ASSIGNMENT 1
DUE THURSDAY JANUARY 22

(1) Let X be a manifold. Show that the wedge product on the differential forms $\Omega^\bullet(X)$ descends to the de Rham cohomology $H^\bullet_{\mathrm{DR}}(X)$.

(2) (a) Let W be a vector space and let $V = W \oplus W^\star$. Define a bilinear form Ω on V by

$$\Omega((w_1, \alpha_1), (w_2, \alpha_2)) = \alpha_2(w_1) - \alpha_1(w_2).$$

Show that Ω is non-degenerate and skew symmetric.

(b) Let (V, Ω) be a symplectic vector space (ie a vector space with a skew-symmetric non-degenerate bilinear form). Let $L \subset V$ be Lagrangian. Show that V/L is canonically isomorphic to $L^\star$.

(c) Show that (V, Ω) is (non-canonically) symplectomorphic to $(V, L \oplus L^\star)$. What extra structure is needed to make this canonical?

(3) (a) Let X be a manifold and let E be a vector bundle on X. Show that there is short exact sequence of vector bundles on E

$$0 \to \pi^*(E) \to TE \to \pi^*(TX) \to 0$$

(Short exact sequence of vector bundles means that we have maps of vector bundles such that at each point we have a short exact sequence of vector spaces. Also, the notation $\pi^*(E)$ means pullback of vector bundles.)

(b) Can we write the symplectic form on T^*X in a simple fashion as in (2a) above? What additional data is needed?

(4) Do Homework 3, question 3 in Cannas da Silva.

(5) (a) Let X be a compact manifold and let f be a smooth function. Show that the graph of df and the zero section X intersect in at least two points inside $T^\star X$.

(b) For which manifolds can you find a function such that they intersect in exactly two points? In general, what is a better bound?