MAT 347
Cyclic groups and cyclic subgroups
October 2, 2015

Definition: A group is *cyclic* when it has a generating set with a single element. In other words, a group G is cyclic when there exists $a \in G$ such that

$$G := \{a^n \mid n \in \mathbb{Z}\}$$

When this happens, we write $G =< a >$.

1. If G is a cyclic group generated by a, what is the relation between $|G|$ and $|a|$? Remember that $|G|$ is the order of G, namely its cardinality. On the other hand, $|a|$ is the order of the element a, which has a different definition.

2. True of False? A group G is cyclic if and only if it contains an element whose order equals $|G|$.

3. Prove that two cyclic groups are isomorphic if and only if they have the same order.

Because of Problem 3, given any positive integer n, we define Z_n to be *the* cyclic group of order n. We normally use a multiplicative notation for it. A presentation of Z_n would be

$$Z_n :=< a \mid a^n = 1 >$$

Notice that Z_n is isomorphic to $\mathbb{Z}/n$ (where the group operation is addition).

On the other hand, we will just write $(\mathbb{Z}, +)$ for *the* cyclic group of infinite order, with additive notation.

4. Let G be a cyclic group generated by a. What are all the generators of G? (Here I am asking, which other single-element subsets of G would generate G?) How many of them are there? You may want to think of the finite and infinite cases separately. If you do not know how to start, consider the specific cases Z_6 and Z_{12} first.

5. If G is any group and $g \in G$, then g generates a subgroup $H =< g >$ of G. If g has order n, then what are the elements of H? What about if g has infinite order?

The subgroup H defined in the previous question is called a *cyclic subgroup* of G.

6. What can you say about all the subgroups of Z_n? This is a long, and somewhat vague question. Here are some ways to make it concrete:

    - Is every subgroup of Z_n cyclic?

- For every $d|n$, how many subgroups of order d does Z_n have?
- For each subgroup of Z_n, who are all its generators? How many are there?
- Which subgroups contains which other subgroups?

Again, if you do not know what to do, study Z_6 and Z_{12} first, then make a conjecture, and then try to prove it.

7. Investigate the same questions as in Problem 6, but this time for the infinite cyclic group.

8. Find all the cyclic subgroups of D_6. Does D_6 have any other proper subgroups?

9. Find a finite group G and a proper subgroup $H \leq G$ such that H is not a cyclic subgroup.