TAKE-HOME FINAL

(1) Let M be a symplectic manifold, let G be a Lie group acting smoothly on M and let $\Phi : M \to \mathfrak{g}^\star$ be a moment map for this action. For $X \in \mathfrak{g}$, let Φ^X be the associated component of the moment map. Show that the Poisson bracket on M is related to the Lie bracket on $\mathfrak{g}$ by the formula

$$\{\Phi^X, \Phi^Y\} = \Phi^{[X,Y]}$$

(2) Consider the set of all 2 dimensional subspaces of $\mathbb{C}^4$, denoted $G(2,4)$. Give this set the structure of symplectic manifold by identifying it with an orbit of Hermitian matrices. Consider the action of $T = (S^1)^3$ acting on this manifold by conjugation with the diagonal matrix with diagonal entries $(t_1, t_2, t_3, 1)$.

Find the fixed points of this action and draw the moment polytope for this action.

Bonus: Pick a generic vector in the Lie algebra of T, thus giving a Morse function on $G(2,4)$. For each fixed point find the attracting set. Use this to find the dimensions of the cohomology groups of $G(2,4)$.

(3) Let M be a Kähler manifold and let $N \subset M$ be a submanifold. Prove or find a counterexample to each of the following statements.

(a) If N is a symplectic submanifold, then it is a complex submanifold.
(b) If N is a complex submanifold, then it is a symplectic submanifold.
(c) If N is a Lagrangian submanifold, then it is a totally real submanifold.
(d) If N is a totally real submanifold, then it is a Lagrangian submanifold.

(A totally real submanifold N of a complex manifold M is a submanifold such that for each point $p \in N$, we have $T_p(N) \oplus JT_p(N) = T_p(M)$.)

(4) Explain how one can distinguish between the complex manifolds $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}i)$ and $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}2i)$ using the interaction between the integral homology and the Hodge decomposition of cohomology.