c_i H_i$, then $\operatorname{ad} H$ acts on V_j by

$$\sum_i c_i \lambda_{ij} = \sum_i c_i \lambda_j(H_i) = \lambda_j(H).$$

In other words, $\operatorname{ad}\mathfrak{a}$ acts in simultaneously diagonal fashion on $\mathfrak{g}$, and the simultaneous eigenvalues are members of the dual vector space $\mathfrak{a}'$. There are finitely many such simultaneous eigenvalues, and we write $\mathfrak{g}_\lambda$ for the eigenspace corresponding to $\lambda \in \mathfrak{a}'$. The nonzero such λ are called *restricted roots.*

Let us summarize. For $\lambda \in \mathfrak{a}'$, let $\mathfrak{g}_\lambda$ be the corresponding simultaneous eigenspace, namely

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid (\operatorname{ad} H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$, then λ is a *restricted root*, and any $X \in \mathfrak{g}_\lambda$ is called a *restricted-root vector.* Let Σ be the set of all restricted roots. The result of the previous paragraph is that we obtain a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda.$$

This is called the *restricted-root space decomposition* of $\mathfrak{g}$.

Examples.

1) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. Then $\Sigma = \{e_i - e_j \mid i \neq j\}$. Here $\mathfrak{g}_{e_i - e_j} = \mathbb{R}E_{ij}$ and $\mathfrak{g}_0 = \{\text{real diagonal, trace } 0\}$.

2) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Then $\Sigma = \{e_i - e_j \mid i \neq j\}$. Here $\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}$ and $\mathfrak{g}_0 = \{\text{complex diagonal, trace } 0\}$.

3) Let $\mathfrak{g} = \mathfrak{su}(p, q)$ with $p > q$. Recall that $\mathfrak{g}$ consists of all $\left(\begin{smallmatrix} a & b \\ b^* & d \end{smallmatrix}\right)$ with the indices grouped into groups of sizes p and q and with a and d skew-Hermitian of total trace 0. It can be shown that

$$\Sigma = \{\pm f_i \pm f_j\} \cup \{\pm f_i\} \cup \{\pm 2f_i\}$$

with f_i defined as follows. One choice of $\mathfrak{a}$ that we can use is to take $a = 0$, $d = 0$, and b equal to 0 except in the bottom $q \times q$ block, where it consists of all real antidiagonal matrices $\left(\begin{smallmatrix} 0 & \cdots & a_q \\ & \cdots & \\ a_1 & \cdots & 0 \end{smallmatrix}\right)$. Then $f_i \in \mathfrak{a}'$ has value a_i on this. The formulas for the restricted-root vectors and the verifications of our formulas for all the restricted roots are too complicated to give here, and we omit them. The exercises ask for a computation in a relatively easy case and in the general case.

Proposition.

1) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.

2) $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$. *Hence* $\lambda \in \Sigma$ *implies* $-\lambda \in \Sigma$.

3) $\mathfrak{g}_\lambda$ *and* $\mathfrak{g}_\mu$ *are orthogonal with respect to* $\langle\,\cdot\,,\,\cdot\,\rangle$ *if* $\lambda \neq \mu$.

4) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, *where* $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ *is the centralizer of* $\mathfrak{a}$ *in* $\mathfrak{k}$. *Moreover the sum is an orthogonal sum.*

To define $\mathfrak{n}$, we introduce a "lexicographic ordering" in $\mathfrak{a}'$. Namely fix an ordered basis $\lambda_1, \dots, \lambda_l$ of $\mathfrak{a}'$. Define $\lambda = \sum c_i \lambda_i$ to be *positive* if the first nonzero c_i is > 0. The ordering comes from saying $\lambda > \mu$ if $\lambda - \mu$ is positive.

Let Σ^+ be the set of positive members of Σ.

Examples. In the examples above, we can arrange that

- $\Sigma^+ = \{e_i - e_j \mid i < j\}$.
- $\Sigma^+ = \{e_i - e_j \mid i < j\}$.
- $\Sigma^+ = \{f_i \pm f_j \mid i < j\} \cup \{f_i\} \cup \{2f_i\}$.

Now define

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

This is a Lie subalgebra of $\mathfrak{g}$ as a consequence of conclusion (1) of the proposition. Let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. We can now state the *Iwasawa decomposition*, first on the level of Lie algebras and then on the level of Lie groups.

Proposition. *For G linear connected reductive,* $\mathfrak{g}$ *is a direct sum* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. *Here* $\mathfrak{a}$ *is abelian,* $\mathfrak{n}$ *is nilpotent,* $\mathfrak{a} \oplus \mathfrak{n}$ *is solvable, and* $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}]$ *equals* $\mathfrak{n}$.

Theorem. *For G linear connected reductive, let A and N be the analytic subgroups with Lie algebras* $\mathfrak{a}$ *and* $\mathfrak{n}$. *Then A, N, and AN are simply connected closed subgroups of G, and the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto.*

We conclude this lecture by discussing "minimal parabolic subgroups". We define $M = Z_K(\mathfrak{a})$ to be the centralizer of $\mathfrak{a}$ in K, i.e., the set of all $k \in K$ such that $\mathrm{Ad}(k) = 1$ on $\mathfrak{a}$. In $SL(n,\mathbb{R})$ or $SL(n,\mathbb{C})$ with A chosen as in the examples above, M is the diagonal subgroup. In $SU(m,n)$, M is nonabelian if $m > n+1$.

Proposition.

1) *M is a closed subgroup of K, hence compact.*
2) *M centralizes $\mathfrak{a}$ and normalizes each $\mathfrak{g}_\lambda$.*
3) *M centralizes A and normalizes N. In fact, $\mathrm{Ad}(m)\mathfrak{g}_\lambda \subseteq \mathfrak{g}_\lambda$ for all λ.*
4) *MAN is a closed subgroup of G.*

Proof of (2). Take $H \in \mathfrak{a}$, $m \in M$, and $X_\lambda \in \mathfrak{g}_\lambda$. Then $[H, \mathrm{Ad}(m)X_\lambda] = \mathrm{Ad}(m)[\mathrm{Ad}(m)^{-1}H, X_\lambda] = \mathrm{Ad}(m)[H, X_\lambda] = \lambda(H)\mathrm{Ad}(m)X_\lambda$.

The subgroup MAN is called a *minimal parabolic subgroup* of G.

Theorem. *The natural inclusion $N_K(\mathfrak{a}) \hookrightarrow N_G(\mathfrak{a})$ induces an isomorphism $N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \cong N_G(\mathfrak{a})/Z_G(\mathfrak{a})$, and these quotients are finite groups.*

The left side, for example, is the set of distinct linear transformations by which members of K act on the vector space $\mathfrak{a}$.

Example. For $G = GL(n,\mathbb{C})$, $N_K(\mathfrak{a})$ consists of all matrices with one nonzero entry in each row and column.

Theorem (Bruhat decomposition). *The double coset space $MAN\backslash G/MAN$ is parametrized in one-one onto fashion by $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$, the double coset corresponding to w in this quotient being $MAN\widetilde{w}MAN$, where $\widetilde{w}$ is any representative of w in $N_G(\mathfrak{a})$.*

We do not make much use of this theorem in these lectures, but the theorem can be used in proving that generic principal series representations are irreducible.

Notes

Examples of semisimple Lie algebras and groups are given in [**K2**], pp. 33–36 and pp. 66–73. Some of this material may be found also in [**K1**], pp. 4–6. Structure theory of the kind in this lecture is discussed for linear groups in [**K1**], pp. 3–4, pp. 7–10, and Chapter V. A more thorough treatment, not limited to linear groups, is in [**K2**], pp. 24–32, pp. 291–318, pp. 379–384, and pp. 397–401. Structure theory may be found also in [**He**].

Exercises

1. Verify that the polar decomposition for any $g \in GL(n,\mathbb{C})$ is unique.
2. For $\mathfrak{su}(2,1)$ with $\mathfrak{a}_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}$, find the restricted root space spaces, and verify that the restricted roots are as given in the lecture.
3. Redo Exercise 2 for the general case of $\mathfrak{su}(p,q)$.

LECTURE 3
Introduction to Representation Theory

Abstract Representation Theory of Compact Groups

A *multiplicative character* of a topological group is a continuous homomorphism of G into $\mathbb{C}^\times$. It may be canonically identified with a one-dimensional representation by identifying $\mathbb{C}^\times$ with $GL(1, \mathbb{C})$. The condition Image $\subseteq \{|z| = 1\}$ is equivalent with the condition that this one-dimensional representation be unitary.

Some authors use the word "quasicharacter" when the image is allowed to be in $\mathbb{C}^\times$, reserving the word "character" for the case that the image is in $\{|z| = 1\}$. Later in this section we shall discuss characters associated to representations of dimension greater than 1; the adjective "multiplicative" is used to stress that the associated representation is one-dimensional.

The setting for this section is that G is compact. Our interest is in the irreducible finite-dimensional representations of G. Only near the very end do we consider any infinite-dimensional representations.

The prototype is the case that G is the circle group, namely $\mathbb{R}/2\pi\mathbb{Z}$. In this case the measure $\frac{1}{2\pi}\,dx$ is invariant under translation (i.e., is a "Haar measure"), and its total mass is 1. The multiplicative characters for the circle group are the functions $x \mapsto e^{inx}$, and there are no others, as is shown in the exercises. These functions have the properties that they are orthogonal and and have norm one in $L^2(\frac{1}{2\pi}\,dx)$. If f is any integrable function on G, its Fourier coefficients are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

Parseval's formula expresses the completeness of the orthonormal set of multiplicative characters of the circle group in $L^2(G)$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

The left side here is nothing more than the L^2 norm of f with respect to our choice of invariant measure.

For a general compact group G, the multiplicative characters are insufficient for an analysis of $L^2(G)$. To see this, let $[G, G]$ be the subgroup of G generated by all elements $xyx^{-1}y^{-1}$. Every multiplicative character is trivial on these elements.

Hence if $G = \overline{[G, G]}$, as is the case with $G = SU(2)$, then G has no nontrivial multiplicative character.

We shall need to develop a general theory as a substitute.

We begin by generalizing $\frac{1}{2\pi}\,dx$ from the circle group to a general compact group G. A *left* or *right Haar measure* on G is a nonzero regular Borel measure on G invariant under left or right translations.

Theorem. *A compact group G has a left Haar measure unique up to a constant, and it is also a right Haar measure.*

Normalize Haar measure to have total mass 1, and write it as dx.

Proposition. *If Φ is a representation of G on a finite-dimensional vector space V, then V admits a Hermitian inner product such that Φ is unitary.*

Sketch of proof. From any given Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on V, define $(\,\cdot\,,\,\cdot\,)$ by

$$(u, v) = \int_G \langle \Phi(x)u, \Phi(x)v\rangle\, dx.$$

One checks readily that $(\,\cdot\,,\,\cdot\,)$ is an inner product and that Φ is unitary relative to it.

This proposition is fundamental, and it is often applied without specific mention. When we write $(\,\cdot\,,\,\cdot\,)$ for a Hermitian inner product on the space of a finite-dimensional representation, we assume that it exhibits the representation as unitary, i.e., that $(\,\cdot\,,\,\cdot\,)$ is invariant under the group action.

Corollary. *If Φ is a representation of G on a finite-dimensional vector space V, then Φ is the direct sum of irreducible representations. In other words, $V = V_1 \oplus \cdots \oplus V_k$, with each V_j an invariant subspace on which Φ acts irreducibly.*

Sketch of proof. As we saw in Lecture 1, the orthogonal complement of an invariant subspace for a unitary representation is an invariant subspace. Decompose the representation, and keep on decomposing the resulting pieces. The finite-dimensionality forces the process to stop with all pieces irreducible.

The corollary has an interesting interpretation in terms of matrices. It says that the space V admits a basis in which all $\Phi(g)$ are simultaneously block diagonal matrices, and each block is an irreducible representation.

Theorem (Schur's Lemma). *Suppose Φ and Ψ are irreducible representations of G on finite-dimensional vector spaces U and V, respectively. If $L : U \to V$ is a linear map such that $\Psi(g)L = L\Phi(g)$ for all $g \in G$, then L is one-one onto or $L = 0$.*

Sketch of proof. $\ker L$ and $\operatorname{image} L$ are invariant subspaces. Sort out the possibilities.

Remark. Note that the alternative in Schur's Lemma that L is one-one onto means that Φ and Ψ are equivalent. Thus the conclusion of the result is that either $L = 0$ or L exhibits Φ and Ψ as equivalent.

Corollary. *Suppose Φ is an irreducible representation of G on a finite-dimensional vector space V. If $L : V \to V$ is a linear map such that $\Phi(g)L = L\Phi(g)$ for all $g \in G$, then L is scalar.*

Proof. Consider $L - \lambda I$ for an eigenvalue λ of L, and apply Schur's Lemma to see that $L - \lambda I = 0$.

Corollary. *For a compact abelian group, every irreducible finite-dimensional representation is one-dimensional and hence is given by a multiplicative character.*

Proof. Take $L = \Phi(g_0)$. Since $\Phi(g)L = L\Phi(g)$ for all $g \in G$, the previous corollary shows that $\Phi(g_0)$ is scalar for each $g_0 \in G$. Any one-dimensional subspace is then invariant, and irreducibility forces the whole space to be one-dimensional.

Theorem (Schur orthogonality relations).

1) *Let Φ and Ψ be inequivalent irreducible unitary representations of G on finite-dimensional vector spaces U and V, respectively, and let the understood invariant Hermitian inner products be denoted $(\,\cdot\,,\,\cdot\,)$. Then*

$$\int_G (\Phi(x)u, v)\overline{(\Psi(x)u', v')}\, dx = 0$$

for all u, v in U and u', v' in V.

2) *Let Φ be an irreducible unitary representation on a finite-dimensional vector space V, and let the understood invariant Hermitian inner product be denoted $(\,\cdot\,,\,\cdot\,)$. Then*

$$\int_G (\Phi(x)u_1, v_1)\overline{(\Phi(x)u_2, v_2)}\, dx = \frac{(u_1, u_2)\overline{(v_1, v_2)}}{\dim V}$$

for u_1, v_1, u_2, $v_2 \in V$.

The functions $(\Phi(x)u, v)$ in the above theorem are called *matrix coefficients*. According to the first conclusion of the theorem, matrix coefficients of inequivalent irreducible finite-dimensional representations are orthogonal. The prototype for this conclusion is the orthogonality of e^{imx} and e^{inx} for the circle when $m \neq n$.

The second conclusion of the theorem generalizes the fact for the circle group that each e^{inx} has L^2 norm 1. It says that $\sqrt{\dim V}\,(\Phi(x)u, v)$ has L^2 norm 1 provided the representation on V is irreducible.

At the beginning of the lecture we promised that we would introduce a notion of character for a finite-dimensional representation of dimension greater than 1, and we now come to that. The *degree* $d = d_\Phi$ of a finite-dimensional representation Φ is the dimension of the underlying vector space. The *global character* of Φ is the function

$$\chi_\Phi(x) = \operatorname{Tr}\Phi(x) = \sum_i (\Phi(x)u_i, u_i),$$

provided the u_i form an orthonormal basis of the vector space.

Proposition. *Global characters of finite-dimensional representations of G satisfy the following properties:*

(a) χ_Φ *depends only on the equivalence class of* Φ
(b) $\chi_\Phi(gxg^{-1}) = \chi_\Phi(x)$
(c) $\chi_\Phi = \chi_{\Phi_1} + \cdots + \chi_{\Phi_n}$ *if* $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_n$
(d) *For contragredients $\Phi^c(x) = \Phi(x^{-1})^t$ and tensor products $(\Phi \otimes \Psi)(x) = \Phi(x) \otimes \Psi(x)$, the characters satisfy $\chi_{\Phi^c} = \overline{\chi_\Phi}$ and $\chi_{\Phi\otimes\Psi} = \chi_\Phi\chi_\Psi$.*

Proposition. *The global character χ of an irreducible finite-dimensional representation has $\|\chi\|_2 = 1$. If χ and χ' are characters of inequivalent irreducible finite-dimensional representations, then χ and χ' are orthogonal in $L^2(G)$.*

Sketch of proof. This follows from Schur orthogonality.

Let Φ be given, and let τ be irreducible. Decompose Φ into a direct sum of irreducible representations, and let m_τ be the number of summands equivalent with τ. From the second proposition and (c) in the first proposition,

$$m_\tau = \int_G \chi_\Phi(x)\overline{\chi_\tau(x)}\,dx.$$

Therefore the *multiplicity* m_τ of τ in Φ is well defined independently of the decomposition of Φ into irreducible representations.

Theorem (Peter-Weyl Theorem). *The linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations of a compact group G is dense in $L^2(G)$.*

This is the generalization of the statement about the circle group that linear combinations of all the e^{inx} are dense in $L^2(G)$.

Corollary (Plancherel formula). *If $\{\Phi^{(\alpha)}\}$ is a maximal set of inequivalent finite-dimensional irreducible unitary representations of G and if $\{(d^{(\alpha)})^{1/2}\Phi^{(\alpha)}_{ij}\}_{i,j,\alpha}$ is a corresponding orthonormal set of matrix coefficients, then $\{(d^{(\alpha)})^{1/2}\Phi^{(\alpha)}_{ij}\}_{i,j,\alpha}$ is an orthonormal basis of $L^2(G)$.*

In the statement, $d^{(\alpha)}$ is understood to be the degree of $\Phi^{(\alpha)}$. This corollary specializes to Parseval's formula in the case of the circle group $G = \mathbb{R}/2\pi\mathbb{Z}$.

Corollary. *Any compact Lie group G has a one-one finite-dimensional representation and hence is isomorphic to a closed linear group.*

This follows by starting with a nontrivial finite-dimensional representation Φ (available from the Peter-Weyl Theorem), passing to $G/\bigcap_{x\in G}\ker\Phi(x)$, and iterating the process. The hypothesis "Lie" is used to ensure that the process terminates, and then a direct sum of the various representations of G that appear in the process is the required one-one representation.

The Peter-Weyl Theorem has important consequences for infinite-dimensional unitary representations of a compact group G. If f is integrable on G and Φ is a unitary representation, we can define a bounded operator $\Phi(f)$ on $L^2(G)$ to be a smear of the actions by the various $\Phi(x)$. The formal definition is

$$\Phi(f) = \int_G f(x)\Phi(x)\,dx.$$

We can make this precise by setting $A(u,v) = \int_G f(x)(\Phi(x)u,v)\,dx$. It is clear from the Schwarz inequality that $|A(u,v)| \le \|f\|_1\|u\|\|v\|$. Since $A(u,v)$ is linear in u and conjugate linear in v, it follows from general Hilbert space theory that A comes from a bounded linear operator whose norm is $\le \|f\|_1$. This operator is what we take as $\Phi(f)$. Thus the precise definition of $\Phi(f)$ is

$$(\Phi(f)u,v) = \int_G f(x)(\Phi(x)u,v)\,dx,$$

and we see that $\|\Phi(f)\| \leq \|f\|_1$. A handy way of using the operator $\Phi(f)$ is set up by the following lemma.

Lemma. *If Φ is a unitary representation of G and $f \geq 0$ is a function of integral 1 on G vanishing off the set N, then*

$$\|\Phi(f)v - v\| \leq \|f\|_1 \sup_{x\in N} \|\Phi(x)v - v\|$$

for all $v \in V$.

