MAT 347
The action of a group on itself by conjugation
October 23, 2015

Recall some definitions. Let G be a group.

- Given $g \in G$, the centralizer of g in G is the subgroup $C_G(g) := \{a \in G \mid ag = ga\}$.
- Given $S \subseteq G$, the centralizer and the normalizer of S are the subgroups $C_G(S) := \{a \in G \mid ag = ga \; \forall g \in S\}$ and $N_G(S) := \{a \in G \mid aSa^{-1} = S\}$.
- Two elements $g, h \in G$ are *conjugate* when there exists $a \in G$ such that $h = aga^{-1}$. The conjugacy class of g in G is the set $K_G(g) := \{aga^{-1} \mid a \in G\}$.
- Two subsets $S, T \subseteq G$ are conjugate when there exists $a \in G$ such that $T = gSg^{-1}$.

Centers and centralizers

1. Given $S \subseteq G$, what is the relation between S, $C_G(S)$, and $N_G(S)$? In other words: which ones are contained in which ones?

   What if $S \leq G$?

2. What is the relation between the cardinalities of the conjugacy class of an element and its centralizer? *Hint:* Use the Orbit-Stabilizer theorem.

3. What is the size of the conjugacy class of 3-cycles in S_5? What is the size of the centralizer of a 3-cycle in S_5 ? Find all the elements in the centralizer of a 3-cycle in S_5? Generalize your answer to k-cycle in S_n.

4. Let $S \subseteq G$. What is the relation between the number of conjugates of S and the order of its centralizer/normalizer?

5. Complete the sentence: "The centre of a group $Z(G)$ is the set of elements whose conjugacy class has cardinality ..."

6. Complete the statement of *the Class Equation* and prove it:

   Theorem. Let G be a finite group and let $g_1, \ldots, g_r$ be [...] Then

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

A lemma

7. Prove or disprove:

   (a) Let G be a group. If $G/Z(G)$ is abelian, then G is abelian.

   (b) Let G be a group. If $G/Z(G)$ is cyclic, then G is cyclic.

   (c) Let G be a group. If $G/Z(G)$ is cyclic, then G is abelian.

p-groups

Definition. Let p be a prime. A p-group is a group whose order is a power of p.

8. Prove that every p-group has non-trivial centre. (This means, the centre is not just the identity.)

9. Prove that every group of order p^2 is abelian.

10. Classify the groups of order p^2 up to isomorphism.