REPRESENTATION THEORY
ASSIGNMENT 3
DUE FRIDAY MARCH 18

(1) Consider the flag variety G/B of the group $G = SO_{2n}(\mathbb{C})$.
   (a) Define an inclusion from G/B into the set of orthogonal flags in $\mathbb{C}^{2n}$ and identify the image.
   (b) Recall that the compact form is $K = SO_{2n}(\mathbb{R})$. Give a linear algebra description of K/T and find a bijection between this set and the set of orthogonal flags described above. (Note that the maximal torus of K consists of 2×2 blocks of rotation matrices placed down the diagonal.)

(2) Take $K = U(n)$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a non-generic point in $\mathfrak{t}^*$ (so some of the λ_i are allowed to be equal).
   (a) Describe $\mathcal{H}_\lambda$ in terms of orthogonal decompositions of $\mathbb{C}^n$ and also in terms of partial flags.
   (b) Find a subgroup $P \subset GL_n(\mathbb{C})$ (depending on λ) such that $G/P \cong \mathcal{H}_\lambda$.
   (c) Specialize to the case where $\lambda = (1, \dots, 1, 0, \dots, 0)$ (there are k 1s and $n - k$ 0s). Describe the B orbits on G/P. In particular find their dimensions.

(3) Let V be a vector space. Consider the line bundle $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$ on $\mathbb{P}(V)$. Show that there is an injective map

$$Sym^k(V^*) \to \Gamma(\mathbb{P}(V), \mathcal{O}(k))$$

and show that this map is an isomorphism when $\dim V = 2$ (i.e. for $\mathbb{P}^1$) by using the usual open cover of $\mathbb{P}^1$ (actually it's always an isomorphism).

(4) Take $G = GL_n$ and take $\lambda = (k, 0, \dots, 0)$. Describe the line bundle $L(\lambda)$ on G/B and the resulting map $G/B \to \mathbb{P}^N$. Describe $\Gamma(G/B, L(\lambda))^*$. Do the same thing when $\lambda = (1, \dots, 1, 0, \dots, 0)$.

(5) Let $\mathbb{G}_a$ denote the complex numbers, viewed as a group under addition (the additive group). A unipotent group is a an algebraic group G such that either $G \cong \mathbb{G}_a$ or G contains a central subgroup Z, with $Z \cong \mathbb{G}_a$, and such that G/Z is also unipotent (it is a recursive definition).
   (a) Show that the group of uni-uppertriangular matrices N is a unipotent group.

(b) Show that the unipotent subgroup N of any reductive group is a unipotent group. (Hint: define subgroups of N using subalgebras of $\mathfrak{n}$.)

(c) Show that every irreducible representation of a unipotent group G is trivial. (Hint: first show that Z acts by a scalar by Schur's lemma and hence trivially and then continue.)

(d) Show that if G is unipotent, then its Lie algebra is nilpotent. Is the converse true?

(e) (optional) Usually, unipotent groups are defined as groups G which admit a filtration $G_0 = \{1\} \subset G_1 \subset \cdots \subset G_n = G$, where each G_i is normal in G and G_{i+1}/G_i is isomorphic to $\mathbb{G}_a$ for all i. Prove or disprove that this matches my definition above.