Proof. For $\|u\| \leq 1$, we observe that

$$\begin{aligned}
\|\Phi(f)v - v, u)\| &\leq \Big| \int_G f(x)[(\Phi(x)v, u) - (v, u)]\, dx \Big| \\
&\leq \int_N |f(x)| \|\Phi(x)v - v\| \|u\|\, dx \\
&\leq \|f\|_1 \sup_{x\in N} \|\Phi(x)v - v\|,
\end{aligned}$$

and we take the supremum over all such u.

The closed neighborhoods of 1 are ordered downward by inclusion, and, for each $v \in V$, $\|\Phi(x)v - v\|$ tends to 0 as N shrinks to 1 as a consequence of the continuity of Φ. The lemma allows us to adapt this statement so that it applies to the operators $\Phi(f)$. We say that a system a functions $f_N \geq 0$ on G, indexed by the closed neighborhoods of 1, is an *approximate identity* if $\int_G f_N(x)\, dx = 1$ for all N and if f_N vanishes outside N. An example is $f_N = |N|^{-1} I_N$, where $|N|$ is the measure of N and I_N is the characteristic function of N. The lemma shows that $\Phi(f_N)v \to v$ for all $v \in V$ if $\{f_N\}$ is an approximate identity.

Let us combine this circle of ideas with the Peter-Weyl Theorem.

Corollary. *Let Φ be a unitary representation of G on a Hilbert space V. Then V is the orthogonal sum of finite-dimensional irreducible invariant subspaces.*

Proof. By Zorn's Lemma choose a maximal orthogonal set of finite-dimensional irreducible invariant subspaces. Let U be the closure of the sum of these. Arguing by contradiction, we suppose that U is not all of V. Then $U^\perp$ is a nonzero closed invariant subspace. We make use of any approximate identity $\{f_N\}$ all of whose members are actually L^2 functions. Fix $v \neq 0$ in $U^\perp$, and form $\Phi(f_N)v$. Direct computation shows that $\Phi(f_N)v$ is in $U^\perp$ for every N. As N shrinks to $\{1\}$, $\Phi(f_N)v$ tends to v by the lemma; hence some $\Phi(f_N)v$ is not 0. Fix such an N.

If h is a linear combination of matrix coefficients of irreducible representations, then h lies in a finite-dimensional subspace S of $L^2(G)$ that is invariant under left translation. Let $h_1, \dots, h_n$ be a basis of this space S. Understanding that the following equalities are to be interpreted as the first entries of equalities of expressions $(\,\cdot\,, w)$, we can write

$$\begin{aligned}
\Phi(g)\Phi(h_i)v = \Phi(g) \int_G h_i(x)\Phi(x)v\, dx &= \int_G h_i(x)\Phi(gx)v\, dx \\
= \int_G h_i(g^{-1}x)\Phi(x)v\, dx &= \sum_{j=1}^n c_{ij} \int_G h_j(x)\Phi(x)v\, dx
\end{aligned}$$

for constants c_{ij} depending on g. Hence the finite-dimensional subspace $\sum_j \mathbb{C}\Phi(h_j)v$ is an invariant subspace for Φ. Consequently we will obtain a contradiction if we show that $\Phi(h)v \neq 0$ for some linear combination h of matrix coefficients.

To do this, choose h by the Peter-Weyl Theorem so that

$$\|f_N - h\|_1 \leq \|f_N - h\|_2 \leq \tfrac{1}{2}\|\Phi(f_N)v\|/\|v\|.$$

Then

$$\|\Phi(f_N)v - \Phi(h)v\| = \|\Phi(f_N - h)v\| \leq \|f_N - h\|_1\|v\| \leq \tfrac{1}{2}\|\Phi(f_N)v\|,$$

and hence

$$\|\Phi(h)v\| \geq \|\Phi(f_N)v\| - \|\Phi(f_N)v - \Phi(h)v\| \geq \tfrac{1}{2}\|\Phi(f_N)v\| > 0.$$

Thus h has the required property.

Corollary. *Every irreducible unitary representation of a compact group G is finite-dimensional.*

Induced Representations

For the group $G = SL(2,\mathbb{R})$ we were able to reformulate the principal series, given in the noncompact picture as $\mathcal{P}^{\pm,iv}$, in an "induced picture". The reformulated representations $P^{\pm,iv}$ act initially in

$$\{F \in C(G) \mid F(xman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}F(x)\}$$

with

$$P^{\pm,iv}(g)F(x) = F(g^{-1}x).$$

It will simplify the notation if we absorb the correspondence $(\pm, iv) \leftrightarrow (\sigma,\nu)$ into the notation by writing $P^{\sigma,\nu}$ for the representation in the induced picture.

Let us generalize. The general setting for "unnormalized" induction will be that G is a Lie group, H is a closed subgroup, and (σ, V) is a representation of H on a Hilbert space V. Let $C(G,V)$ denote the space of continuous functions from G into V. Initially the space and action of the *unnormalized induced representation* are given by

$$\{F \in C(G,V) \mid F(xh) = \sigma(h)^{-1}F(x)\}$$
$$U^\sigma(g)F(x) = F(g^{-1}x).$$

We notice that if F is in $C(G,V)$, then $U^\sigma(g)F$ is again in $C(G,V)$.

This definition may be regarded also as a generalization of the case that (σ, V) is the one-dimensional trivial representation. Then $C(G,V)$ reduces to the lifts to G of the continuous functions on G/H, and unnormalized induction reduces essentially to the instance of Example 2 in Lecture 1 with $X = G/H$.

Example.

Let $G = O(2)$ and $H = SO(2)$. The subgroup H of G is actually normal in G, of index 2. For $h_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in H$, define $\sigma(h_\theta) = e^{in\theta}$ on the vector space $V = \mathbb{C}$.

Our interest is in the representation U^σ obtained by inducing σ from H to G. Any F in the induced space is determined by its values on coset representatives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. So dim(induced space) $= 2$, and a basis consists of two functions F_+ and F_- satisfying

$$F_+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{and} \qquad F_+\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

$$F_-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0 \qquad \text{and} \qquad F_-\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Using that H is normal, we easily compute that the restriction of the action of U^σ to H is given by

$$\begin{aligned} U^\sigma(h_\theta)F(x) &= F(h_\theta^{-1}x) = F(x(x^{-1}h_\theta^{-1}x)) \\ &= \sigma(x^{-1}h_\theta^{-1}x)^{-1}F(x) = \sigma(x^{-1}h_\theta x)F(x). \end{aligned}$$

Therefore

$$U^\sigma(h_\theta)F_+ = e^{in\theta}F_+ \qquad \text{and} \qquad U^\sigma(h_\theta)F_- = e^{-in\theta}F_-.$$

Consequently $U^\sigma|_H = \mathbb{C}e^{in\theta} \oplus \mathbb{C}e^{-in\theta}$.

If $n \neq 0$, U^σ is irreducible. In fact, the only nontrivial invariant subspaces under H are $\mathbb{C}F_+$ and $\mathbb{C}F_-$, and these are not invariant under $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which interchanges F_+ and F_-.

If $n = 0$, U^σ is reducible. In fact, U^σ is trivial on H and therefore descends to a representation of G/H. Since G/H is abelian, a two-dimensional representation cannot be irreducible.

Let us mention that unnormalized induction has a geometric interpretation in terms of vector bundles. Historically, induced representations of Lie groups predate vector bundles, and the interpretation by means of vector bundles provided an additional perspective to the subject of induction. It is a little easier to go backwards from the historical direction, and thus let us first construct the relevant vector bundle.

In the product space $G \times V$, let us define an equivalence relation by $(gh, v) \sim (g, \sigma(h)v)$ for $h \in H$. Then set $G \times_H V = \{(g, v)/\sim\}$. Projection of the G coordinate to G/H makes $G \times_H V \to G/H$ into a vector bundle, possibly infinite-dimensional. The group G acts on sections γ of this bundle by

$$(g_0\gamma)(gH) = g_0(\gamma(g_0^{-1}gH)).$$

For instance, when (σ, V) is the trivial representation of H on $\mathbb{C}$, then $G \times_H \mathbb{C}$ may be identified with the trivial line bundle $(G/H) \times \mathbb{C}$ over G/H. The sections in the latter case are all numerical-valued functions on G/H.

In the general case let us identify the sections of $G \times_H V \to G/H$ with elements of the space of an induced representation. Let $[(g, v)]$ be the class of (g, v) in $G \times_H V$. To any section γ we can associate $\varphi_\gamma : G \to V$ by

$$\gamma(gH) = [(g, \varphi_\gamma(g))] \in G \times_H V.$$

Apart from continuity conditions, the space of φ_γ's is the space of the unnormalized induced representation, and the action of G on the γ's yields the action on the induced representation.

Let us return to a direct discussion of induced representations. An important consideration is that we would like σ unitary to imply U^σ unitary. If G/H has an invariant measure $d(xH)$, restrict F to have compact support modulo H and define

$$\|F\|^2 = \int_{G/H} |F(x)|_V^2 \, d(xH),$$

where $|\cdot|_V$ denotes the norm in V. This is well defined since $|F(x)|_V$ is well defined on G/H:

$$|F(xh)|_V^2 = |\sigma(h)^{-1} F(x)|_V^2 = |F(x)|_V^2.$$

The invariant measure exists for passing from $SO(2)$ to $O(2)$, and in this case unnormalized induction (followed by a step of completion) is what we take as induction. But it does not exist for passing from MAN to $SL(2, \mathbb{R})$, and a normalization is needed.

Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. A *left Haar measure* on G is a nonzero left-invariant Borel measure on G.

Theorem. *Any Lie group G has a left Haar measure, and any two left Haar measures are proportional.*

A similar definition and theorem apply to right Haar measures. When a left Haar measure is also a right Haar measure, we say G is *unimodular*. Not every G is unimodular, as is noted in the exercises. The theorem allows us to define the *modular function* $\Delta_G : G \to \mathbb{R}$ of G by the formula

$$d_l(\,\cdot\, t) = \Delta_G(t)^{-1} d_l(\,\cdot\,).$$

In particular, G is unimodular if and only if $\Delta_G \equiv 1$.

Proposition. $\Delta_G(t) = |\det \mathrm{Ad}(t)|$.

The proof uses top-degree differential forms on G. An important property of Δ_G is that it is a smooth homomorphism. Thus the proposition has the following corollary:

Corollary.
1) *Any compact G is unimodular.*
2) *Any semisimple G is unimodular.*

Proof of (2). Otherwise the kernel in $\mathfrak{g}$ of the differential of Δ_G would be an ideal of codimension 1 and would have a complementary ideal of dimension 1.

With more effort, one sees that any reductive or nilpotent Lie group is unimodular. In the exercises it is noted that the subgroup AN of $SL(2, \mathbb{R})$ is not unimodular. Similarly MAN is not unimodular.

Proposition. *Let H be a closed subgroup of the Lie group G. Then G/H has a nonzero left invariant Borel measure if and only if the restriction of Δ_G to H coincides with Δ_H.*

We shall note a more general result below.

Example. $SL(2,\mathbb{R})$ is unimodular, while MAN is not. Thus $SL(2,\mathbb{R})/MAN$ has no nonzero left invariant Borel measure.

When the condition of the proposition is not satisfied, the appropriate objects to integrate on G/H are not functions but instead objects called "densities" whose transformation properties incorporate a substitute for an invariant measure on G/H. The notation is as above: G is a Lie group, H is a closed subgroup, d_lg is a left Haar measure on G, and d_lh is a left Haar measure on H. Fix a continuous homomorphism $\omega : H \to \mathbb{R}^+$, and consider continuous functions $F : G \to \mathbb{C}$ having

$$F(gh) = \omega(h)^{-1}F(g) \tag{$*$}$$

Examples of such functions may be obtained by taking $f \in C_{\mathrm{com}}(G)$ and putting

$$F(g) = \int_H f(gh)\omega(h)\, d_lh. \tag{$**$}$$

Then F satisfies $(*)$ and is compactly supported modulo H.

Define $C_{\mathrm{com}}(G/H,\omega)$ as the space of continuous F on G satisfying $(*)$ and having compact support modulo H.

Proposition. *The above map $f \to F$ carries $C_{\mathrm{com}}(G)$ onto $C_{\mathrm{com}}(G/H,\omega)$.*

We omit the proof, which uses a partition of unity.

A natural attempt at defining invariant integration on $C_{\mathrm{com}}(G/H,\omega)$ is to define

$$\int_{G/H} F = \int_G f(g)\, d_lg \qquad \text{if } f \text{ maps to } F \text{ as above.}$$

If invariant integration is well defined for given ω, it is linear and left-invariant, and nonnegative F leads to $\int_{G/H} F \geq 0$.

Theorem. *Invariant integration on $C_{\mathrm{com}}(G/H,\omega)$, given as above, is well defined if and only if $\omega(h) = \Delta_G(h)^{-1}\Delta_H(h)$.*

In particular, invariant integration on $C_{\mathrm{com}}(G/H) = C_{\mathrm{com}}(G/H,1)$ exists if and only if $\Delta_G|_H = \Delta_H$. Thus the theorem generalizes an earlier proposition.

A *density* is a function $F : G \to \mathbb{C}$ satisfying $(*)$ for $\omega = \Delta_G^{-1}\Delta_H$.

Armed with the notion of a density, we can now define *normalized induction* in such a way that unitary representations induce to unitary representations. Let G be a Lie group, let H be a closed subgroup, and let (σ, V) be a unitary representation of H.

Let $F \in C(G,V)$ have compact support modulo H and satisfy $F(xh) = \omega(h)^{-1/2}\sigma(h)^{-1}F(x)$ with $\omega(h) = \Delta_G(h)^{-1}\Delta_H(h)$. Then $|F|^2$ is a density. So $\int_{G/H} |U^{\omega^{1/2}\sigma}(g)F|^2 = \int_{G/H} |F|^2$, and $U^{\omega^{1/2}\sigma}$ is unitary (after completion of the space and extension of the operators to the completion). We thus define $\mathrm{Ind}_H^G(\sigma) = U^{\omega^{1/2}\sigma}$, and the operation of normalized induction, given by Ind, carries unitary representations of H to unitary representations of G.

Normalized induction is what we use to define the principal series for general linear connected reductive groups. Thus let G be linear connected reductive, and let $Q = MAN$ be a minimal parabolic subgroup. We have $\Delta_G(x) = 1$ since G is reductive, and the general formula for the modular function gives $\Delta_Q(man) = \det \mathrm{Ad}(a)|_{\mathfrak{n}}$. Let $2\rho = \sum_{\lambda\in\Sigma^+} m_\lambda \lambda$ be the sum of the positive restricted roots, counting multiplicities. An easy computation of the determinant shows that $\Delta_Q(man) = e^{2\rho \log a}$. Then we have

$$\omega(man) = \Delta_G(man)^{-1}\Delta_Q(man) = e^{2\rho\log a}.$$

The data for a member of the principal series are an irreducible unitary representation (σ, V) of M, necessarily finite-dimensional, and a member ν of $i\mathfrak{a}'$. Then e^ν is a unitary multiplicative character of A. We define a representation $\sigma\otimes e^\nu\otimes 1$ of Q by

$$(\sigma\otimes e^\nu\otimes 1)(man) = e^\nu\sigma(m).$$

The representation obtained by normalized induction from Q to G is the corresponding member of the principal series.

Specifically the induced space initially consists of all $F \in C(G, V)$ with

$$F(xman) = e^{-(\rho+\nu)\log a}\sigma(m)^{-1}F(x).$$

The norm squared is defined by

$$\|F\|^2 = \int_{G/Q} |F|_V^2.$$

The integral here refers to integration of a density and is well defined. The action is

$$U^{\sigma,\nu}(g)F(x) = F(g^{-1}x).$$

These operators carry the initial induced space boundedly into itself, and hence they extend to the completion. The completed space, together with the extended operators, is the normalized induced representation, and it is unitary.

We write it as $U^{\sigma,\nu} = \mathrm{Ind}_{MAN}^G(\sigma\otimes e^\nu\otimes 1)$ or sometimes, by abuse of notation, as $U^{\sigma,\nu} = \mathrm{Ind}_{MAN}^G(\sigma\otimes\nu\otimes 1)$.

The formula for the norm of the principal series representation can be rewritten in terms of ordinary measures, but then it is less apparent why the representation is unitary. To do the rewriting, we make use of a general identity from measure theory: If $G = KQ$ with G unimodular and $K\cap Q$ compact, then

$$\int_G f(x)\,dx = \int_{K\times Q} f(kq)\,d_lk\,d_rq$$

in obvious notation. Let $b\in C_{\mathrm{com}}(G)$ map onto the density B. Using the above measure-theoretic identity gives

$$\begin{aligned}\int_{G/Q} B = \int_G b(x)\,dx &= \int_{K\times Q} b(kq)\,dk\,d_rq\\ &= \int_K\Big[\int_Q b(kq)\Delta_Q(q)\,d_lq\Big]dk = \int_K B(k)\,dk,\end{aligned}$$

the last equality following from $(**)$. Thus

$$\|F\|^2 = \int_K |F(k)|_V^2 \, dk.$$

For the compact picture of the principal series representation, restrict to K and use as initial space

$$\{F : C(K,V) \mid F(km) = \sigma(m)^{-1} F(k)\},$$

The norm squared is $\|F\|^2 = \int_K |F(k)|_V^2 \, dk$, but the action by G is complicated. In fact, it is already complicated for $G = SL(2,\mathbb{R})$, where the components of the Iwasawa decomposition play a role in the formula.

For the noncompact picture of the principal series representation, restrict to $\overline{N} = \Theta N$. The space is $L^2(\overline{N}, V)$, and again the action is complicated. The verification that $L^2(\overline{N}, V)$ gives the correct norm uses a variant of the above measure-theoretic identity appropriate for a decomposition $G \doteq \overline{N}MAN$. This decomposition follows from the Bruhat decomposition.

We shall now introduce "parabolic subgroups" of G, which can be used to form induced representations that generalize those in the principal series. Let us change notation and write $Q_{\mathfrak{p}} = M_{\mathfrak{p}} A_{\mathfrak{p}} N_{\mathfrak{p}}$ for the constructed minimal parabolic subgroup. A *standard parabolic subgroup* is any closed subgroup Q containing $M_{\mathfrak{p}} A_{\mathfrak{p}} N_{\mathfrak{p}}$.

Example. For $G = SL(n,\mathbb{R})$, the standard parabolic subgroups Q are all the block upper triangular subgroups, with the diagonal blocks of arbitrary sizes. The number of Q's is 2^{n-1}.

If G is semisimple, the number of standard parabolic subgroups is $2^{\dim A_{\mathfrak{p}}}$.

