Groups

1. Let $A, B, C$ be finite abelian groups. Suppose that $A \times B \cong A \times C$. Prove that $B \cong C$. Can you find a counterexample if you remove the conditions on $A, B, C$?

2. (a) Construct a transitive action of $\mathbb{Z}_n$ on $\{1, \ldots, n\}$.
   (b) Are there any other groups of size $n$ which act transitively on $\{1, \ldots, n\}$?
   (c) Find all the orbits for the action of $\mathbb{Z}_n$ on the set of 2-element subsets of $\{1, \ldots, n\}$.
   (d) Count the number of orbits for the action of $\mathbb{Z}_n$ on the set of 2-element subsets of $\{1, \ldots, n\}$ using Burnside’s Lemma.

3. For every integer $n$, find how many elements of order $n$ there are in $S_5$.

4. Give an example of a group $G$ and a homomorphism $\phi : G \to G$ such that
   (a) $\phi$ is injective but not surjective
   (b) $\phi$ is surjective but not injective

5. Obtain a formula for the number of different necklaces that can be made with $n$ stones, if we have stones of $k$ different colours.
   Hint: I suggest you try to solve the cases $n = 5$ and $n = 6$ first.

6. Prove the a subgroup of index 2 is always normal.

7. Find two different decomposition series of $D_{12}$ that do not produce the same composition factors in the same order. How many different decomposition series does $D_{12}$ have?

8. Can you express $Q_8$ as a semi-direct product?

9. Find a presentation for $S_4$. (Warning: proving that you have a complete set of relations may be tedious)

10. Let $G$ be a nonabelian simple group. Prove that $|G|$ is divisible by at least two prime numbers.
11. Let $p$ be a prime. Let $\sigma \in S_p$ be a $p$-cycle and let $\tau \in S_p$ be a transposition. Prove that $\sigma$ and $\tau$ generate $S_p$. What if $p$ is not prime?

12. Let $G$ be a finite group and let $H$ be a proper subgroup. Prove that $\bigcup_{g \in G} gHg^{-1} \neq G$.

13. Let $G$ be a finite group. Prove that $G$ acts transitively on $\{1, \ldots, n\}$ if and only if $G$ contains a subgroup $H$ such that $[G : H] = n$.

Fields

1. Let $K$ be a finite field. Prove the product of all the non-zero elements of $K$ is equal to $-1$.

2. Give an example of a homomorphism $\phi : F \to F$ from a field to itself which is not an automorphism.

3. Determine the splitting field of the polynomial $x^p - x - 1$ over $F_p$. Show directly that its Galois group is cyclic.

4. Determine $[\mathbb{Q}(\sqrt[2k]{7 + 4\sqrt{3}}) : \mathbb{Q}]$.

5. Let $K/F$ be an extension of degree 2. Suppose that the characteristic of $F$ is not 2. Show that $K = F(\alpha)$ for some $\alpha \in K$ such that $\alpha^2 \in F$. What happens if the characteristic of $F$ is equal to 2? Find an example of a degree 3 extension $K/F$ such that $K$ is not equal to $F(\alpha)$ for any $\alpha \in K$ such that $\alpha^3 \in F$.

6. Let $\Phi_n(x) \in \mathbb{Q}[x]$ denote the $n$th cyclotomic polynomial. Show that $\Phi_{2k}(x) = x^{2^{k-1}} + 1$.

7. For each $n \geq 1$, find a finite extension $K/F$ such that $K \neq F(\alpha_1, \ldots, \alpha_n)$ for any $\alpha_1, \ldots, \alpha_n \in K$.

8. Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Let $\alpha, \beta$ be two roots of $f(x)$ (in some bigger field). Prove that there exists an isomorphism $F(\alpha) \cong F(\beta)$ (you may not use Theorem A). Give an example to show that the irreducibility of $f(x)$ is a necessary assumption.

9. Let $F$ be an algebraically closed field of characteristic $p$, and let $K = F(x, y)$. Let $a$ and $b$ satisfy $a^p = x$ and $b^p = y$. Prove that $K(a, b)/K$ is a finite extension with infinitely many intermediate fields.
Rings

1. Let $f(x), g(x) \in \mathbb{Z}[x]$. Prove that $f(x)$ and $g(x)$ are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal generated by $f(x)$ and $g(x)$ in $\mathbb{Z}[x]$ contains a nonzero integer.

2. Let $R$ be any ring with $1$. Let $a \in R$ and suppose that $a^n = 0$ for some $n$. Prove that $1 + a$ is a unit in $R$.

3. Let $R$ be a UFD in which every non-zero prime ideal is maximal. Prove that $R$ is a principal ideal domain.

4. Let $X$ be any set. We denote its power set (that is, the set of subsets of $X$) by $\mathcal{P}(X)$. We define two operations in $\mathcal{P}(X)$; given $A, B \subseteq X$:

   $A \setminus B = \{x \in A \mid x \notin B\}$
   $A \triangle B = (A \setminus B) \cup (B \setminus A)$

Notice that $(\mathcal{P}(X), \triangle, \cap)$ is an commutative ring with identity for any set $X$. (You do not need to prove this, but I recommend that you persuade yourself that this is true.)

(a) Assume that $X$ is finite. Then $(\mathcal{P}(X), \triangle)$ is a finite abelian group. By the Fundamental Theorem of Finite Abelian Groups, we know that it has to be isomorphic to a direct product of finite cyclic groups. Find positive integers $n_1, \ldots, n_r$ such that

$(\mathcal{P}(X), \triangle) \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$

and explicitly construct one such isomorphism.

(b) Find all zero-divisors and all units of the ring $(\mathcal{P}(X), \triangle, \cap)$.

(c) Describe all the principal ideals of the ring $(\mathcal{P}(X), \triangle, \cap)$.

(d) Assume $X$ is finite. Prove that every ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$ is principal.

(e) Construct an explicit example of a set $X$ and a non-principal ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$.

(f) Assume $X$ is finite. Find all prime ideals of the ring $(\mathcal{P}(X), \triangle, \cap)$.

(g) Assume $X$ is infinite. Prove that there exists a non-principal prime ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$.

5. Factor $4004$ into primes in $\mathbb{Z}[i]$.

6. Prove that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is an integral domain but not a field.