MAT 347
Joins
October 9, 2015

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Let G be a group and let H, K be subgroups. Recall the set $HK = \{hk : h \in H, k \in K\}$. We call this the *product* of H and K. In general, this is not a subgroup! It is just a subset of G. Do not confuse this product with the abstract construction of the direct product. We define the *join* of H and K as the smallest subgroup of G containing both H and K. In other words, the join of H and K is $\langle H \cup K \rangle$.

1. Show that $HK \subseteq \langle H \cup K \rangle$.
2. Explore the relation between the following statements (which ones imply which ones)?
   (a) $HK = \langle H \cup K \rangle$.
   (b) $HK \leq G$.
   (c) $HK = KH$.
   (Notice that this does not mean that the elements of H and the elements of K commute with each other! It only means that HK and KH are the same set.)
   (d) $H \trianglelefteq G$.
3. Fine two subgroups H, K of D_8, each of order 2, such that $HK = \langle H \cup K \rangle$.
4. Find two subgroups $H, K$of D_8, each of order 2, such that $HK \neq \langle H \cup K \rangle$.
5. Recall that $H \times K = \{(h, k) : h \in H, k \in K\}$ is a group with multiplication defined component-wise. There is always a map of sets $H \times K \to \langle H \cup K \rangle$ given by $(h, k) \mapsto hk$. When is this map injective? When is this map a homomorphism of groups? When is this map an isomorphism of groups?