Any standard parabolic subgroup has a *Langlands decomposition* $Q = MAN$ obtained as follows. In considering these formulas it is helpful to bear in mind the natural decomposition of block upper triangular matrices within $SL(n,\mathbb{R})$. We define

$$\begin{aligned}
MA &= Q \cap \Theta Q \\
A &= Z_{MA} \\
\mathfrak{a} &= \text{Lie algebra of } A \\
\mathfrak{m} &= \text{orthocomplement of } \mathfrak{a} \text{ in } \mathfrak{m} \oplus \mathfrak{a} \text{ relative to } \langle \cdot\,, \cdot \rangle \\
M_0 &= \text{analytic subgroup corresponding to } \mathfrak{m} \\
M &= Z_K(A) M_0 \quad (\text{noncompact if } Q \neq Q_{\mathfrak{p}}).
\end{aligned}$$

Next we repeat the construction for restricted roots, but we use this $\mathfrak{a}$ instead of $\mathfrak{a}_{\mathfrak{p}}$. Let $\mathfrak{n}$ be the sum of the eigenspaces in $\mathfrak{q}$ for eigenvalues $\neq 0$. (No $\bar{\mathfrak{n}}$ is present, so that we may think of all eigenvalues as ≥ 0.) Let

$$\begin{aligned}
\rho &= \text{half the sum of eigenvalues with multiplicities} \\
N &= \text{analytic subgroup corresponding to } \mathfrak{n}.
\end{aligned}$$

The Langlands decomposition of a parabolic subgroup allows us to define a continuous series of representations relative to that parabolic subgroup: We start from an irreducible unitary representation (σ, V) of M and a member ν of $i\mathfrak{a}'$. The continuous series representation is the normalized induced representation $U^{\sigma,\nu} =$

$\mathrm{Ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$, which we write also as $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes \nu \otimes 1)$. Specifically we consider all $F \in C(G, V)$ with

$$F(xman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}F(x),$$

and we define

$$U^{\sigma,\nu}(g)F(x) = F(g^{-1}x).$$

Going through the step of completion, we obtain a unitary representation of G.

Notes

The abstract representation theory of compact groups is discussed in [**K2**], pp. 186–195, and in [**K1**], pp. 14–21. The two corollaries at the end of the first section of the lecture appear only in [**K1**]. One may see also [**Wal**] and [**War**] for this abstract theory.

Induced representations are discussed concretely in the context of the principal series in [**K1**], pp. 167–172. For a broader discussion, see [**War**]. The material on densities is taken from [**KV**], pp. 660–664. See also [**K2**], pp. 470–471. For Haar measure, see [**K2**], pp. 463–471. For parabolic subgroups, see [**K2**], pp. 411–421.

Exercises

1. Let χ be a multiplicative character of the circle group $\mathbb{R}/2\pi\mathbb{Z}$, and suppose χ is differentiable somewhere. Prove that χ is differentiable everywhere and that $\chi'(\theta) = \chi'(0)\chi(\theta)$. Conclude that $\chi(\theta) = e^{i\theta\chi'(0)}$.

2. Let χ be a multiplicative character of the circle group $\mathbb{R}/2\pi\mathbb{Z}$. Put $X(\theta) = \int_0^\theta \chi(t)\,dt$ and show that $X(\theta') - X(\theta) = \chi(\theta)X(\theta' - \theta)$. Conclude from this formula that χ is differentiable somewhere.

3. Find all multiplicative characters of the cyclic group $\mathbb{Z}/m$. Using the ordinary theory of Fourier series on $L^2(S^1)$ as a template, develop a Fourier theory for functions on $\mathbb{Z}/m$.

4. Find left and right Haar measures $f_l(a,b)\,da\,db$ and $f_r(a,b)\,da\,db$ for the group of all $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with real entries by using the change-of-variables formula for double integrals.

LECTURE 4
Cartan Subalgebras and Highest Weights

Roots for Compact Connected Lie Groups

Let G be a compact connected Lie group. Such a group, as we saw after the Peter-Weyl Theorem, is linear connected reductive. We may therefore regard it as a closed subgroup of some unitary group $U(n)$.

Let $\mathfrak{g}_0$ be its algebra, and let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. The sets of matrices $\mathfrak{g}_0$ and $i\mathfrak{g}_0$ are contained in the sets of skew-Hermitian matrices and Hermitian matrices, respectively, and therefore meet in 0. Consequently we can identify $\mathfrak{g}$ canonically with the set of matrices $\mathfrak{g}_0 + i\mathfrak{g}_0$ within $\mathfrak{gl}(n, \mathbb{C})$.

Using this identification, we can define the *trace form* $\langle X, Y\rangle = \operatorname{Tr}(XY^*)$ as a Hermitian inner product on $\mathfrak{g}$. This form is *invariant* under G in the sense that

$$\langle \operatorname{Ad}(g)X, \operatorname{Ad}(g)Y\rangle = \langle X, Y\rangle$$

because $g \in U(n)$ forces $(gXg^{-1})(gYg^{-1})^* = gXg^{-1}gY^*g^{-1} = gXY^*g^{-1}$.

Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{g}_0$. A little later this will be our first example of a "Cartan subalgebra". Let T be the corresponding analytic subgroup; this is a maximal torus (subgroup) in G.

We shall now repeat a construction similar to that of restricted roots. The set $\operatorname{Ad}(T)$ consists of commuting unitary transformations on $\mathfrak{g}$. Therefore $\operatorname{ad}(\mathfrak{t}_0)$ consists of commuting skew-Hermitian transformations on $\mathfrak{g}$. Let $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_0$, and let $\mathfrak{t}$ be the complexification. Then $\operatorname{ad}(\mathfrak{t}_{\mathbb{R}})$ consists of commuting Hermitian transformations on $\mathfrak{g}$. For $\alpha \in \mathfrak{t}'_{\mathbb{R}}$, let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{t}_{\mathbb{R}}\}.$$

A *root* is a nonzero α for which $\mathfrak{g}_\alpha \neq 0$. We write Δ for the set of roots. The result of our construction is that $\mathfrak{g}$ has a *root-space decomposition* given by

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

Example. Let $G = U(n)$, $\mathfrak{g}_0 = \mathfrak{u}(n)$, $\mathfrak{t}_0 = \{\text{imaginary diagonal}\}$, $\mathfrak{t} = \{\text{diagonal}\}$, $\mathfrak{t}_{\mathbb{R}} = \{\text{real diagonal}\}$. Define e_i to be evaluation of the i^{th} diagonal entry on $\mathfrak{t}_{\mathbb{R}}$. Then $\Delta = \{e_i - e_j \mid i \neq j\}$ and $\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}$.

We can transfer $\langle\,\cdot\,,\,\cdot\,\rangle$ from $\mathfrak{t}_{\mathbb{R}}$ to $\mathfrak{t}'_{\mathbb{R}}$, and we use the same notation for the inner product on $\mathfrak{t}'_{\mathbb{R}}$. In particular, $\langle\alpha,\beta\rangle$ is defined if α and β are in Δ.

Proposition. *Roots have the following properties:*

1) $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta}$.
2) *If α is in Δ, then so is $-\alpha$.*
3) *If α is in Δ, then* $\dim\mathfrak{g}_\alpha=1$.
4) *If α is in Δ, then $n\alpha$ is not in Δ for any integer ≥ 2.*
5) *Let α be in Δ, and let β be in $\Delta\cup\{0\}$. Then the α string containing β (i.e., the subset of all $\beta+n\alpha$ in $\Delta\cup\{0\}$) has the form $\beta+n\alpha$ for $-p\leq n\leq q$ with $p\geq 0$ and $q\geq 0$. There are no gaps. Also $p-q=\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$.*

Example, continued. We continue with $G=U(n)$. Then $\langle e_i,e_j\rangle=\delta_{ij}$. The last four assertions of the proposition are as follows: Conclusion (2) says that $e_i-e_j\in\Delta$ implies $e_j-e_i\in\Delta$, (3) says that $\dim\mathbb{C}E_{ij}=1$, and (4) says that $n(e_i-e_j)$ is not in Δ for $n\geq 2$. To illustrate (5), let us take $\alpha=e_i-e_j$ and $\beta=e_j-e_k$. The α string containing β is $\{\beta,\ \beta+\alpha\}$, so that $p=0$ and $q=1$. Then $p-q=-1$, and indeed

$$\frac{2\langle e_j-e_k,e_i-e_j\rangle}{|e_i-e_j|^2}=-1.$$

Let us define the notion of *root reflection* in $\mathfrak{t}'_{\mathbb{R}}$. Let α be in Δ, and define

$$s_\alpha(\varphi)=\varphi-\frac{2\langle\varphi,\alpha\rangle}{|\alpha|^2}\,\alpha$$

for $\varphi\in\mathfrak{t}'_{\mathbb{R}}$. This is an orthogonal transformation, and it follows from the above proposition that s_α carries Δ into itself. The *Weyl group* $W(\Delta)$ is the group of linear transformations of $\mathfrak{t}'_{\mathbb{R}}$ generated by the s_α. This group is finite since the permutation of Δ induced by a member of $W(\Delta)$ completely determines the member of $W(\Delta)$.

In analogy with what we did with restricted roots, we introduce a "lexicographic ordering" in $\mathfrak{t}'_{\mathbb{R}}$. Fix an ordered basis $\varphi_1,\ldots,\varphi_l$ of $\mathfrak{t}'_{\mathbb{R}}$, and say that $\varphi=\sum c_i\varphi_i$ is *positive* if the first nonzero c_i is >0. Then define $\psi>\varphi$ if $\psi-\varphi$ is positive. Let Δ^+ be the set of positive members of Δ. This is sometimes called the *positive system* for this ordering. For suitable definitions, we can arrange that

- in $SU(n)$, $\Delta^+=\{e_i-e_j\mid i<j\}$,
- in $SO(2n+1)$, $\Delta^+=\{e_i\pm e_j\mid i<j\}\cup\{e_i\}$,
- in $Sp(n)$, $\Delta^+=\{e_i\pm e_j\mid i<j\}\cup\{2e_i\}$,
- in $SO(2n)$, $\Delta^+=\{e_i\pm e_j\mid i<j\}$.

Weights for Finite-Dimensional Representations

We continue with the situation that G is a compact connected Lie group, and the notation $\mathfrak{g}_0$, $\mathfrak{g}$, $\mathfrak{t}_0$, $\mathfrak{t}_{\mathbb{R}}$, $\mathfrak{t}$, and Δ remains as above. Fix a lexicographic ordering, and let Δ^+ be the associated positive system.

Let Φ be a finite-dimensional representation of G on V. From Lecture 3 we know that we may assume that Φ is unitary without loss of generality. Let φ be the corresponding representation of $\mathfrak{g}_0$ on V. Since V is complex, we may extend φ to a complex-linear representation of $\mathfrak{g}$ on V by the definition $\varphi(X+iY)v=\varphi(X)v+i\varphi(Y)v$.

By the same argument as in the case of roots, $\Phi(T)$ unitary yields $\varphi(\mathfrak{t}_0)$ skew-Hermitian and $\varphi(\mathfrak{t}_{\mathbb{R}})$ Hermitian. For $\lambda \in \mathfrak{t}'_{\mathbb{R}}$, let

$$V_\lambda = \{v \in V \mid \varphi(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{t}_{\mathbb{R}}\}.$$

A *weight* is an element $\lambda \in \mathfrak{t}'_{\mathbb{R}}$ (possibly 0) for which $V_\lambda \neq 0$. The *highest weight* is the largest weight in the lexicographic ordering. The result of our analysis is the *weight space decomposition*

$$V = \bigoplus_\lambda V_\lambda.$$

An important elementary property of roots, weights, and weight spaces is that $\varphi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$, which follows from the Jacobi identity and the definitions.

Example 1. Let V be the complex vector space of polynomial functions in z_1 and z_2 homogeneous of degree N. This has $\{z_1^{N-k}z_2^k\}_{k=0}^N$ as basis. The group $SL(2,\mathbb{C})$ acts by

$$\left(\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right).$$

Restrict this action to $G = SU(2)$, and consider the corresponding Lie algebra action of $\mathfrak{t}_0 = \mathbb{R}i\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \mathbb{R}ih$. We can see that the basis vectors $z_1^{N-k}z_2^k$ are in fact weight vectors, as follows. Write $T = \{t_\theta\} = \left\{\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)\right\}$. Then

$$\begin{aligned}\Phi(t_\theta)(z_1^{N-k}z_2^k)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= z_1^{N-k}z_2^k\begin{pmatrix} e^{-i\theta}z_1 \\ e^{i\theta}z_2 \end{pmatrix} \\ &= e^{-i(N-k)\theta+ik\theta}z_1^{N-k}z_2^k = e^{-i(N-2k)\theta}z_1^{N-k}z_2^k.\end{aligned}$$

Differentiation gives

$$\varphi\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(z_1^{N-k}z_2^k) = -i(N-2k)(z_1^{N-k}z_2^k).$$

Thus $z_1^{N-k}z_2^k$ is a weight vector with weight $h \mapsto 2k-N$. If we take $\Delta^+ = \{e_1-e_2\}$, the highest weight is $h \mapsto N$.

Let us examine the structure of this representation further. For $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, the identity $[h,e] = 2e$ implies that $\varphi(e)$ sends a weight vector of one weight $2k-N$ to the next higher weight $2(k+1)-N$; in fact, if v has weight $2k-N$, then

$$\begin{aligned}\varphi(h)(\varphi(e)v) &= \varphi(e)\varphi(h)v + [\varphi(h),\varphi(e)]v \\ &= (2k-N)\varphi(e)v + \varphi[h,e]v = (2(k+1)-N)\varphi(e)v.\end{aligned}$$

Similarly $\varphi(f)$ sends the weight vector to a weight vector of the next lower weight since $[h,f] = -2f$. It is easy to see from this that φ is irreducible. Hence so is Φ.

One can prove that there are no other irreducible finite-dimensional complex-linear representations of $\mathfrak{sl}(2,\mathbb{C})$. Consequently the irreducible finite-dimensional complex-linear representations of $\mathfrak{sl}(2,\mathbb{C})$ are determined by their highest weights, which are of the form $h \mapsto N$ with $N \geq 0$ integral.

Example 2. Let V be the complex vector space of polynomial functions in $z_1, \ldots, z_n$ homogeneous of degree N. This has the vectors $z_1^{k_1} \cdots z_n^{k_n}$ with $\sum_j k_j = N$ as basis. The group $GL(n, \mathbb{C})$ acts in a fashion similar to that in Example 1. Restrict this action to $U(n)$, and consider the corresponding Lie algebra action by $\mathfrak{t}_0 = \{\text{imaginary diagonal}\}$. The basis vectors $z_1^{k_1} \cdots z_n^{k_n}$ with $\sum_j k_j = N$ can be seen to be weight vectors, with respective weights $-\sum_j k_j e_j$. With Δ^+ taken as $\{e_i - e_j \mid i < j\}$, the highest weight is $-Ne_n$. For each N, this representation can be shown to be irreducible.

These examples are special cases of the following general result, whose proof will be sketched below.

Theorem (Theorem of the Highest Weight). *An irreducible finite-dimensional representation of G is characterized up to equivalence by its highest weight, say λ, which is dominant (i.e., $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+$) and analytically integral (i.e., λ is the differential of a multiplicative character of T). The weight space for the highest weight has dimension 1, and all other weights are obtained from the highest weight by subtracting nonnegative integer combinations of positive roots from it. Conversely any dominant, analytically integral member of $i\mathfrak{t}'_{\mathbb{R}}$ is the highest weight of some irreducible finite-dimensional representation of G.*

Universal Enveloping Algebra

The universal enveloping algebra is a tool we can use to understand the mechanism for the Theorem of the Highest Weight. To construct it, we proceed as follows. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra The tensor algebra of $\mathfrak{g}$ is

$$T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots .$$

This is an associative algebra with identity, the product being simply tensor product. The *universal enveloping algebra* is defined as the quotient $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the two-sided ideal generated by all $X \otimes Y - Y \otimes X - [X, Y]$ with X and Y in $\mathfrak{g}$. The universal enveloping algebra is an associative algebra with identity; the tensor sign is dropped in the multiplication. We let ι be the canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ obtained by inclusion into first-order tensors and then passage to the quotient modulo I. It is not really clear at first that $U(\mathfrak{g})$ contains anything other than the constants and in particular that ι is one-one.

Proposition (universal property of $U(\mathfrak{g})$ and ι). *Whenever A is a complex associative algebra with identity and $\pi : \mathfrak{g} \to A$ is a linear mapping such that $\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi[X, Y]$ for all X and Y in $\mathfrak{g}$, then there exists a unique algebra homomorphism $\widetilde{\pi} : U(\mathfrak{g}) \to A$ such that $\widetilde{\pi}(1) = 1$ and $\widetilde{\pi} \circ \iota = \pi$.*

This is fairly easy to prove, starting from the corresponding universal property of the tensor algebra.

Corollary. *Complex-linear representations of $\mathfrak{g}$ on complex vector spaces stand in one-one correspondence with unital left $U(\mathfrak{g})$ modules. Here unital means that 1 acts as 1.*

Sketch of proof. If π is a representation of $\mathfrak{g}$ on V, apply the universal property to $\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}} V$ and define $uv = \widetilde{\pi}(u)v$.

Proposition. *There exists a unique antiautomorphism* $u \mapsto u^t$ *of* $U(\mathfrak{g})$ *such that* $\iota(X)^t = -\iota(X)$ *for all* $X \in \mathfrak{g}$.

This antiautomorphism is called *transpose*. It enables one to convert any left $U(\mathfrak{g})$ module into a right $U(\mathfrak{g})$ module by $vu = u^t v$ for $u \in U(\mathfrak{g})$ and $v \in V$. In a later lecture we shall see that it plays a role in the subject of differential equations.

Theorem (Poincaré-Birkhoff-Witt Theorem). *Let* $\{X_1, \dots, X_n\}$ *be an ordered basis of* $\mathfrak{g}$. *Then the set of all monomials* $(\iota X_1)^{j_1} \cdots (\iota X_n)^{j_n}$ *with all* $j_k \geq 0$ *is a basis of* $U(\mathfrak{g})$. *In particular the canonical map* ι *is one-one (and can be dropped from the notation).*

This is the serious result, the linear independence being the hard part. It follows that if $\mathfrak{h}$ is a complex Lie subalgebra of $\mathfrak{g}$, then $U(\mathfrak{h})$ can be regarded as an associative subalgebra of $U(\mathfrak{g})$. This identification is very important for being able to work effectively with $U(\mathfrak{g})$.

Mechanism for Theorem of the Highest Weight

Now let us consider the idea behind the Theorem of the Highest Weight. We continue with the notation G, $\mathfrak{g}_0$, $\mathfrak{g}$, $\mathfrak{t}$, Δ, Δ^+, Φ on V, and φ on V.

We make the following construction: Let $\beta_1, \dots, \beta_k$ be an enumeration of Δ^+, and let $H_1, \dots, H_l$ be a basis of $\mathfrak{t}$. The Poincaré-Birkhoff-Witt Theorem implies that all elements of the form

$$(*) \qquad E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l} E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k}$$

form a basis of $U(\mathfrak{g})$. Using φ, we make V into a unital left $U(\mathfrak{g})$ module.

Now let λ be the highest weight of φ, and let v be a nonzero highest weight vector. Let us see the main ideas behind the proof of the Theorem of the Highest Weight.

We begin with the easy parts. Assume that φ irreducible, and apply $(*)$ to v. Any $E_{\beta_j}^{p_j}$ gives 0 if $p_j > 0$. If all $p_j = 0$, H_j gives a factor $\lambda(H_j)$. Then the $E_{-\beta_j}$ push weights down. So the only vectors of weight λ in $U(\mathfrak{g})v$ are $\mathbb{C}v$, and the only weights are λ minus combinations of positive roots. But $U(\mathfrak{g})v = V$ by irreducibility. So our conclusions about $U(\mathfrak{g})v$ are valid for V.

