MAT347Y1 HW4 Marking Scheme + Tips!

Friday, October 16

Total: 24 points.

2.2.10: 4 points.

2.4.7: 5 points.

- (1) Let $H = \langle (1\,2), (1\,3)(2\,4) \rangle$. Define a map $\phi : D_8 \to H$.
- (2) Check that the map respects the relations of D_8 (depending on how you defined ϕ, this would fall under either proving it's a homomorphism or checking that it's well-defined)
- (2) At least two of: injective, surjective, or computing the order of the subgroup.

(There are other ways to do this as well; for example, you could use the fact that D_8 acts faithfully on the vertices of a square to get an injective homomorphism $D_8 \to S_4$, and prove that the image is H)

Tip: it's in general a bad idea to use the technique of "these two groups have the same presentations and are therefore isomorphic." Why? Because proving that generators satisfy some relations is easy; proving that you've found a *complete* set of relations - necessary for it to be a presentation - is a lot harder. Better to construct a homomorphism (which only uses the relations of D_8, which you know), and use injectivity/surjectivity/order to show it's an isomorphism.

3.1.32: 5 points. And yes, the question asks about *every* subgroup so you do need to mention $\{1\}$ and Q_8.

- (3) Proofs of normality
- (2) Derivation of quotients

Tip: If you're trying to prove something by checking every element of a group, you're probably doing more work than you need to. For example in this case, you needed to check that $gHg^{-1} \subseteq H$ for all $g \in G$. Rather than checking ghg^{-1} for every $g \in G$ and $h \in H$, it's enough to check it on the generators of G and the generators of H. (Don't know why? Prove it!) Other common things a lot of students do in assignments: proving a map $\phi : G \to H$ is surjective by checking where every single element of H comes from (it's enough to show that generators of H are in the image of ϕ, because every other element is a product of these), or checking that ϕ is a homomorphism by computing it on every possible product of two things in G (it's enough to plug the relations of G into ϕ, because every other product can be computed from these).

3.1.35: 6 points. This problem is pretty easy if you use the first isomorphism theorem, but for those of you who didn't use it:

- (2) $\mathrm{SL}_n(F)$ is normal
- (2) Define a map from $\mathrm{GL}_n(F)/\mathrm{SL}_n(F)$ and show that it's well-defined and a homomorphism
- (2) Injectivity and surjectivity

Tip: Remember that being the kernel of a homomorphism is enough to show that a subgroup is normal - some of you did more work than you needed to by showing $M\mathrm{SL}_n(F)M^{-1} \subseteq \mathrm{SL}_n(F)$, and then defining a homomorphism with kernel $\mathrm{SL}_n(F)$. The second step makes the first one totally redundant.

3.2.11: 4 points. Note that if the problem specifically says "do not assume G is a finite group," you should not use notation like $i \in \{1, 2, \ldots, n\}$ unless you specifically state that this set is finite. Better to use "$i \in I$ for some index set I."

- (2) Every coset in G/H can be written using a representative of a coset in G/K and a representative of a coset in K/H
- (2) Each pair of coset representatives (one from G/K, one from K/H) gives a *unique* coset of G/H

Note: choosing representatives from each coset is necessary here. Defining a map $G/K \times K/H \to G/H$ by $(gK, kH) \mapsto gkH$, while it seems like an obvious thing to do, actually isn't well-defined! For example, let $G = Z_4 = \langle a \rangle$, $H = \{1\}$, $K = \langle a^2 \rangle$. Then $1K = a^2K$, but according to the map above, $(1K, 1H) \mapsto \{1\}$ while $(a^2K, 1H) \mapsto \{a^2\}$, a contradiction.