**REPRESENTATION THEORY**
**ASSIGNMENT 1**
**DUE FRIDAY JANUARY 28**

(1) Let $G$ be a finite group acting on a finite set $X$. Explain how to construct a representation of $G$ on $V = \mathbb{C}[X]$. Prove that $\chi_V(g)$ is the number of fixed points of $g$ acting on $X$.
(2) Let $V$ be the 2-dimensional irreducible representation of $S_3$. Using the character table computed in class, decompose $V^{\otimes n}$ as a representation of $S_3$.
(3) Find the character table of $S_4$.
(4) Let $G$ be a finite group and $V$ be an irreducible representation. Prove that the dimension of $V$ divides the size of $G$.
(5) Prove that if $G$ is a finite group, then it is impossible to find a proper subgroup $T$, such that every element of $G$ is conjugate into $T$.

Use this to prove that if $G$ is a finite group and $T$ is a proper subgroup, then the map $Rep(G) \to Rep(T)$ (given by restriction of representations) is not injective.
(6) Take $T = U(1)^2$, thought of as $2 \times 2$ unitary diagonal matrices. $T$ acts on $\mathbb{C}^2$ in the obvious manner. Decompose $(\mathbb{C}^2)^{\otimes n}$ as a representation of $T$. (This means find all the weight spaces and their dimensions.) Do the same thing for $Sym^n\mathbb{C}^2$.
(7) Consider $\mathbb{C}^\times$ and its coordinate ring

$$R = \mathcal{O}(\mathbb{C}^\times) = \mathbb{C}[z, z^{-1}].$$

Define a $\mathbb{C}$-antilinear ring homomorphism $\sigma : R \to R$ by setting $\sigma(z^n) = z^{-n}$, and extending "antilinearly", so that

$$\sigma(\sum_n a_n z^n) = \sum_n \overline{a_n} z^{-n}$$

where $\bar{\ }$ denotes complex conjugation.

Prove that $R^\sigma = \{f \in R : f^\sigma = f\}$ is isomorphic to

$$\mathbb{R}[x, y]/(x^2 + y^2 - 1).$$

Now generalize this result. If $T$ is a compact torus and $T_\mathbb{C}$ is its complexification, construct an analog of $\sigma$ and compute its invariants.