Next we consider the dominance of λ. If α is in Δ^+, then E_α, $E_{-\alpha}$, and $[E_\alpha, E_{-\alpha}]$ generate a copy of $\mathfrak{sl}(2,\mathbb{C})$, and $\frac{2\langle\lambda,\alpha\rangle}{|\alpha|^2}$ is the weight of v under the element h of $\mathfrak{sl}(2,\mathbb{C})$. One uses that the weights of any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ are closed under negatives. Then $\langle\lambda,\alpha\rangle \geq 0$.

Now let us consider uniqueness. Suppose (φ_1, V_1) and (φ_2, V_2) are given irreducible representations, λ is the common highest weight, and v_1 and v_2 are nonzero highest weight vectors. Form $S = (\varphi_1 \oplus \varphi_2)(U(\mathfrak{g}))(v_1 \oplus v_2)$. One uses $U(\mathfrak{g})$ to check that this is irreducible. Apply Schur's Lemma to the projection of S to V_1 to see that S is equivalent with V_1. In similar fashion S is equivalent with V_2. Therefore V_1 and V_2 are equivalent with each other.

The main step in the proof of the Theorem of the Highest Weight is to prove existence. We give the idea of the proof when G is simply connected, omitting the general case. Define

$$\begin{aligned}
\mathfrak{n}^+ &= \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \\
\mathfrak{n}^- &= \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \\
\mathfrak{b} &= \mathfrak{t}\oplus\mathfrak{n}^+ \\
\delta &= \tfrac{1}{2}\sum_{\alpha\in\Delta^+}\alpha.
\end{aligned}$$

Here $\mathfrak{n}^+$, $\mathfrak{n}^-$, and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{g}$, and $\mathfrak{b}$ is called a *Borel subalgebra* of $\mathfrak{g}$. Form the *Verma module*

$$V(\lambda+\delta) = U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_\lambda.$$

Under the action on the left of the first factor, this is an infinite-dimensional $U(\mathfrak{g})$ module with a basis of weight vectors (namely all $E_{-\beta_1}^{q_1}\cdots E_{-\beta_k}^{q_k}\otimes 1$), and $1\otimes 1$ is a weight vector for the highest weight λ. The sum of all $U(\mathfrak{g})$ submodules not meeting $\mathbb{C}(1\otimes 1)$ is a maximal proper $U(\mathfrak{g})$ submodule $M(\lambda+\delta)$, and $L(\lambda+\delta) = V(\lambda+\delta)/M(\lambda+\delta)$ is thus irreducible. For λ dominant integral, one proves that $L(\lambda+\delta)$ is finite-dimensional. Restricting the action from $\mathfrak{g}$ to $\mathfrak{g}_0$ and using simple connectivity of G, we obtain the required representation of G.

Role of Complex Semisimple Lie Algebras

The above theory really concerns irreducible finite-dimensional representations of arbitrary complex semisimple Lie algebras $\mathfrak{g}$. We saw how to pass from a compact G to its complexified Lie algebra, and this passage is valid in particular when G is semisimple. To see the reverse direction, we have only to apply the following two theorems.

Theorem (Cartan). *If $\mathfrak{g}$ is complex semisimple, then there exists a compact Lie group G whose complexified Lie algebra is isomorphic to $\mathfrak{g}$.*

Theorem (Weyl's Theorem). *The universal covering group of a compact semisimple Lie group is compact.*

Thus the group in Cartan's theorem may, without loss of generality, be taken to be simply connected.

There is, however, a direct approach. One can start with a complex semisimple Lie algebra $\mathfrak{g}$ and define a *Cartan subalgebra* to be a nilpotent subalgebra that is equal to its own normalizer. With some effort one shows that Cartan subalgebras exist, are all conjugate, are abelian, and act diagonalizably in every finite-dimensional representation. The theory proceeds by exploiting copies of $\mathfrak{sl}(2,\mathbb{C})$ lying in $\mathfrak{g}$. We omit the details.

Cartan Subalgebras in the Noncompact Case

We conclude this lecture by discussing Cartan subalgebras within the Lie algebra of a general linear connected reductive group.

Let G be a linear connected reductive, let $\mathfrak{g}_0 = \mathfrak{k}_0\oplus\mathfrak{p}_0$ be a Cartan decomposition of its Lie algebra, and let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$.

For orientation we first consider the case that G is compact. In this case a *Cartan subalgebra* is nothing more than a maximal abelian subspace of $\mathfrak{g}_0$. Any two are conjugate via $\mathrm{Ad}(G)$. The complexification $\mathfrak{t}$ of $\mathfrak{t}_0$ is then a Cartan subalgebra of $\mathfrak{g}$.

Now we consider the case of a general G. A *Cartan subalgebra* $\mathfrak{h}_0$ is a subalgebra of $\mathfrak{g}_0$ whose complexification in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$. We are interested at the moment only in those that are stable under the Cartan involution. Then $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$.

In view of what happens with compact groups, it may be an unpleasant surprise to realize that in a noncompact reductive group, the Cartan subalgebras are not necessarily all conjugate via $\mathrm{Ad}(G)$. For example, in the case of $SL(2,\mathbb{R})$, $\mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are nonconjugate Cartan subalgebras.

On the other hand, the Cartan subalgebras stable under the Cartan involution have the following helpful properties:

- Any two have the same dimension (since their complexifications are conjugate within $\mathfrak{g}$).
- There are only finitely many, up to conjugacy by $\mathrm{Ad}(K)$ (or equivalently $\mathrm{Ad}(G)$).
- There is a unique conjugacy class for which $\dim \mathfrak{t}_0$ is maximal. Namely let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$, and put $\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$. We say $\mathfrak{h}_0$ is *maximally compact*. If $\mathfrak{a}_0 = 0$, we say $\mathfrak{h}_0$ is *compact*.
- There is a unique class for which $\dim \mathfrak{a}_0$ is maximal. Namely let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$, and let $\mathfrak{t}_0$ be maximal abelian in $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$. We say $\mathfrak{h}_0$ is *maximally noncompact*.

Notes

The theory of Cartan subalgebras, roots, and weights, done directly in the context of compact connected Lie groups, may be found in [**K1**], Chapter IV. Many books develop roots and weights in the way described in the section of the lecture called "Role of Complex Semisimple Lie Algebras"—namely by starting from a complex semisimple Lie algebra with no compact group in sight. This approach may be found in [**K2**], Chapters II and V, as well as in [**J**] and [**V**]. A variation of this approach to roots and weights may be found in [**Hu**], and a different variation, used to develop roots but not weights, appears in [**He**].

The book [**K2**] obtains the theory of roots and weights for compact connected Lie groups as a consequence of the abstract theory for complex semisimple Lie algebras. See Chapter IV and pp. 277–283.

The theory of universal enveloping algebras is developed in detail in Chapter III of [**K2**], as well as in [**He**], [**Hu**], [**J**], and [**V**].

For Cartan subalgebras in the noncompact real case, see [**K2**], pp. 326–330 and p. 396. See also [**K1**], pp. 128–132.

Exercises

1. Let Φ be a finite-dimensional representation of a compact group G with highest weight λ. Show that the contragredient Φ^c of Φ, defined by $\Phi^c(g) = \Phi(g^{-1})^t$, has lowest weight $-\lambda$.

2. For $G = U(3)$, show that every irreducible finite-dimensional representation is of the form $(\det)^n \otimes \Phi_N$ or $(\det)^n \otimes \Phi_N^c$ for some integer n and natural number N, where Φ_N is the change-of-coordinates representation on homogeneous polynomials of degree N in 3 variables.

3. For $G = SL(n, \mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R})$, find a maximally compact θ stable Cartan subalgebra, and a maximally noncompact one.

4. Repeat Exercise 3 for $SL(n, \mathbb{C})$.

5. Suppose that $\mathfrak{g}$ is an abelian complex Lie algebra, and suppose A is a commutative associative algebra with identity. Prove that any linear map $\lambda : \mathfrak{g} \to A$ extends uniquely to an algebra homomorphism of $U(\mathfrak{g})$ into A carrying 1 into 1. In particular, any linear map $\lambda : \mathfrak{g} \to U(\mathfrak{g})$ extends uniquely to an algebra homomorphism of $U(\mathfrak{g})$ into itself carrying 1 into 1.

LECTURE 5
Action by the Lie Algebra

Harish-Chandra Isomorphism

In an irreducible finite-dimensional representation, Schur's Lemma implies that operators commuting with the representation will act by scalars. An infinite-dimensional generalization of Schur's Lemma due to Dixmier will be given later in this lecture. That generalization leads us to expect that suitable kinds of operators commuting with an infinite-dimensional representation will act by scalars as well. Accordingly we seek an understanding of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

We are going to take advantage of the fact noted in Lecture 4 that the theory of roots can be developed for complex semisimple Lie algebras without the use of an underlying Lie group. Our notation for this section is as follows: Let $\mathfrak{g}$ be a complex semisimple (or even reductive) Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra. Let $\mathfrak{h}_{\mathbb{R}}$ be a real form of $\mathfrak{h}$ such that roots lie in $\mathfrak{h}'_{\mathbb{R}}$; this is uniquely determined if $\mathfrak{g}$ is semisimple. We assume that $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}'_{\mathbb{R}}$ are identified by an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ built from $\mathfrak{g}$; if $\mathfrak{g}$ is semisimple, this can be obtained from the Killing form, for example. Let $\mathcal{H} = U(\mathfrak{h})$. This coincides with the symmetric algebra $S(\mathfrak{h})$ of $\mathfrak{h}$, and we may identify $U(\mathfrak{h})$ with a subalgebra of $U(\mathfrak{g})$ by the Poincaré-Birkhoff-Witt Theorem, as we noted in Lecture 4. We continue with the notation Δ, $W = W(\Delta)$, Δ^+, $\mathfrak{n}^+$, $\mathfrak{n}^-$, $\mathfrak{b}$, and δ as in Lecture 4.

The group W acts on $\mathfrak{h}'$, hence on $\mathfrak{h}$ via the definition $w\lambda(H) = \lambda(w^{-1}H)$. The action of any member of W, regarded as carrying $\mathfrak{h}$ to $\mathcal{H}$, extends to carry $\mathcal{H}$ to $\mathcal{H}$ because the universal property of the commutative $U(\mathfrak{h})$ says that any linear $w : \mathfrak{h} \to \mathcal{H}$ extends to an algebra homomorphism. The result is a group action of W on $\mathcal{H}$ by algebra automorphisms carrying 1 to 1. Let $\mathcal{H}^W$ be the subalgebra of W invariants in $\mathcal{H}$.

Let us consider the effect of $z \in Z(\mathfrak{g})$ on representations. Let (φ, V) be an irreducible (complex-linear) finite-dimensional representation of $\mathfrak{g}$. Extend the action to $U(\mathfrak{g})$. Then z acts as a scalar, by Schur's Lemma. We compute the scalar by examining how z acts on a nonzero highest weight vector, say v of weight λ. Apply the Poincaré-Birkhoff-Witt Theorem, and expand z in terms of the basis

$$E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l} E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k}.$$

For a term T as above that occurs in the expansion of z with nonzero coefficient,

$$HT - TH = \big(\sum (p_j - q_j)\beta_j(H)\big)T$$

for all $H \in \mathfrak{h}$. Since the T's are independent and since z central forces $Hz - zH = 0$, we must have

$$(*) \qquad \sum (p_j - q_j)\beta_j(H) = 0 \qquad \text{for all } H.$$

Now apply z and each of its monomials T to the highest weight vector v. If some p_j is $\neq 0$, we get 0 since $E_{\beta_i}v = 0$ for all i. If all p_j are 0, then $(*)$ says that all q_j are 0. So the monomial reduces to $H_1^{m_1} \cdots H_l^{m_l}$, which acts on v by

$$\lambda(H_1)^{m_1} \cdots \lambda(H_l)^{m_l}.$$

Thus to understand the action of $z \in Z(\mathfrak{g})$ on a highest weight vector, we want to pick out the terms with just H's present.

To have notation for doing so, we introduce

$$\mathcal{P} = \sum_{\alpha \in \Delta^+} U(\mathfrak{g})E_\alpha,$$
$$\mathcal{N} = \sum_{\alpha \in \Delta^+} E_{-\alpha}U(\mathfrak{g})$$

Proposition.

1) $U(\mathfrak{g}) = \mathcal{H} \oplus (\mathcal{P} + \mathcal{N})$
2) *Any member of* $Z(\mathfrak{g})$ *has its* $\mathcal{P} + \mathcal{N}$ *component in* $\mathcal{P}$.

Sketch of proof. (1) is basically the Poincaré-Birkhoff-Witt Theorem, and (2) was shown above.

We define $\gamma'_{\mathfrak{n}^+}$ to be the projection of $Z(\mathfrak{g})$ into the $\mathcal{H}$ term in (1).

Interpretation 1. $\lambda(\gamma'_{\mathfrak{n}^+}(z))$ is the scalar by which z acts in an irreducible finite-dimensional representation with highest weight λ.

Interpretation 2. $\lambda(\gamma'_{\mathfrak{n}^+}(z))$ is the scalar by which z acts in the Verma module

$$V(\lambda + \delta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

Proofs. In fact, the computation above shows that z acts on a highest weight vector of weight λ by the scalar $\lambda(\gamma'_{\mathfrak{n}^+}(z))$. Since z acts by a scalar in any irreducible representation, we arrive at Interpretation 1. A similar argument leads to Interpretation 2.

Example. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the element $z = \frac{1}{2}h^2 + ef + fe$ is seen by direct calculation to be in $Z(\mathfrak{g})$. Apart from a scalar factor, this element is called the *Casimir element* of $U(\mathfrak{sl}(2, \mathbb{C}))$. Let us take $\Delta^+ = \{e_1 - e_2\}$. Then $ef = fe + [e, f] = fe + h$. So

$$z = (\tfrac{1}{2}h^2 + h) + 2fe \in \mathcal{H} \oplus \mathcal{P}$$

and

$$\gamma'_{\mathfrak{n}^+}(z) = \tfrac{1}{2}h^2 + h.$$

A slight adjustment to $\gamma'_{\mathfrak{n}^+}$ results in an object with better symmetry properties. Define a linear map $\tau_{\mathfrak{n}^+} : \mathfrak{h} \to \mathcal{H}$ by

$$\tau_{\mathfrak{n}^+}(H) = H - \delta(H)1,$$

and extend it to an algebra automorphism of $\mathcal{H}$ by the universal property of $\mathcal{H}$. Then define

$$\gamma = \tau_{\mathfrak{n}^+} \circ \gamma'_{\mathfrak{n}^+}$$

as a map of $Z(\mathfrak{g})$ into $\mathcal{H}$.

Example. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, let $z = \frac{1}{2}h^2 + ef + fe$ as above, so that $\gamma'_{\mathfrak{n}^+}(z) = \frac{1}{2}h^2 + h$. Then $\delta(h) = \frac{1}{2}\alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1$, and so

$$\tau_{\mathfrak{n}^+}(h) = h - 1.$$

Thus

$$\gamma(z) = \tfrac{1}{2}(h-1)^2 + (h-1) = \tfrac{1}{2}h^2 - \tfrac{1}{2}.$$

The improved symmetry property is that this is symmetric under the Weyl group action $h \mapsto -h$.

Theorem (Harish-Chandra). *The mapping γ is an algebra isomorphism of $Z(\mathfrak{g})$ onto the algebra $\mathcal{H}^W$ of Weyl-group invariants, and it does not depend on the choice of the positive system Δ^+.*

The map $\gamma : Z(\mathfrak{g}) \to \mathcal{H}^W$ is called the *Harish-Chandra isomorphism*. The hard part of the proof is to show that γ is onto $\mathcal{H}^W$. This requires, one way or another, the production of many elements of $Z(\mathfrak{g})$. After the theorem has been proved, one can regard the statement that $\gamma : Z(\mathfrak{g}) \to \mathcal{H}^W$ is onto as encoding the fact that $Z(\mathfrak{g})$ is large.

Infinitesimal Character

Here is the result mentioned at the beginning of the lecture.

Proposition (Dixmier, generalizing Schur). *If V is an irreducible $U(\mathfrak{g})$ module (possibly infinite-dimensional), then the only linear maps of V to itself commuting with $U(\mathfrak{g})$ are the scalars.*

Corollary. *If V is an irreducible $U(\mathfrak{g})$ module, then $Z(\mathfrak{g})$ acts by scalars in V. Say the element z of $Z(\mathfrak{g})$ acts by $\chi(z)$. Then χ is an algebra homomorphism of $Z(\mathfrak{g})$ into $\mathbb{C}$ sending 1 into 1.*

We can construct a concrete family of homomorphisms of $Z(\mathfrak{g})$ into $\mathbb{C}$ as follows. Fix $\lambda \in \mathfrak{h}'$, and let $\chi_\lambda(z) = \lambda(\gamma(z))$, where γ is the Harish-Chandra isomorphism. On the right side it is understood that λ has been extended, via the universal property, to an algebra homomorphism of $\mathcal{H}$ into $\mathbb{C}$. Then χ_λ is the composition of homomorphisms and hence is a homomorphism.

Example. In $\mathfrak{sl}(2, \mathbb{C})$ with $z = \frac{1}{2}h^2 + ef + fe$, we know that $\gamma(z) = \frac{1}{2}h^2 - \frac{1}{2}$. Then $\chi_\lambda(z) = \frac{1}{2}\lambda(h)^2 - \frac{1}{2}$. A special case is that $\chi_\delta(z) = 0$.

Proposition. *If λ_1 and λ_2 are in $\mathfrak{h}'$, then $\chi_{\lambda_1} = \chi_{\lambda_2}$ if and only if λ_1 and λ_2 are in the same orbit under the Weyl group W.*

Theorem. *Every homomorphism χ of $Z(\mathfrak{g})$ into $\mathbb{C}$ sending 1 into 1 is of the form $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}'$.*

If $Z(\mathfrak{g})$ acts by scalars χ in a $U(\mathfrak{g})$ module V, the *infinitesimal character* of V is defined to be the parameter $\lambda \in \mathfrak{h}'$ such that $\chi = \chi_\lambda$. This parameter is determined up to the action of W.

The significance is that the infinitesimal character is a nontrivial invariant for an irreducible $U(\mathfrak{g})$ module. It provides a small step toward classification.

Smooth and Analytic Vectors

In Lecture 1 we touched on smooth and analytic vectors for the principal series of $SL(2,\mathbb{R})$. In these sections we generalize these notions. Let G be any Lie group, let $\mathfrak{g}_0$ be its Lie algebra, and let $\mathfrak{g}$ be the complefication of $\mathfrak{g}_0$.

Let Φ be a representation of G on a Hilbert space $\mathcal{V}$. We say that $v \in \mathcal{V}$ is a C^∞ *vector* if $x \mapsto \Phi(x)v$ is a C^∞ function. More precisely, the assumption is that in local coordinates, vector-valued partial derivatives exist for all orders. It is known that it is sufficient to assume that $x \mapsto (\Phi(x)v, w)$ is a C^∞ function for all $w \in \mathcal{V}$.

We say that $v \in \mathcal{V}$ is an *analytic vector* if $x \mapsto \Phi(x)v$ is a real analytic function. More precisely, the assumption is that $x \mapsto \Phi(x)v$ has, for each $x \in G$, a locally convergent vector-valued power series expansion in some coordinate neighborhood of x. It is known that it is sufficient to assume that $x \mapsto (\Phi(x)v, w)$ is a real analytic function for all $w \in \mathcal{V}$.

Examples.

1) In a finite-dimensional representation, every vector is C^∞ and actually analytic.

