Recall some definitions. Let $G$ be a group.

- Given $g \in G$, the centralizer of $g$ in $G$ is the subgroup $C_G(g) := \{ a \in G \mid ag = ga \}$.
- Given $S \subseteq G$, the centralizer and the normalizer of $S$ are the subgroups $C_G(S) := \{ a \in G \mid ag = ga \ \forall g \in S \}$ and $N_G(S) := \{ a \in G \mid aSa^{-1} = S \}$.
- Two elements $g, h \in G$ are conjugate when there exists $a \in G$ such that $h = a ga^{-1}$. The conjugacy class of $g$ in $G$ is the set $K_G(g) := \{ a ga^{-1} \mid a \in G \}$.
- Two subsets $S, T \subseteq G$ are conjugate when there exists $a \in G$ such that $T = gSg^{-1}$.

**Centers and centralizers**

1. Given $S \subseteq G$, what is the relation between $S$, $C_G(S)$, and $N_G(S)$? In other words: which ones are contained in which ones? What if $S \leq G$?

2. What is the relation between the cardinalities of the conjugacy class of an element and its centralizer? *Hint:* Use the Orbit-Stabilizer theorem.

3. What is the size of the conjugacy class of 3-cycles in $S_5$? What is the size of the centralizer of a 3-cycle in $S_5$? Find all the elements in the centralizer of a 3-cycle in $S_5$. Generalize your answer to $k$-cycle in $S_n$.

4. Let $S \subseteq G$. What is the relation between the number of conjugates of $S$ and the order of its centralizer/normalizer?

5. Complete the sentence: “The centre of a group $Z(G)$ is the set of elements whose conjugacy class has cardinality ...”

6. Complete the statement of *the Class Equation* and prove it:

   **Theorem.** Let $G$ be a finite group and let $g_1, \ldots, g_r$ be [$\ldots$] Then

   $$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$
A lemma

7. Prove or disprove:

(a) Let $G$ be a group. If $G/Z(G)$ is abelian, then $G$ is abelian.
(b) Let $G$ be a group. If $G/Z(G)$ is cyclic, then $G$ is cyclic.
(c) Let $G$ be a group. If $G/Z(G)$ is cyclic, then $G$ is abelian.

$p$-groups

Definition. Let $p$ be a prime. A $p$-group is a group whose order is a power of $p$.

8. Prove that every $p$-group has non-trivial centre. (This means, the centre is not just the identity.)

9. Prove that every group of order $p^2$ is abelian.

10. Classify the groups of order $p^2$ up to isomorphism.