2) We know that the principal series representation $\mathrm{Ind}_{MAN}^G(\sigma\otimes\nu\otimes 1)$ is unitary if (σ, V) is unitary and if ν is imaginary-valued on $\mathfrak{a}$. More generally the definition of the induced space makes sense for any complex-valued ν on $\mathfrak{a}$, and we get a (continuous) representation in the Hilbert space

$$\{F \in L^2(K, V) \mid F(km) = \sigma(m)^{-1}F(k)\}.$$

This is called a *nonunitary principal series* representation. For this representation the C^∞ vectors are the C^∞ functions on K within the space, and the analytic vectors are the real analytic functions.

We denote the vector spaces of C^∞ and analytic vectors in $\mathcal{V}$ by $C^\infty(\mathcal{V})$ and $C^\omega(\mathcal{V})$, respectively. For general representations it is not immediately clear whether these are 0. It will turn out that they are actually dense.

Let us define corresponding representations of $\mathfrak{g}_0$ on $C^\infty(\mathcal{V})$ and $C^\omega(\mathcal{V})$. If v is in $C^\infty(\mathcal{V})$ and X is in $\mathfrak{g}_0$, set

$$\varphi(X)v = \frac{d}{dt}\,\Phi(\exp tX)v|_{t=0}.$$

Then $\varphi(X)v$ is in $C^\infty(\mathcal{V})$ and φ is a representation of $\mathfrak{g}_0$ on $C^\infty(\mathcal{V})$. No topology is needed on $C^\infty(\mathcal{V})$. The subspace $C^\omega(\mathcal{V})$ is an invariant subspace. Thus $C^\infty(\mathcal{V})$ and $C^\omega(\mathcal{V})$ become $U(\mathfrak{g})$ modules.

Let us discuss elementary properties of these representations. If $\mathcal{U}$ is a closed G invariant subspace, then $C^\infty(\mathcal{U}) = \mathcal{U} \cap C^\infty(\mathcal{V})$ and $C^\omega(\mathcal{U}) = \mathcal{U} \cap C^\omega(\mathcal{V})$.

These identities tell how to pass from Lie group representations to Lie algebra representations.

We will be interested in knowing how much information is lost in this process. For example, if it could happen that $C^\infty(\mathcal{V})$ or $C^\omega(\mathcal{V})$ is 0, then everything is lost. In any event we do know that if S is a $U(\mathfrak{g})$ invariant subspace of $C^\infty(\mathcal{V})$, then $\overline{S}$ need not be G invariant. A counterexample was given in Lecture 1.

By contrast, a relatively simple argument shows that if S is a $U(\mathfrak{g})$ invariant subspace of $C^\omega(\mathcal{V})$, then $\overline{S}$ is G invariant. This is the special feature, for our purposes, of analyticity.

Theorem (Gårding). *$C^\infty(\mathcal{V})$ is dense in $\mathcal{V}$.*

This is a relatively easy theorem with the tools we have. First let us assume that the group representation is unitary. For arbitrary $v \in V$ and $f \in C^\infty_{\text{com}}(G)$, define

$$\Phi(f)v = \int_G f(x)\Phi(x)v\,dx$$

as in Lecture 3. An easy computation shows that $\Phi(f)v$ is in $C^\infty(\mathcal{V})$. As f runs through an approximate identity, $\Phi(f)v$ tends to v. Hence $C^\infty(\mathcal{V})$ is dense. If the group representation is not unitary, one reviews this argument to see that it is still valid.

Theorem (Harish-Chandra, Nelson). *$C^\omega(\mathcal{V})$ is dense in $\mathcal{V}$.*

This theorem is much harder. The tool in the unitary case is invariant elliptic differential operators on G.

Before applying these theorems to reductive groups, we make some preliminary remarks about a compact group K. Let Φ be a unitary representation of K on a Hilbert space V. We saw as a corollary of the Peter-Weyl Theorem that V is an orthogonal Hilbert space direct sum of finite-dimensional K invariant subspaces. Denote

$$\widehat{K} = \left\{\begin{array}{l}\text{equivalence classes of irreducible}\\ \text{finite-dimensional representations}\\ \text{of } K\end{array}\right\}.$$

For each $\tau \in \widehat{K}$, let V_τ be the sum of all irreducible invariant subspaces of type τ. Then $V = \sum V_\tau$ orthogonally as a Hilbert space sum, and a relatively easy computation shows that the orthogonal projection on V_τ is given by

$$E_\tau v = d_\tau \int_K \overline{\chi_\tau(k)}\Phi(k)v\,dk.$$

Therefore the *multiplicity* of τ in V is well defined as $\dim V_\tau / \dim \tau$.

Theorem (Frobenius reciprocity). *Let K be a compact group, let L be a closed subgroup, let σ be an irreducible unitary representation of L, and let τ be an irreducible unitary representation of K. Then the multiplicity of τ in $\operatorname{Ind}_L^K(\sigma)$ equals the multiplicity of σ in the restriction of τ to L.*

Sketch of proof. The first multiplicity equals the dimension of the space of K commuting linear maps from the space for τ into the induced space, and the second multiplicity equals the dimension of the space of L commuting linear maps from the space of σ into the space of τ. Composition of maps of the first kind with evaluation at the identity yields maps of the second kind, and this correspondence can be shown to be one-one onto.

Application to Reductive Groups

Now let us specialize the above setting. We return to the situation that G is linear connected reductive, and we let $\mathfrak{g}_0$, $\mathfrak{g}$, K, $\mathfrak{k}_0$, etc., be as earlier. All representations of G will be assumed to be in Hilbert spaces with K acting unitarily.

Theorem. *If (Φ, V) is an irreducible unitary representation of G, then every $\tau \in \widehat{K}$ has finite multiplicity in V.*

A representation (Φ, V) is said to be *admissible* if every $\tau \in \widehat{K}$ has finite multiplicity in V. In this terminology the above theorem says that irreducible unitary representations are admissible.

There is a second class of representations that we can identify right away as admissible:

Proposition. *If MAN is parabolic in G and σ is an admissible representation of M (with $K \cap M$ acting unitarily), then $\operatorname{Ind}_{MAN}^G(\sigma \otimes \nu \otimes 1)$ is an admissible representation of G.*

This is an easy consequence of Frobenius reciprocity as stated above. We work with the compact picture, so that the restriction to K of the given induced representation can be identified with an induced representation from $K \cap M$ to K. Then we put $L = K \cap M$, apply the theorem to each irreducible summand of the restriction of σ to $K \cap M$, and add the results.

Proposition.

1) *For each $\tau \in \widehat{K}$, $C^\infty(V) \cap V_\tau$ is dense in V_τ and $C^\omega(V) \cap V_\tau$ is dense in V_τ.*

2) *Write $V_K = \bigoplus_{\tau \in \widehat{K}} V_\tau$ for the algebraic direct sum of the V_τ. Then the spaces $C^\infty(V) \cap V_K$ and $C^\omega(V) \cap V_K$ are invariant under $\mathfrak{g}_0$ and hence under $U(\mathfrak{g})$.*

This proposition follows readily from the theorems of Garding and Harish-Chandra–Nelson about denseness of $C^\infty(V)$ and $C^\omega(V)$. The members of $V_K = \bigoplus_{\tau \in \widehat{K}} V_\tau$ are called *K finite vectors.*

Corollary 1. *If (Φ, V) is admissible, then every K finite vector is analytic.*

This is immediate from (1) in the proposition.

Corollary 2. *If (Φ, V) is admissible, then the closed G invariant subspaces W of V stand in one-one correspondence with the $U(\mathfrak{g})$ invariant subspaces of V_K, the correspondence $W \leftrightarrow S$ being*

$$S = W_K \quad \textit{and} \quad W = \overline{S}.$$

This is a powerful result, saying that the Lie algebra representation captures the full information of the group representation in the admissible case.

For an admissible representation (Φ, V), one often works with only the space V_K of K finite vectors. This carries a $U(\mathfrak{g})$ module action and a K action, which satisfy

(a) every member of V_K lies in a finite-dimensional space on which K acts by a (continuous) representation,
(b) the differentiated version of the K action is the restriction to $\mathfrak{k}_0$ of the $\mathfrak{g}$ action,
(c) $(\operatorname{Ad}(k)u)v = k(u(k^{-1}x))$ for $k \in K$, $u \in U(\mathfrak{g})$, $v \in V$.

A complex vector space with a left $U(\mathfrak{g})$ module structure and a K action such that (a), (b), and (c) hold is called a $(\mathfrak{g}, K)$ *module.* (Condition (c) is automatic for G connected and is included so that the definition will be applicable in the disconnected case.)

Infinitesimal equivalence of two representations of G means algebraic equivalence of their underlying $(\mathfrak{g}, K)$ modules. This notion is of interest only in the admissible case.

We conclude this lecture with results special to the case of unitary representations. Suppose that (Φ, V) is an admissible unitary representation of G with inner product $\langle \cdot, \cdot \rangle$. Then the underlying $(\mathfrak{g}, K)$ module has

(a) $\langle X v_1, v_2 \rangle = -\langle v_1, X v_2 \rangle$ for $X \in \mathfrak{g}_0$,
(b) $\langle k v_1, k v_2 \rangle = \langle v_1, v_2 \rangle$ for $k \in K$.

A Hermitian form $\langle \cdot, \cdot \rangle$ on a $(\mathfrak{g}, K)$ module is said to be *invariant* if (a) and (b) hold. A $(\mathfrak{g}, K)$ module is *infinitesimally unitary* if it admits a positive definite invariant Hermitian form. The next theorem in principle reduces the study of irreducible unitary representations to the study of infinitesimally unitary irreducible $(\mathfrak{g}, K)$ modules.

Theorem.

1) *Any irreducible admissible infinitesimally unitary* $(\mathfrak{g}, K)$ *module is the underlying* $(\mathfrak{g}, K)$ *module of an irreducible unitary representation of* G *on a Hilbert space.*

2) *Two irreducible unitary representations of* G *on Hilbert spaces are unitarily equivalent if and only if they are infinitesimally equivalent.*

Part (2) breaks down if it is assumed only that the two irreducible representations of G are infinitesimally unitary. For example, a principal series for $SL(2, \mathbb{R})$ in the induced picture can be initially constructed in a space of smooth functions. Then it can be completed in the L^2 norm or the L^2 norm of the function and the derivative. The resulting representations are infinitesimally unitary and infinitesimally equivalent, but they are not equivalent by a bounded linear operator with a bounded inverse.

Notes

Full details for the material concerning the Harish-Chandra isomorphism and infinitesimal characters may be found in [**K2**], pp. 246–258. A different approach, not pursued completely, is in [**K1**], pp. 218–226. This material appears also in a number of other places, including [**Wal**] and [**War**]. A proof of Dixmier's generalization of Schur's Lemma appears on page 236 of [**K2**].

Much of the material on smooth and analytic vectors, together with the application to reductive groups, may be found in [**K1**], pp. 51–57 and pp. 205–213. For more precise statements of results and detailed references, see [**Bal**]. A summary of this material that puts $(\mathfrak{g}, K)$ modules into context appears in the introduction of [**KV**], pp. 3–7.

For Frobenius reciprocity, see [**K1**], pp. 22–23.

LECTURE 6
Cartan Subgroups and Global Characters

Cartan Subgroups in the Compact Case

Before discussing Cartan subgroups for general reductive groups, we discuss the compact case for orientation. Let $G \subseteq U(n)$ be a compact connected Lie group, and let $\mathfrak{g}_0$ be its Lie algebra.

We introduced global characters for finite-dimensional representations of G in Lecture 3. Recall that the character of Φ is defined to be $\chi_\Phi(x) = \operatorname{Tr}\Phi(x)$. This satisfies $\chi_\Phi(1) = \dim \Phi$ and in particular is not the zero function.

Characters are invariant under group conjugation, and the character of Φ depends only on the equivalence class of Φ. Moreover, characters of inequivalent irreducible representations are orthogonal, hence linearly independent.

Since characters are invariant under conjugation, finding formulas for characters may be expected to involve information about the conjugacy classes in G. This is where the idea of a "Cartan subgroup" comes in. Thus let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{g}_0$, and let T be the corresponding analytic subgroup; this is a maximal torus. First we consider an analog in $\mathfrak{g}_0$ of what we seek. This is an easy theorem.

Theorem 1. *Any two maximal abelian subspaces of $\mathfrak{g}_0$ are conjugate.*

Since any member of $\mathfrak{g}_0$ is contained in a maximal abelian subspace, Theorem 1 implies that any member of $\mathfrak{g}_0$ is conjugate to one in a fixed maximal abelian subspace.

Corollary. *Any two maximal tori of G are conjugate.*

The next theorem is a little harder.

Theorem 2. *Let S be a torus in G. If $g \in G$ centralizes S, then there is a torus S' in G containing both S and g.*

Corollary 1. *The centralizer in G of a torus is connected.*

Corollary 2. *The centralizer $Z_G(\mathfrak{t}_0)$ of $\mathfrak{t}_0$ in G, i.e.,*

$$\{g \in G \mid \operatorname{Ad}(g)H = H \text{ for all } H \in \mathfrak{t}_0\},$$

is connected. In other words, $T = Z_G(\mathfrak{t}_0)$.

The significance of Corollary 2 is not immediately apparent. We shall observe in a moment that T meets every conjugacy class. But, by means of Corollary 2, we should think of the group $Z_G(\mathfrak{t}_0)$ as the one that is meeting every conjugacy class; this is the result that will generalize better, though not perfectly, to noncompact reductive groups.

The group $Z_G(\mathfrak{t}_0)$, which here equals T, is called a *Cartan subgroup* of G. Here is the hard theorem.

Theorem 3. *Each element of G is conjugate to a member of T.*

Corollary 1. *Each element of G lies in some maximal torus.*

Corollary 2. *The center Z_G lies in every maximal torus.*

Corollary 3. *The exponential map is onto G.*

The theorem says that every conjugacy class meets T. This fact accounts on the group level for the relatively simple description of the irreducible representations of a compact connected Lie group. For a finite group, no such conjugacy theorem is available, and indeed the representation theory of finite groups is much more complicated.

Another way of writing the conclusion of Theorem 3 is that $G = \bigcup_{g\in G} gTg^{-1}$. Characters for our compact connected G are therefore determined by their values on T. For $SU(2)$, T is all $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{smallmatrix}\right)$, and this particular matrix acts in the irreducible representation of highest weight n with eigenvalues $e^{in\theta}, e^{i(n-2)\theta}, \dots, e^{-in\theta}$, each eigenvalue having multiplicity one. Thus the character is

$$e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

The generalization to arbitrary compact connected Lie groups G is known as the *Weyl character formula.* Formally it says that

$$\chi_\lambda(t) = \frac{\sum_{w\in W} (\operatorname{sgn} w) e^{w(\lambda+\delta)\log t}}{\prod_{\alpha\in\Delta^+} (e^{\frac{1}{2}\alpha\log t} - e^{-\frac{1}{2}\alpha\log t})}.$$

However, δ might not be integral, and this formula is therefore not necessarily rigorous. To get a rigorous formula, we factor $e^{\delta\log t}$ from numerator and denominator. Then $w(\lambda+\delta)-\delta$ is integral, and the result is

$$\chi_\lambda(t) = \frac{\sum_{w\in W} (\operatorname{sgn} w) e^{(w(\lambda+\delta)-\delta)\log t}}{\prod_{\alpha\in\Delta^+} (1 - e^{-\alpha\log t})}.$$

For many purposes, the expression for $\chi_\lambda(t)$ is just as useful for working with a representation with highest weight λ as a concrete version of the representation itself.

Cartan Subgroups in the Noncompact Case

Now let us turn to the noncompact case. Let G be linear connected reductive, let θ be the Cartan involution, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0$, and let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. We begin with some material that is partly a review from the end of Lecture 4 and partly generalizes those results a little.

A *Cartan subalgebra* $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}_0$ whose complexification in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

Theorem. *Any Cartan subalgebra of $\mathfrak{g}_0$ is $\mathrm{Ad}(G)$ conjugate to one that is stable under the Cartan involution (i.e., is a θ stable Cartan subalgebra).*

Here are some facts about θ stable Cartan subalgebras, some of which we noted in Lecture 4:

- They are not necessarily all conjugate. (For example, in $\mathfrak{sl}(2,\mathbb{R})$, they are not.)
- There are only finitely many, up to conjugacy.
- Any two have the same dimension (because the Cartan subalgebras of $\mathfrak{g}$ are all conjugate).
- $\mathrm{Ad}(K)$ conjugacy is equivalent with $\mathrm{Ad}(G)$ conjugacy. (For this statement, the hypothesis "θ stable" is essential.)

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. The corresponding *Cartan subgroup* is $H = Z_G(\mathfrak{h}_0)$. Recall from Theorem 2 in the previous section that Cartan subgroups are necessarily connected when G is compact connected. But for noncompact G, Cartan subgroups need not be connected, as the following example shows.

Example. Let $G = SL(2,\mathbb{R})$. For $\mathfrak{h}_0$ equal to $\mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, H is $\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, r \in \mathbb{R}^\times\right\}$ in the respective cases. The second of these is not connected.

Proposition. *Let $\mathfrak{h}_0$ be a θ stable Cartan subalgebra of $\mathfrak{g}_0$, and decompose $\mathfrak{h}_0$ according to θ as $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$.*

1) *If $\mathfrak{h}_0$ is maximally compact, then H is connected.*
2) *$H = TA$, where $T = Z_K(\mathfrak{h}_0)$ and $A = \exp \mathfrak{a}_0$. Here T has Lie algebra $\mathfrak{t}_0$.*

Conclusion (1) generalizes Theorem 2 in the previous section. For an example of the decomposition in Conclusion (2), let $G = SL(2,\mathbb{R})$, and take the second of the two Cartan subgroups listed in the example above. Then $T = \{\pm 1\}$ and $A = \left\{\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, r > 0\right\}$.

Good examples to study for a further understanding of Cartan subgroups are the cases that $G = SL(n,\mathbb{R})$ and $G = Sp(2,\mathbb{R})$.

Problem. Choose a complete set of representatives of conjugacy classes of θ stable Cartan subalgebras in $\mathfrak{g}_0$, and let $H_1, \ldots, H_r$ be the corresponding Cartan subgroups. What can be said about

$$(*) \qquad \bigcup_{i=1}^{r} \bigcup_{g \in G} g H_i g^{-1}\,?$$

It is not quite true that this union is all of G. For example, in $SL(2,\mathbb{R})$, $\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ for $x \neq 0$ is not conjugate to any $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or to any $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$. The role of the union $(*)$ is more subtle.

To understand $(*)$, we introduce the notion of "regular elements" in G. For motivation, we begin on the level of Lie algebras. It turns out that Cartan subalgebras may be produced by considering the generalized eigenvalue 0 of $\mathrm{ad}\, X$ for $X \in \mathfrak{g}_0$ (i.e., the multiplicity of 0 as a root of the characteristic polynomial). Elements X for which the dimension of the 0 generalized eigenspace is as small as possible are said to be regular elements of the Lie algebra. For a regular element, the 0 generalized eigenspace is a Cartan subalgebra.

On the level of Lie groups, we consider the generalized eigenvalue 1 for $\mathrm{Ad}(x)$ for $x \in G$. This is an analog of the generalized eigenvalue 0 for $\mathrm{ad}\, X$ for $X \in \mathfrak{g}_0$ since $\mathrm{Ad}(\exp X) = e^{\mathrm{ad}\, X}$. Consider

$$\det((\lambda+1)1 - \mathrm{Ad}(x)) = \lambda^n + \sum_{j=0}^{n-1} D_j(x)\lambda^j.$$

For a single element x, the minimum index r with $D_r(x) \neq 0$ is the multiplicity of 1 as a generalized eigenvalue for $\mathrm{Ad}(x)$. We let x vary and let l be the minimum index with $D_l(x) \not\equiv 0$ on G. One can show that l is the common dimension of all Cartan subalgebras. A *regular element* of G is an $x \in G$ for which $D_l(x) \neq 0$. Let G' be the set of regular elements. This is *open* and *dense* in G, and its complement is of lower dimension.

Example. For $G = SL(n, \mathbb{R})$, G' is the set of elements with n distinct eigenvalues in $\mathbb{C}$.

Theorem.

1) $G' \subseteq \bigcup_{i=1}^r \bigcup_{g \in G} gH_ig^{-1}$.
2) *Each member of G' lies in just one Cartan subgroup of G.*

In $SL(n, \mathbb{R})$, the particular conjugacy class of the Cartan subgroup in (2) is determined by the number of complex-conjugate pairs of eigenvalues.

The theorem is suggestive that if irreducible global characters can be made meaningful for infinite-dimensional representations, their behavior on each H_i should practically determine them completely. This idea turns out to be correct, but making it rigorous is much more difficult than it would seem at first.

Global Characters

In this section let G be linear connected reductive with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ for the Lie algebra, with corresponding Cartan involution θ, and with complexified Lie algebra $\mathfrak{g}$. Let K be the analytic subgroup corresponding to $\mathfrak{k}_0$. We work with representations (π, V) such that V is a Hilbert space and $\pi|_K$ is unitary.

Here is the first basic difficulty in defining global characters for infinite-dimensional representations. Think of a unitary π in an orthonormal basis $\{v_i\}$. The diagonal matrix entries of $\pi(x)$ are $(\pi(x)v_i, v_i)$. We cannot expect $\sum_i (\pi(x)v_i, v_i)$ to converge and give a good analog of a trace. Even if it does converge occasionally, there will be a question of what order to use for computing the sum.

This difficulty is resolved as follows. First average $\pi(x)$ by a function f in the usual way, obtaining $\pi(f)$ with $\pi(f)v = \int_G f(x)\pi(x)v\, dx$. Then compute the trace of $\pi(f)$. The result is a map $f \mapsto \pi(f)$.

First we must make sense of the notion of "trace". A *trace class* operator L on a Hilbert space V is a bounded linear operator for which $\sum |(B^{-1}LBv_i, v_i)| < \infty$ for every orthonormal basis $\{v_i\}$ and every bounded invertible linear B. In this case $\sum (B^{-1}LBv_i, v_i)$ is independent of B and is called the *trace* of L.

An admissible representation π of G has a *global character* if $\pi(f)$ is of trace class for all $f \in C^\infty_{\text{com}}(G)$ and if $f \mapsto \operatorname{Tr}\pi(f) = \Theta(f)$ is a *distribution* (i.e., a continuous linear functional on $C^\infty_{\text{com}}(G)$).

In this case the distribution Θ is *invariant* in the sense of agreeing on $f(x)$ and any conjugate $f(gxg^{-1})$.

Theorem. *Every admissible representation π of G whose decomposition $\pi|_K = \sum_{\tau\in\hat{K}} n_\tau \tau$ has $n_\tau \le C \dim\tau$ has a global character.*

Example. For $G = SL(2,\mathbb{R})$, consider any nonunitary principal series representation. The K multiplicities are 1 or 0. Therefore it makes sense to speak of the global character $\Theta_{\sigma,\nu}$ of $\operatorname{Ind}^G_{MAN}(\sigma\otimes\nu\otimes 1)$. One can compute that $\Theta_{\sigma,\nu} = \theta_{\sigma,\nu}(x)\,dx$ for a locally integrable function on G. Let us describe this function. The function $\theta_{\sigma,\nu}(x)$ is determined a.e. by its values on $k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ and $\pm a_t$ with $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$. The formula is

$$\begin{aligned}\theta_{\sigma,\nu}(k_\theta) &= 0\\ \theta_{\sigma,\nu}(\pm a_t) &= \sigma(\pm 1)\,\frac{e^{\nu\log a_t} + e^{-\nu\log a_t}}{|e^t - e^{-t}|}\end{aligned}$$

as a result of a computation with integrals. The idea is that the operator obtained by applying the representation to a function $f \in C^\infty_{\text{com}}(G)$ in the manner of Lecture 3 can be realized as an integral operator over K. The kernel in the integrand is a function on $K\times K$, and the trace of the operator is given by the integral of the kernel over the diagonal of $K\times K$.

Let us apply the theorem to nonunitary principal series for general reductive groups. Let $\pi = \operatorname{Ind}^G_{MAN}(\sigma\otimes\nu\otimes 1)$, with σ acting in V^σ and MAN minimal parabolic. Then $\pi|_K = \operatorname{Ind}^K_M(\sigma)$. If τ is in $\widehat{K}$, then Frobenius reciprocity shows that the multiplicity of τ in $\pi|_K$ equals the multiplicity of σ in $\tau|_M$. The latter is $\le \dim\tau$. Hence any $\tau\in\widehat{K}$ occurs with multiplicity $\le\dim\tau$ in any nonunitary principal series, and the theorem says that the nonunitary principal series representation has a global character.

It is only a little harder to prove the following theorem.

Theorem. *For any irreducible unitary representation π, each $\tau\in\widehat{K}$ occurs in π with multiplicity $\le\dim\tau$. Hence π has a global character.*

We shall see that this result is still valid with "irreducible unitary" replaced by "irreducible admissible". The improved theorem is considerably harder to prove.

Recall that infinitesimal equivalence for admissible representations refers to algebraic equivalence on the underlying space of K finite vectors. The tool for improving the above theorem is the Subrepresentation Theorem.

Theorem (Subrepresentation Theorem, to be discussed in Lecture 8). *Any irreducible admissible representation of G is infinitesimally equivalent with a subrepresentation of a member of the nonunitary principal series.*

Since infinitesimal equivalence respects dimensions of K types, this gives

Corollary 1. *If π is an irreducible admissible representation of G, then $\pi|_K = \sum_{\tau\in\hat{K}} n_\tau\tau$ has $n_\tau\le\dim\tau$ for all $\tau\in\widehat{K}$.*

In other words, the estimate in the theorem about irreducible unitary representations extends by means of the Subrepresentation Theorem to all irreducible admissible representations. Applying the existence theorem for global characters gives the following corollary.

Corollary 2. *Every irreducible admissible representation of G has a global character.*

To complete our initial discussion of global characters, we state two further results. They are the reason global characters are useful in our setting.

Proposition. *If π and π' are infinitesimally equivalent admissible representations of G and both have global characters, then the characters of π and π' are equal.*

Theorem. *Let $\pi_1, \dots, \pi_n$ be mutually infinitesimally inequivalent, irreducible admissible representations of G with global characters $\Theta_1, \dots, \Theta_n$. Then $\Theta_1, \dots, \Theta_n$ are linearly independent.*

Differential Equations Satisfied by Characters

We turn now to an investigation of the main analytic tool in the subject—the differential equations that come from $Z(\mathfrak{g})$. These equations will repeatedly play a role from now on.

We begin with some remarks about differential operators on manifolds, starting with Euclidean space. Let D be a linear partial differential operator on a connected open set $U \subset \mathbb{R}^n$. We define a "transposed" operator D^t as follows: The map $D \mapsto D^t$ is $\mathbb{C}$ linear, it reverses order in composition, it sends $a(x)I$ into itself, and it sends $\partial/\partial x_j$ into $-\partial/\partial x_j$. From integration by parts it follows that

$$\int_U (Df_1)(x) f_2(x)\, dx = \int_U f_1(x) D^t f_2(x)\, dx$$

whenever f_1 and f_2 are smooth and at least one has compact support. The validity of the integration-by-parts formula determines this map $D \mapsto D^t$ completely, and this is the fact that we rely on for the passage to Lie groups.

To begin with, let us extend this construction to a smooth manifold. Fix a measure that is a smooth invertible function times Lebesgue measure in any chart. In any chart we can form a map $D \mapsto D^t$ for which the integration-by-parts formula holds. By the uniqueness in local coordinates, these maps are consistent on intersections of coordinate neighborhoods, and thus we can piece these together by a partition of unity.

If D is given as a partial differential operator on our smooth manifold and if $\Theta = f\, dx$ is a distribution given by a function, we want to have $D\Theta$ be $Df(x)\, dx$. Then the integration-by-parts formula suggests defining $D\Theta$ in general by

$$(D\Theta)(f) = \Theta(D^t f) \qquad \text{for } f \in C^\infty_{\mathrm{com}}(G).$$

Now let us relate these matters to our group G.

Proposition. *Let $X \in \mathfrak{g}_0$ act by left-invariant differentiation, and let f_1 and f_2 be smooth functions on G with at least one of them of compact support. Then*

$$\int_G (Xf_1)(x) f_2(x)\, dx = -\int_G f_1(x) X f_2(x)\, dx$$

Hence $X^t = -X$ *for* $X \in \mathfrak{g}_0$.

The left-invariant (linear) differential operators on G with complex coefficients may be identified with $U(\mathfrak{g})$. Hence we may restrict the map $D \to D^t$ to $U(\mathfrak{g})$.

Corollary. *The restriction of the map* $D \mapsto D^t$ *to* $U(\mathfrak{g})$ *is an associative algebra antiautomorphism of* $U(\mathfrak{g})$ *that extends the Lie algebra antiautomorphism* $X \mapsto -X$ *of* $\mathfrak{g}$. *The map* $D \mapsto D^t$ *carries* $Z(\mathfrak{g})$ *to itself.*

In Lecture 4 we constructed an abstract transpose map of $U(\mathfrak{g})$ that was determined by the conditions in the corollary. Thus we find that transpose of differential operators, when specialized to left-invariant differential operators, coincides with our earlier notion of transpose on $U(\mathfrak{g})$.

Application to Global Characters

Let π be an irreducible admissible representation of G. By the theorem on the correspondence of closed G invariant subspaces with $U(\mathfrak{g})$ invariant subspaces of K finite vectors, the underlying $(\mathfrak{g}, K)$ module is irreducible. Therefore π has an infinitesimal character.

Proposition. *Let* π *be an admissible representation of* G *having an infinitesimal character, say* $\pi(z) = \chi(z)I$ *for* $z \in Z(\mathfrak{g})$. *If* π *has a global character* Θ *and if* $z \in Z(\mathfrak{g})$ *is considered as a left-invariant differential operator, then* $z\Theta = \chi(z)\Theta$.

Sketch of proof. One calculates that $(\pi(z^t f)v_i, v_i) = \chi(z)(\pi(f)v_i, v_i)$, where $f \in C^\infty_{\text{com}}(G)$ and $\{v_i\}$ is an orthonormal basis of K finite vectors for the space on which π acts. Summing on i gives $\Theta(z^t f) = \chi(z)\Theta(f)$, and the result follows.

When $z\Theta = \chi(z)\Theta$, we say that the global character is an *eigendistribution* (of $Z(\mathfrak{g})$). We now know that global characters of irreducible admissible representations are invariant eigendistributions.

Now let us make use of the differential equations obtained from $Z(\mathfrak{g})$ as in the above proposition.

Theorem 1. *Suppose that* Θ *is an invariant eigendistribution on* G *with* $z\Theta = \chi(z)\Theta$ *for* $z \in Z(\mathfrak{g})$. *The restriction of* Θ *to smooth functions compactly supported in the regular set* G' *is a real analytic function invariant under conjugation. That is,* Θ *is of the form* $\Theta(x)\,dx$ *on* G' *with* dx *equal to Haar measure and with* Θ *real analytic and invariant under conjugation.*

To have a formula for this function $\Theta(x)$, it is enough to have a formula on each of the standard Cartan subgroups $H = TA$. On the conjugates of the regular set H' in H, let us write $\Theta(h) = \tau_H(h)/D_H(h)$, where $D_H(h)$ is a standard denominator as in the Weyl character formula.

We want to define $D_H(h)$, modulo technicalities. For this we use the linearity of G: We think of G as embedded in a complex group $G^{\mathbb{C}}$ and let $H^{\mathbb{C}}$ be the subgroup corresponding to $\mathfrak{h}$. For any root α, let ξ_α be the multiplicative character of $H^{\mathbb{C}}$ with differential α, and restrict ξ_α back to H. We do the same thing for δ, the half sum of the positive roots in some order (ignoring that δ may not be integral). Then $D_H(h)$ is given by $D_H(h) = \xi_\delta(h)\prod_{\alpha\in\Delta^+}(1 - \xi_{-\alpha}(h))$.

Theorem 2. *Suppose that $\chi(z) = \chi_\lambda(z)$ relative to H. Then the numerator τ_H of Θ on H satisfies $\gamma(z)\tau_H = \chi_\lambda(z)\tau_H$ for $z \in Z(\mathfrak{g})$, where γ is the Harish-Chandra isomorphism.*

The system $\gamma(z)\tau_H = \chi_\lambda(z)\tau_H$ is a system of partial differential equations with constant coefficients on H. Here H, apart from disconnectedness, is the product of a torus and a Euclidean space. The system can be solved, and here is the result.

Corollary. *Under the assumptions of Theorem 2, let $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$. Fix $h \in H$ and let $\mathfrak{h}_1$ be a connected component of the set of all X in $\mathfrak{h}_0$ such that $D_H(\exp X) \neq 0$. Then there exist uniquely determined polynomial functions p_w on $\mathfrak{h}_0$ for $w \in W$ such that $p_{ws} = p_w$ for $s \in W_\lambda$ and such that the numerator τ_H satisfies*

$$\tau_H(\exp X) = \sum_{w \in W} p_w(X) e^{w\lambda(X)}$$

for all $X \in \mathfrak{h}_1$. Moreover, the degrees of the polynomials p_w are all less than $|W_\lambda|$.

Qualitatively we can think of the polynomials as constant. Then the numerator is a linear combination of exponentials, with the allowable exponents being the Weyl group transforms of λ. It turns out that the assumption that the polynomials are constants is indeed valid if Θ is a global character.

Theorem 3 (Harish-Chandra, 1963). *An invariant eigendistribution Θ on G is given on all of G by a locally integrable function (whose restriction to G' is a real analytic function).*

That is, $\Theta = \Theta(x)\,dx$ on all of G. This theorem is much harder than Theorem 1.

Notes

The structure-theoretic results about compact connected groups may be found with full proofs in [**K2**], pp. 196–206, and also in [**V**]. For an alternative treatment with some additional results but with some proofs discussed only in examples, see [**K1**], 86–89.

The Weyl character may be found in one form in [**K2**], pp. 259–269, and in another form in [**K2**], pp. 280–283. See also [**K1**], [**Hu**], [**J**], and [**V**].

The conjugacy of any Cartan subalgebra to a θ stable one is proved on p. 328 of [**K2**]. The results about Cartan subgroups in the noncompact case are all proved in [**K2**], pp. 424–435.

Global characters are defined and shown to exist under certain conditions in [**K1**], pp. 333–336. One of the proofs in those pages has a small gap, and a full proof appears in [**De**]. A proof of the theorem about admissibility of irreducible unitary representations for linear connected reductive groups may be found in [**K1**], pp. 205–207. The other basic results about global characters that do not use differential equations are proved in [**K1**], pp. 336–338. Calculations of some characters are in the pages afterward in [**K1**].

Theorems 1 and 2 in the section "Application to Global Characters", as well as the corollary, are proved in Chapter X of [**K1**]. Theorem 3 is too difficult to prove in that book, and some discussion of the theorem appears in pp. 371–374 of [**K1**]. Detailed references appear in the chapter "Notes" at the end of the book.

Exercises

1. Write down three nonconjugate Cartan subalgebras in $SL(4, \mathbb{R})$.
2. Write down four nonconjugate Cartan subalgebras in $Sp(2, \mathbb{R})$.
3. Prove that

$$\int_G (Xf_1)(x) f_2(x)\, dx = -\int_G f_1(x)(Xf_2)(x)\, dx$$

for any Lie group G, any $X \in \mathfrak{g}_0$, and any two smooth functions f_1 and f_2 on G, at least one of which has compact support.

4. Let H be a θ stable Cartan subgroup of G. Define the Weyl group of H to be $W(H) = N_G(H)/H$. Show that $W(H) \cong N_K(H)/(H \cap K)$. (Hint: First show that $N_G(H) = N_K(H) \exp(\mathfrak{a}_0)$.)

LECTURE 7
Discrete Series and Asymptotics

Discrete Series for $SL(2,\mathbb{R})$

The principal series gave us our first infinite-dimensional irreducible unitary representations of $SL(2,\mathbb{R})$, and now we introduce some others. It will be more convenient to work with the group

$$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \;\middle|\; |\alpha|^2 - |\beta|^2 = 1 \right\},$$

which is isomorphic to $SL(2,\mathbb{R})$.

We write

$$K = T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}.$$

The group G acts on $\Omega = \{|z| < 1\}$ by $g(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$.

For f analytic on Ω and $n \geq 2$, put

$$\mathcal{D}_n \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = (-\bar{\beta} z + \alpha)^{-n} f\left(\frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha} \right)$$

$$\|f\|^2 = \int_\Omega |f(z)|^2 (1 - |z|^2)^{n-2}\, dx\, dy.$$

As with the principal series, we have constructed this representation from a transitive action of G and a "multiplier" $(-\bar{\beta} z + \alpha)^{-n}$. The new feature is that we use only analytic functions.

The action by $\mathcal{D}_n$ respects multiplication of elements in G, the space of analytic f's of finite norm can be shown to be complete, and the action can be shown to be continuous. To see that the action is unitary, one makes a simple change of variables and uses that $(1 - |z|^2)^{-2}\, dx\, dy$ is invariant under G in the action of G on Ω. Thus $\mathcal{D}_n$ is a unitary representation.

The functions z^N are orthogonal in the Hilbert space, and the fact that an analytic function has a power series expansion readily implies that only 0 is orthogonal to all z^N. Direct computation gives

$$\mathcal{D}_n \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} z^N = e^{-(n+2N)i\theta} z^N,$$

and it follows that the only K types that appear are of the form $e^{-(n+2N)i\theta}$ with $N \geq 0$ and that these have multiplicity one.

Now let us prove irreducibility. The K finite elements are the linear combinations of the functions z^N, and we are interested in $U(\mathfrak{g})$ invariant subspaces in the space of K finite elements. Computing $\mathcal{D}_n\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ and differentiating yields the action of the Lie algebra element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The result is that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z^N = (n+N)z^{N+1} - Nz^{N-1}.$$

An invariant subspace must be generated by its K types. Say z^N is in an invariant subspace. Then the presence of the right side in the invariant subspace forces z^{N+1} and z^{N-1} to be present unless $N = 0$. Consequently all z^N are present in any nonzero $U(\mathfrak{g})$ invariant subspace. This proves irreducibility.

The representation $\mathcal{D}_n$ has a new property that is not shared by the principal series reprsentations. This is known as "square integrability", and we say that $\mathcal{D}_n$ is in the *discrete series* of G. The easiest aspect of square integrability to see directly is given in the following proposition. We shall see in the first theorem below that the finiteness of the integral in the proposition implies the finiteness of other related integrals.

Proposition. $\int_G |(\mathcal{D}_n(g)1, 1)|^2 \, dg < \infty$.

Proof. In fact, $\mathcal{D}_n(g)1(z) = (-\bar{\beta}z + \alpha)^{-n}$, where 1 denotes the constant function. Thus

$$\begin{aligned} (\mathcal{D}_n(g)1, 1) &= \int_\Omega (\bar{\beta}z + \alpha)^{-n}(1 - |z|^2)^{n-2} \, dx \, dy \\ &= \alpha^{-n} \int_\Omega (1 - |z|^2)^{n-2} \, dx \, dy \\ &= c_n \alpha^{-n}, \text{ say,} \end{aligned}$$

the second equality following by expanding $(\bar{\beta}z + \alpha)^{-n}$ in binomial series with constant term α^{-n}. Thus we obtain

$$|(\mathcal{D}_n(g)1, 1)|^2 = c_n^2 |\alpha|^{-2n}.$$

Now $g(0) = \frac{\beta}{\bar{\alpha}}$ implies

$$1 - |g(0)|^2 = 1 - \tfrac{|\beta|^2}{|\alpha|^2} = \tfrac{|\alpha|^2 - |\beta|^2}{|\alpha|^2} = |\alpha|^{-2}.$$

So $|(\mathcal{D}_n(g)1, 1)|^2 = c_n^2(1 - |g(0)|^2)^n$. For a suitable normalization of dg, whenever a function h on G is of the form $h(g) = h_0(g(0))$, then

$$\int_G h(g) \, dg = \int_\Omega h_0(z)(1 - |z|^2)^{-2} \, dx \, dy.$$

We apply this fact with $h_0(z) = c_n^2(1 - |z|^2)^n$ and get

$$\int_G |(\mathcal{D}_n(g)1, 1)|^2 \, dg = c_n^2 \int_\Omega (1 - |z|^2)^{n-2} \, dx \, dy = c_n^3 < \infty,$$

as asserted.

A function $g \mapsto (\pi(g)v_1, v_2)$ is called a *matrix coefficient* of π, just as was the case for compact groups. For $\mathcal{D}_n$, we have just seen that one of the nonzero matrix coefficients is in $L^2(G)$.

Theorem (Godement). *For an irreducible unitary representation π of a unimodular Lie group G, the following three conditions are equivalent:*

(a) *Some nonzero matrix coefficient is in $L^2(G)$.*
(b) *All matrix coefficients are in $L^2(G)$.*
(c) *π is equivalent with a direct summand of the right regular representation of G on $L^2(G)$.*

A representation satisfying these three equivalent conditions is said to be in the *discrete series* of G.

Problem. Find all discrete series representations when G is linear connected reductive.

Here is the complete answer for $G = SU(1,1)$:

$$\text{All } \mathcal{D}_n, \text{ as well as all } \mathcal{D}_n \circ \begin{pmatrix} \text{complex} \\ \text{conjugation} \end{pmatrix} \text{ for } n \geq 2.$$

In fact, the classification of all irreducible unitary representations of $SU(1,1) \cong SL(2,\mathbb{R})$ is due to Bargmann (1947). This assertion about discrete series is one of the things that Bargmann shows.

To generalize, we give a group-theoretic formulation of the construction of $\mathcal{D}_n$ for $G = SU(1,1)$. Let $G^{\mathbb{C}} = SL(2,\mathbb{C})$, and let B be the subgroup $\left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \right\}$ of $SL(2,\mathbb{C})$. It is not hard to verify the following facts:

- Elements of GB are in one-one correspondence with all products
$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \quad \text{with} \quad \zeta \in \mathbb{C},\ \gamma \in \mathbb{C}^\times,\ |z| < 1.$$
- GB is open in $G^{\mathbb{C}}$, and its product complex structure from the above formula is the same as its complex structure as an open subset of $G^{\mathbb{C}}$.

Now we can construct the representation. Put $\xi_n \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^{-n}$. Let V_n consist of all $F : GB \to \mathbb{C}$ such that

(a) F is holomorphic,
(b) $F(xb) = \xi_n(b)^{-1} F(x)$ for $x \in GB,\ b \in B$,
(c) $\|F\|^2 = \int_G |F(g)|^2 \, dg < \infty$,

and let $L(g)F(x) = F(g^{-1}x)$ for $F \in V_n,\ g \in G,\ x \in GB$. Then L is a unitary representation, the only nontrivial part of the verification being that the underlying space is complete.

We can directly exhibit a unitary equivalence of this representation with $\mathcal{D}_n$. In fact, the correspondence is that $F \leftrightarrow f$, where

$$f(z) = F(z, 1, 0)$$
$$F(z, \gamma, \zeta) = \gamma^{-n} f(z).$$

This realization of $\mathcal{D}_n$ has a vector-bundle interpretation. We have a holomorphic bundle $G^{\mathbb{C}} \to G^{\mathbb{C}}/B$ and an open subbundle $GB \to GB/B$. We can identify the base with G/T, where $T = K$ is a compact Cartan subgroup. Form the associated holomorphic line bundle relative to $\xi_n : B \to \mathbb{C}^\times$ and restrict to the part of the bundle over G/T, obtaining

$$G \times_T \mathbb{C}^\times \to G/T.$$

Then V_n consists of the square-integrable holomorphic sections.

General Discrete Series for Reductive Groups

The group-theoretic construction of $\mathcal{D}_n$ generalizes to yield the "holomorphic discrete series". The setting is as follows. Let G be linear connected reductive, and let other notation be as usual. Assume that G/K has a G invariant complex structure. If $\mathfrak{c}_0$ is the center of $\mathfrak{k}_0$, this condition can be shown to be equivalent with the assumption $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. Under this hypothesis, $\mathfrak{g}_0$ always has a compact Cartan subalgebra $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$. Let $T = \exp \mathfrak{t}_0$.

The roots relative to the complexification $\mathfrak{t}$ of $\mathfrak{t}_0$ are of two kinds, called "compact" and "noncompact". To define these terms, form $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$. For $H \in \mathfrak{t}_0$, $\alpha \in \Delta$, and $X \in \mathfrak{g}_\alpha$, we have $[H, \theta X] = \theta[\theta H, X] = \theta[H, X] = \theta(\alpha(H)X) = \alpha(H)\theta X$. Thus θX is in $\mathfrak{g}_\alpha$, and θ is a linear transformation of $\mathfrak{g}_\alpha$ to itself with $\theta^2 = 1$. Since $\dim \mathfrak{g}_\alpha = 1$, $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$. Call α *compact* or *noncompact* accordingly. Let $\Delta_K = \Delta(\mathfrak{k}, \mathfrak{t})$ be the subset of compact roots.

Proposition. *The compact roots are exactly those vanishing on* $\mathfrak{c}_0$.

We use this proposition to introduce a "good ordering". We have $i\mathfrak{t}_0 = i\mathfrak{c}_0 \oplus i(\mathfrak{t}_0 \cap [\mathfrak{k}_0, \mathfrak{k}_0])$. Introduce an ordering that takes $i\mathfrak{c}_0$ before the rest. Observe by the proposition that every noncompact positive root is greater than every compact root. Hence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is 0 if α and β are positive noncompact.

We can now define the subgroup B of $G^{\mathbb{C}}$. Let $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$. This is a Borel subalgebra built from the *negative* roots. Let $B \subseteq G^{\mathbb{C}}$ be the analytic subgroup corresponding to $\mathfrak{b}$. Then GB may be shown to be open in $G^{\mathbb{C}}$.

To construct a representation, let $\lambda \in \mathfrak{t}'$ be integral and satisfy $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta_K^+$. Define V_λ to consist of all $F : GB \to \mathbb{C}$ such that

(a) F is holomorphic,
(b) $F(xb) = \xi_\lambda(b)^{-1} F(x)$ for $x \in GB$ and $b \in B$,
(c) $\|F\|^2 = \int_G |F(g)|^2 \, dg < \infty$,

and let $L(g)F(x) = F(g^{-1}x)$ for $F \in V_\lambda$, $g \in G$, and $x \in GB$.

Theorem (Harish-Chandra, 1956). *If* $\langle \lambda + \delta, \alpha \rangle < 0$ *for all positive noncompact roots* α, *then* (L, V_λ) *is a (nonzero) irreducible unitary representation in the discrete series of* G.

Since these representations and their twists by complex conjugation exhaust the discrete series for $SL(2, \mathbb{R})$, one might hope that a similar thing happens for all linear connected reductive groups. The following example indicates that this is not the case.

Example (Dixmier). The group $G = SO_0(4, 1)$, in which G/K does not have an invariant complex structure, has discrete series.

Theorem (Harish-Chandra, 1966). *The linear connected reductive group G has no discrete series unless G has a compact Cartan subgroup. If G has a compact Cartan subgroup $T \subseteq K$ and if $\lambda \in \mathfrak{t}'$ has the property that $\lambda + \delta$ is integral and $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, then there exists a unique invariant eigendistribution Θ_λ on G such that*

(a) $z\Theta_\lambda = \chi_\lambda(z)\Theta_\lambda$,

(b) *the numerator $\tau_{T,\lambda}$ of Θ_λ on T is given by*

$$\tau_{T,\lambda}(\exp X) = \sum_{w \in W(\Delta_K)} (\operatorname{sgn} w) e^{w\lambda(X)}$$

for $X \in \mathfrak{t}_0$,

(c) *the numerator $\tau_{H,\lambda}$ of Θ_λ on each Cartan subgroup H is bounded.*

Up to a sign, Θ_λ is the global character of a discrete series representation π_λ of G. The representations π_λ exhaust the discrete series, and two such π_λ and $\pi_{\lambda'}$ are equivalent if and only if λ' is in $W(\Delta_K)\lambda$.

The representation π_λ is called the discrete series with *Harish-Chandra parameter* λ. The infinitesimal character of π_λ is λ. The theorem in a sense solves the problem in the previous section of finding all the discrete series of G. But we can ask for more.

Problem. Find a global realization of π_λ.

The four solutions to this problem that have continued to be useful are the following, in chronological order of completion.

1) (Langlands, Schmid) By L^2 cohomology. This realization was conjectured by Langlands and proved by Schmid.

2) (Zuckerman, Vogan) By cohomological induction. Zuckerman gave a construction and an outline of a proof, and Vogan gave a full proof.

3) (Flensted-Jensen) By techniques of analysis on semisimple symmetric spaces.

4) (Kostant, Schmid, Aguilar-Rodriguez) By sheaf cohomology. This realization was roughly conjectured by Kostant and partly proved by Schmid. Aguilar-Rodriguez completed Schmid's proof.

Let us make some remarks about the Langlands-Schmid solution and its relation to other lecture series at this conference. The construction intersects most directly with Roger Zierau's course. One clue to a global realization comes from specializing the holomorphic discrete series to $G = K$ compact. Note for compact G that every irreducible representation is in the discrete series. For a compact group Harish-Chandra's construction of holomorphic discrete series specializes to the Borel-Weil Theorem, which was discovered independently of Harish-Chandra's construction at about the same time. (Tits also discovered the Borel-Weil Theorem at that time.)

Borel-Weil Theorem. *Let G be compact, and let T be a maximal torus. Introduce a positive system in $\Delta(\mathfrak{g}, \mathfrak{t})$, and build a Borel subgroup B from the negative roots. If λ is dominant integral, then the space of holomorphic functions F on GB (which equals $G^{\mathbb{C}}$) such that*

$$F(xb) = \xi_\lambda(b)^{-1} F(x) \quad \text{for } x \in GB,\ b \in B,$$

relative to the left regular representation of G, is an irreducible representation of G with highest weight λ.

A better clue to the Langlands-Schmid construction comes from a generalization of the Borel-Weil Theorem by Bott. Let us put this is the context of the bundle interpretation. The quotient $G/T = G^{\mathbb{C}}/B$ is a complex manifold. Let $\mathbb{C}_\lambda$ denote $\mathbb{C}$ with an action of T by ξ_λ. Then we have a holomorphic line bundle $G \times_T \mathbb{C}_\lambda \to G/T$, and the representation space in question is the space of all holomorphic sections. Bott asked what happens if λ is integral but not dominant. The space of holomorphic sections is then zero. He reinterpreted the space of holomorphic sections as sheaf cohomology in degree 0 and asked what the sheaf cohomology is in other degrees. The sheaf cohomology turns out to be nonzero in at most one degree, and there it is irreducible. He identified the highest weight in the nonzero case.

Hodge theory over a compact base allows one to reformulate sheaf cohomology in terms of harmonic differential forms. Langlands, making a formal calculation with characters, conjectured specifically how to obtain discrete series by using L^2 harmonic forms over G/T in the noncompact case. One of Schmid's accomplishments was to prove this conjecture of Langlands.

Asymptotic Expansions

Let G be a linear connected reductive group, and let other notation be as usual. The matrix coefficients of a representation of G having an infinitesimal character satisfy a system of differential equations that gives some control over the possibilities for the representation.

First let π be an admissible representation of G with no assumption about an infinitesimal character. If v_1 and v_2 are K finite vectors in the representation space and if X is in $\mathfrak{g}_0$, then the action of X as a left-invariant differential operator gives

$$X(\pi(g)v_1, v_2) = (\pi(g)\pi(X)v_1, v_2)$$

because the left side is

$$= \frac{d}{dt}(\pi(g \exp tX)v_1, v_2)\,|_{t=0} = \Big(\frac{d}{dt}\pi(\exp tX)v_1, \pi(g)^* v_2\Big)\Big|_{t=0},$$

and this equals the right side. Iterating, we obtain

$$u(\pi(g)v_1, v_2) = (\pi(g)\pi(u)v_1, v_2)$$

for every $u \in U(\mathfrak{g})$, with u regarded on the left side as a left-invariant differential operator on G.

If π has infinitesimal character χ, then $\pi(z)v_1 = \chi(z)v_1$ for $z \in Z(\mathfrak{g})$. Thus the above equation yields

$$\boxed{z(\pi(g)v_1, v_2) = \chi(z)(\pi(g)v_1, v_2)}$$

for $z \in Z(\mathfrak{g})$.

We seek to use this boxed equation to get information about $(\pi(g)v_1, v_2)$ as $g \to \infty$. We illustrate matters by taking $G = SL(2, \mathbb{R})$ for the remainder of this section.

We begin by introducing the KAK decomposition of $G = SL(2, \mathbb{R})$. In this G, we have $G = K \exp \mathfrak{p}_0$, and every member of $\exp \mathfrak{p}_0$ is K conjugate to $\mathfrak{a}_0$, by the Spectral Theorem. Thus $G = KAK$. Let $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. In the KAK decomposition, the A component can be taken to be some a_t with $t \geq 0$. The idea will be to reduce the boxed differential equation on G to one on A, i.e., one on the real line $\mathbb{R}$.

Let π be an admissible representation of $SL(2,\mathbb{R})$ with infinitesimal character χ, and let $k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. We work at first with a matrix coefficient of π of the form $F(x) = (\pi(x)v_1, v_2)$, where $\pi(k_{\theta'})v_1 = e^{im\theta'}v_1 = \tau_2(k_{\theta'})v_1$ and $\pi(k_\theta)v_2 = e^{in\theta}v_2 = \tau_1(k_\theta)v_2$. Then

$$\boxed{F(k_\theta x k_{\theta'}) = e^{i(n\theta+m\theta')}F(x) = \tau_1(k_\theta)F(x)\tau_2(k_{\theta'}).}$$

Actually we shall work with any $F \in C^\infty(G)$ satisfying this transformation law and the formula $zF = \chi(z)F$ for $z \in Z(\mathfrak{g})$.

We introduce the "radial part" of a differential operator on $SL(2,\mathbb{R})$. For any fixed $a \neq 1$, we can check that

$$\mathfrak{g}_0 = (\mathrm{Ad}(a^{-1})\mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}_0$$

for $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$. By the Poincaré-Birkhoff-Witt Theorem, every member of $U(\mathfrak{g})$ is a linear combination of terms $(\mathrm{Ad}(a^{-1})X)HY$ with $X \in U(\mathfrak{k})$, $Y \in U(\mathfrak{k})$, $H \in U(\mathfrak{a})$. The particular linear combination typically depends on a. For F as above, we can compute that

$$(\mathrm{Ad}(a^{-1})X)HYF(a) = \tau_1(X)(HF)(a)\tau_2(Y).$$

We consider H, followed by left by $\tau_1(X)$ and right by $\tau_2(Y)$, to be a differential operator on functions on $A^+ = \{a_t \mid t > 0\}$. Then to each $u \in U(\mathfrak{g})$ we can associate a differential operator $D_{\tau_1,\tau_2}(u)$ on A^+ such that

$$(uF)|_{A^+} = D_{\tau_1,\tau_2}(u)(F|_{A^+}).$$

The operator $D_{\tau_1,\tau_2}(u)$ is the *radial part* of u.

Let us compute the radial part of the Casimir operator $u = \Omega = \frac{1}{2}h^2 + ef + fe$. The Casimir operator is a member of $Z(\mathfrak{g})$. Let $\alpha = e_1 - e_2$ be the positive restricted root, let $\xi = \xi_\alpha(a) = e^{\alpha \log a}$, and let $Y = \frac{1}{2}(e + \theta e) = \frac{1}{2}(e - f)$. Then

$$\begin{aligned} D_{\tau_1,\tau_2}(\Omega)F(a) = {} & \frac{1}{2}h^2F(a) + \frac{\xi^2+1}{\xi^2-1}\,hF(a) \\ & + \frac{8\xi^2}{(\xi^2-1)^2}(F(a)\tau_2(Y)^2 + \tau_1(Y)^2F(a)) \\ & - \frac{8\xi(\xi^2+1)}{(\xi^2-1)^2}\,\tau_1(Y)F(a)\tau_2(Y). \end{aligned}$$

With $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, we have $\xi = \xi_\alpha(a_t) = e^t$, and the differential operator h becomes $2\,\frac{d}{dt}$. Substitution gives

$$\begin{aligned} \tfrac{1}{2}D_{\tau_1,\tau_2}(\Omega)F(a_t) = {} & \frac{d^2F}{dt^2} + (\coth t)\,\frac{dF}{dt} \\ & + \frac{1}{(\sinh t)^2}(F(a_t)\tau_2(Y)^2 + \tau_1(Y)^2F(a_t)) \\ & - \frac{2\cosh t}{\sinh^2 t}\,\tau_1(Y)F(a_t)\tau_2(Y). \end{aligned}$$

This formula is a differential equation satisfied by our function F, and a change of variables will enable us to obtain an asymptotic expansion for $F(a_t)$ as $t \to \infty$. The left side of the above equation, under our assumption, is $\frac{1}{2}\chi(\Omega)F(a_t)$. Substitute $z = e^{-t}$ and $\frac{d}{dt} = -z\frac{d}{dz}$, and the result is

$$\begin{aligned}\tfrac{1}{2}\chi(\Omega)F = \Big(z\,\frac{d}{dz}\Big)^2 F - \frac{1+z^2}{1-z^2}\Big(z\,\frac{d}{dz}\Big)F \\ + \frac{4z^2}{(1-z)^2}(R_{\tau_2(Y))^2} + L_{\tau_1(Y)^2})F \\ - \frac{4(z+z^3)}{(1-z^2)^2}(R_{\tau_2(Y)}L_{\tau_1(Y)})F.\end{aligned}$$

Expansion in series about $z = 0$ gives

$$z^2\,\frac{d^2F}{dz^2} + O(z)z\,\frac{dF}{dz} + (-\tfrac{1}{2}\chi(\Omega) + O(z))F = 0$$

with each instance of $O(z)$ representing an analytic function for $|z| < 1$ with no constant term. This is an ordinary differential equation with a regular singular point at $z = 0$, and one looks for solutions z^s(power series). The power s must satisfy the "indicial equation" $s(s-1) - \frac{1}{2}\chi(\Omega) = 0$. For $\chi = \chi_\lambda$, we have seen that $\chi(\Omega) = \frac{1}{2}\lambda(h)^2 - \frac{1}{2}$. Then $s = \frac{1}{2}(1 \pm \lambda(h))$. The leading terms of the asymptotic expansion are thus $e^{-\frac{1}{2}(1\pm\lambda(h))t}$, possibly with a factor of t.

The possible factor of t comes about as follows. When the two solutions s of the indicial equation do not differ by an integer, there exist respective solutions of the form z^s times a power series. When the two solutions do differ by an integer, one solution can involve a factor $\log z$, and this becomes essentially a factor of t when we restore the variable t.

From this analysis we can see a connection between the infinitesimal character and the possible square integrability of the matrix coefficient on G. In the KAK decomposition, one can check that Haar measure is given by $(\sinh t)\, d\theta'\, dt\, d\theta$. The product of the leading terms, namely of $e^{-\frac{1}{2}(1+\lambda(h))t}$ and $e^{-\frac{1}{2}(1-\lambda(h))t}$ is e^{-t}, and so one of these leading terms fails to be square integrable. Thus one leading term must have coefficient 0 for a discrete series representation. For the unitary principal series $\mathrm{Ind}_{MAN}^G(\sigma \otimes \nu \otimes 1)$, the infinitesimal character is ν. Then $\nu(h)$ is imaginary, and the magnitude of both leading terms is $e^{-t/2}$, which just misses square integrability.

Notes

The material in the lecture concerning the discrete series of $SL(2,\mathbb{R})$ may be found in [**K1**], pp. 39–41, p. 142, and pp. 150–152. The Godement theorem, with a small amount of restriction, is proved in [**K1**], pp. 284–286. For holomorphic discrete series in general, see [**K1**], pp. 153–164. The relevant structure theory is carried out in more detail in [**K2**], pp. 435–449.

Harish-Chandra's 1966 theorem parametrizing the discrete series in general has four approaches that yield global realizations, as is noted in the lecture. Approaches (1) and (4) are discussed in some detail in [**SB**], approach (2) is in [**KV**] and also [**Wal**], and approach (3) is in [**K1**], Chapter IX.

The material on asymptotic expansions is taken largely from [**K1**], pp. 215–218. Chapter VIII of that book discusses the generalization to other groups. See also [**Wal**].

Exercises

1. Prove that the correspondence $F \mapsto f$ between the group theoretic formulation of $\mathcal{D}_n$ and the original formulation preserves norms, up to a global constant.
2. Verify the calculation of $D_{\tau_1,\tau_2}(\Omega)$ for $SL(2,\mathbb{R})$.

LECTURE 8
Langlands Classification

Subrepresentation Theorem

In this section we shall give an idea of where the Subrepresentation Theorem comes from, and we shall state the result precisely. The device for obtaining the theorem is $\bar{\mathfrak{n}}_0$ invariant linear functionals on a representation space. In order to isolate the ideas clearly, we shall work just with $G = SL(2,\mathbb{R})$, but the theory works for any linear connected reductive G. Write $\overline{N} = \Theta N$, and let $\bar{\mathfrak{n}}_0$ be its Lie algebra. Let π be an irreducible admissible representation of G on a Hilbert space V, and let V_K be the space of K finite vectors in V.

A *leading exponential* of π is an exponential $e^{(\nu-\rho)\log a}$ occurring with nonzero coefficient in some $(\pi(a)v_1, v_2)$ such that no $e^{(\nu-\rho+n\alpha)\log a}$ occurs as an exponential in any matrix coefficient for $n > 0$. Leading exponentials exist because of the analysis of asymptotic expansions of matrix coefficients given in Lecture 7.

Fix a leading exponential, say $e^{(\nu-\rho)\log a}$, of π. (If we write $a = a_t$, a factor t may occur in front of the exponential, but we ignore this small complication.) We define a linear function $l : V_K \to \mathbb{C}$ depending on certain choices. Say the leading exponential $e^{(\nu-\rho)\log a}$ occurs in the asymptotic expansion of the matrix coefficient $(\pi(a)v_1, v_2)$. We take $l(v)$ to be the numerical coefficient of $e^{(\nu-\rho)\log a}$ in $(\pi(a)v, v_2)$. Then $l(v_1) \neq 0$, so that l is not the 0 linear functional.

Lemma. *$l(Xv) = 0$ for all $X \in \bar{\mathfrak{n}}_0$ and $v \in V_K$.*

Proof. Let $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We can check that

$$f = \frac{-2\xi^{-1}}{\xi^{-2}-1}\,(\xi^{-1}Y - \mathrm{Ad}(a_t)^{-1}Y)$$

with $\xi = \xi_\alpha(a_t) = e^t$. Then

$$\begin{aligned}
(\pi(a_t)fv, v_2) &= -\frac{2\xi^{-2}}{\xi^{-2}-1}\,(\pi(a_t)Yv, v_2) \qquad \text{by substitution} \\
&\quad + \frac{2\xi^{-1}}{\xi^{-2}-1}\,(\pi(a_t)(\mathrm{Ad}(a_t)^{-1}Y)v, v_2) \\
&= -\frac{2\xi^{-2}}{\xi^{-2}-1}\,(\pi(a_t)v, v_2)\tau_2(Y) + \frac{2\xi^{-1}}{\xi^{-2}-1}\,\tau_1(Y)(\pi(a_t)v, v_2),
\end{aligned}$$

the last equality holding because of the way that the matrix coefficient transforms with τ_1 and τ_2. The first term on the right side begins with $e^{(\nu-2\alpha-\rho)\log a_t}$, and the second begins with $e^{(\nu-\alpha-\rho)\log a_t}$. For both terms the coefficient of $e^{(\nu-\rho)\log a_t}$ is 0 since our exponent was a leading one. Thus $l(fv) = 0$.

Let us now see how we can make use of $\bar{\mathfrak{n}}_0$ invariant linear functionals. The linear functional l above satisfies

$$\begin{aligned} l(Xv) &= 0 && \text{for } X \in \bar{\mathfrak{n}}_0 \\ l(Hv) &= (\nu-\rho)(H)l(v) && \text{for } H \in \mathfrak{a}_0. \end{aligned}$$

Define $L : V_K \to \{\text{complex-valued functions on } K\}$ by

$$L(v)(k) = l(\pi(k)^{-1}v).$$

Regard the range space as the compact picture of $\operatorname{Ind}_{A\overline{N}}^G(e^\nu \otimes 1)$ by extending functions to G according to $f(ka\bar{n}) = e^{(-\nu+\rho)\log a} f(k)$. A little calculation shows that $L(Xv) = X(L(v))$ for all $X \in \mathfrak{g}_0$. Thus L is an infinitesimal equivalence of V_K into a subrepresentation of

$$\operatorname{Ind}_{A\overline{N}}^G(e^\nu \otimes 1) \cong \operatorname{Ind}_{MA\overline{N}}^G(+\otimes e^\nu \otimes 1) \oplus \operatorname{Ind}_{MA\overline{N}}^G(-\otimes e^\nu \otimes 1).$$

One of the two projections thus may be used to embed π infinitesimally into a nonunitary principal series representation (but built from $\overline{N}$ instead of N).

If we had considered a_t with $t \to -\infty$, $\overline{N}$ would get replaced by N. Anyway, the general result is as follows.

Theorem. *Each irreducible admissible representation of a linear connected reductive G is infinitesimally equivalent with a subrepresentation of some nonunitary principal series.*

Irreducible Tempered Representations

We return to the situation that G is a linear connected reductive group. We shall assume that G has compact center. This assumption implies that Σ^+, the system of positive restricted roots lying in $\mathfrak{a}'_{0,\mathfrak{p}}$, actually spans $\mathfrak{a}'_{0,\mathfrak{p}}$. Let $M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$ be a minimal parabolic subgroup of G.

We introduce the notion of "simple restricted roots". Call $\alpha \in \Sigma^+$ *simple* if α is not the sum of two members of Σ^+.

Proposition. *The simple restricted roots form a vector-space basis of $\mathfrak{a}'_{0,\mathfrak{p}}$. In the expansion of any member of Σ^+ in terms of the simple restricted roots, all the coefficients are integers ≥ 0.*

Example. For $G = SL(n,\mathbb{R})$, take Σ^+ to consist of all $e_i - e_j$ with $i < j$. Then the simple restricted roots are $e_1 - e_2$, $e_2 - e_3$, $\ldots$, $e_{n-1} - e_n$.

Let $\alpha_1, \ldots, \alpha_l$ be the simple restricted roots in $\mathfrak{a}'_{0,\mathfrak{p}}$. Define $\omega_1, \ldots, \omega_l$ by $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$. The elements $\omega_1, \ldots, \omega_l$ form a basis of $\mathfrak{a}'_{0,\mathfrak{p}}$.

The theory of asymptotic expansions of matrix coefficients, which we discussed only for $SL(2,\mathbb{R})$, is still valid. One works with a block of matrix coefficients $F(x) = \{(\pi(x)v_1^{(i)}, v_2^{(j)})\}_{ij}$ with i running over all indices corresponding to one or more full K types and j doing the same, possibly for different K types. In order words, F is to be matrix-valued instead of scalar-valued. In any event, $F(k_1xk_2) = \tau_1(k_1)F(x)\tau_2(k_2)$ for suitable representations (τ_1, V_1) and (τ_2, V_2) of K. This function F has an asymptotic expansion. The exponentials are of the form $e^{(\nu-\rho)\log a}$ with ν and ρ in $\mathfrak{a}'_{0,\mathfrak{p}}$, and the coefficients are linear transformations from V_2 to V_1. Polynomial terms in $\log a$ may also be involved, but we ignore them. Leading exponentials of π may then be defined.

Theorem (Langlands). *For an irreducible admissible representation π of G, the following are equivalent:*

1) *All K finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$.*
2) *Every leading exponential $e^{(\nu-\rho)\log a}$ of π satisfies $\mathrm{Re}\langle\nu, \omega_j\rangle \leq 0$ for $1 \leq j \leq l$.*
3) *π is infinitesimally equivalent with a subrepresentation of a standard induced representation $\mathrm{Ind}_{MAN}^G(\sigma_0 \otimes \nu_0 \otimes 1)$ for some parabolic subgroup $MAN \supseteq M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$, some discrete series representation σ_0 of M, and some imaginary parameter ν_0.*

If the equivalent conditions in the theorem are satisfied, π is said to be an irreducible *tempered* representation of G.

Example. $G = SL(2,\mathbb{R})$. We examine which representations are already known to satisfy the respective conditions (1) to (3) above.

1) We initially have no idea what all these representations are. However, we know some examples: any discrete series, or any irreducible constituent of a unitary principal series. Also we have some clues about examples of nontempered representations. The trivial representation has matrix coefficient identically 1. This is not in $L^{2+\varepsilon}(G)$. For an *irreducible* nonunitary principal series $\mathrm{Ind}_{M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}}^G(\sigma \otimes \nu \otimes 1)$ with ν nonimaginary, the candidates for leading exponentials are $e^{(\nu-\rho)\log a}$ and $e^{(-\nu-\rho)\log a}$.

One of these is not in every $L^{2+\varepsilon}(G)$. So if both have nonzero coefficient (which they always do for irreducible nonunitary principal series), the representation is not tempered.

2) We know that this condition is necessary for discrete series. It is also satisfied for irreducible constituents of unitary principal series. For the trivial representation, the value of ν is ρ, and the condition is not satisfied. We initially have no idea whether other irreducible representations might satisfy this condition.

3) Here MAN is G or $M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$. So the representations in question are exactly the discrete series and the irreducible constituents of unitary principal series.

What is hard in the theorem? The example suggests that (1) $\iff$ (2) and (3) $\implies$ (1) are fairly easy. Actually there are some technical problems that need attention when $\dim \mathfrak{a}'_{0,\mathfrak{p}} > 1$, but we ignore these. We examine further the hard step (2) $\implies$ (3).

Take a leading exponential $e^{(\nu-\rho)\log a}$. There may be more than one leading exponential, and we specify which one to use in a moment. Let

$$\mathcal{F} = \{j \mid 1 \leq j \leq l \text{ and } \mathrm{Re}\,\langle\nu, \omega_j\rangle < 0\}.$$

The set $\mathcal{F}$ corresponds to indices where $e^{(\nu-\rho)\log a}$ decreases at an L^2 rate. Write $\mathrm{Re}\,\nu = \sum_{i=1}^{l} c_i\alpha_i$. Take the inner product with ω_j to see that $c_j < 0$ for $j \in \mathcal{F}$ and

$c_j = 0$ (by (2)) for $j \notin \mathcal{F}$. Now

$$\{\alpha_j \mid j \in \mathcal{F}\} \cup \{\omega_j \mid j \notin \mathcal{F}\}$$

is a basis of $\mathfrak{a}'_{0,\mathfrak{p}}$. The reason is that each set is independent, the two sets are orthogonal to each other, and the total number of elements in the set is l.

Expand ν in terms of this basis to get

$$\nu = \sum_{j \notin \mathcal{F}} b_j \omega_j - \sum_{j \in \mathcal{F}} a_j \alpha_j$$

with $\operatorname{Re} b_j = 0$ for $j \notin \mathcal{F}$ and $\operatorname{Re} a_j > 0$ for $j \in \mathcal{F}$. To do so, first expand $\operatorname{Re} \nu$, using only the second term. Then expand $\operatorname{Im} \nu$, and combine the two expansions.

We define $\mathfrak{a}_0 = \sum_{j \notin \mathcal{F}} \mathbb{R} H_{\omega_j}$, and we use the indices $j \in \mathcal{F}$ to generate the roots defining M. The proof of the Subrepresentation Theorem gives an infinitesimal embedding

$$\pi \subseteq \operatorname{Ind}_{M_\mathfrak{p} A_\mathfrak{p} \overline{N}_\mathfrak{p}}^G (\sigma \otimes e^\nu \otimes 1)$$

for some $\sigma \in \widehat{M_\mathfrak{p}}$. We rewrite the right side by double induction, the inside term being induced up to M and the character ν_0 induced from A being $e^{\sum_{j \notin \mathcal{F}} b_j \omega_j}$. Denote by $\widetilde{\sigma}_0$ the representation of M that is obtained. The desired discrete series σ_0 is a certain subrepresentation of $\widetilde{\sigma}_0$. Making the proof go through uses a restriction on $\nu - \rho$, namely that the set $\mathcal{F}$ of indices is minimal under inclusion, as a function of ν. This is the extra condition that we impose at the start on the leading exponential with which we work.

The result is that irreducible tempered representations are characterized as in (3)—they are the irreducible constituents of standard induced representations with discrete series on M and imaginary parameters on A. Also the reducibility of these induced representations is completely understood, though we omit discussion of it. Therefore the irreducible tempered representations are completely understood.

Langlands Classification

We continue with the setting of the previous section: G is linear connected reductive and has compact center. The system Σ^+ of positive restricted roots then spans $\mathfrak{a}'_{0,\mathfrak{p}}$. Let $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$ be a minimal parabolic subgroup of G.

Theorem (Langlands). *The equivalence classes (under infinitesimal equivalence) of irreducible admissible representations of G stand in one-one correspondence with all triples $(MAN, [\sigma_0], \nu_0)$ such that*

MAN is a parabolic subgroup of G containing $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$

σ_0 is an irreducible tempered (unitary) representation of M and $[\sigma_0]$ is its equivalence class

ν_0 is a member of $\mathfrak{a}'$ such that $\operatorname{Re}\langle \nu_0, \alpha \rangle > 0$ for every positive restricted root that does not vanish on $\mathfrak{a}_0$.

The correspondence is that $(MAN, [\sigma_0], \nu_0)$ corresponds to the class of the unique irreducible quotient of $\operatorname{Ind}_{MAN}^G (\sigma_0 \otimes \nu_0 \otimes 1)$.

The parameters $(MAN, [\sigma_0], \nu_0)$ are called the *Langlands parameters* of the given irreducible admissible representation.

Example. $G = SL(2,\mathbb{R})$. Here MAN is G or is $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$. When $MAN = G$, we get the irreducible tempered representations of G, which we know from the previous section are the discrete series and the irreducible constituents of the unitary principal series. When $MAN = M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$, we get the "Langlands quotients" of the members of the nonunitary principal series of G having $\operatorname{Re}\nu > 0$.

It is instructive to note some things that the theorem does *not* do in the example. It does not tell us which parameters of the nonunitary principal series correspond to irreducibility, and it does not tell us much about what irreducible admissible representations are infinitesimally unitary.

There is a supplementary statement to the theorem: The irreducible quotient is explicitly obtained as the image of a specific integral operator sending

$$\operatorname{Ind}_{MAN}^G(\sigma_0 \otimes \nu_0 \otimes 1) \quad \text{equivariantly to} \quad \operatorname{Ind}_{MA\overline{N}}^G(\sigma_0 \otimes \nu_0 \otimes 1).$$

We omit the details.

We conclude by making some comments about the proof. The hard part is to show that every irreducible admissible representation π is realized as a Langlands quotient. A duality argument shows that it is enough to show that π admits an infinitesimal embedding into some $\operatorname{Ind}_{MA\overline{N}}^G(\sigma_0 \otimes \nu_0 \otimes 1)$ with MAN, σ_0, and ν_0 as in the statement of the theorem. The formalism for exhibiting the infinitesimal embedding is rather similar to the argument identifying irreducible tempered representations. We start with a suitable leading exponential $e^{(\nu-\rho)\log a}$ of π. A subtle geometric argument shows that it is possible to find a set $\mathcal{F}$ of indices so that if we write

$$\nu = \sum_{j \notin \mathcal{F}} b_j \omega_j - \sum_{j \in \mathcal{F}} a_j \alpha_j,$$

then $\operatorname{Re} b_j > 0$ for $j \notin \mathcal{F}$ and $\operatorname{Re} a_j \geq 0$ for $j \in \mathcal{F}$. We form $\mathfrak{a}_0 = \sum_{j \notin \mathcal{F}} \mathbb{R} H_{\omega_j}$ and build M from the α_j with $j \in \mathcal{F}$. The same double induction as before, combined with passing to a subrepresentation of M, yields the desired parameters.

Notes

For further motivation and a proof of the Subrepresentation Theorem, see [**K1**], pp. 203–204 and pp. 238–239.

Simple restricted roots form an elementary topic, and their properties follow from Corollary 6.53 and Proposition 2.49 of [**K2**].

Irreducible tempered representations are the subject of pp. 258–266 of [**K1**]. These representations have been classified. See [**K1**], Chapter XIV, for the statement and the idea of classification. The Langlands classification is the subject of pp. 266–276 of [**K1**]. The Langlands classification and the classification of irreducible tempered representations may be combined to give a tidy statement. See Chapter XIV of [**K1**] for details.

A more streeamlined proof of the Langlands classification may be found in [**Wal**]. The exposition [**Ban**] sketches this argument in some detail.

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