Actions

Definition. Let $G$ be a group and let $A$ be a set. An action of $G$ on $A$ is a map

$$G \times A \to A$$

$$(g, a) \mapsto g \cdot a$$

that satisfies the following two properties:

- $g \cdot (h \cdot a) = (gh) \cdot a$ for all $a \in A$, $g, h \in G$.
- $1 \cdot a = a$ for all $a \in A$.

Definition. An action of $G$ on $A$ is called faithful if for all $g \in G$, there exists $a \in A$ such that $g \cdot a \neq a$.

An action of $G$ on $A$ is called transitive if for all $a, b \in A$, there exists $g \in G$ such that $g \cdot a = b$.

1. Explain how the following groups act on the following sets. Which of these actions are faithful? Which are transitive?

   (a) $A$ is any set, $G = S_A$.
   (b) $G = D_{2n}$, $A$ is the set of vertices of a regular $n$-gon.
   (c) $G = D_{2n}$, $A$ is the set of diagonals of a regular $n$-gon.
   (d) $G$ is any group, $A = G$ as a set. (There are a few possibilities here.)
   (e) $A$ is the set of $k$-element subsets of $\{1, \ldots, n\}$ and $G$ is the group $S_n$. (Here $k, n$ are natural numbers with $0 \leq k \leq n$.)

2. Assume we have an action of the group $G$ on the set $A$. For each $g \in G$, let us define a map $\phi_g : A \to A$ by the equation $\phi_g(a) := g \cdot a$. Show that $\phi_g$ is a bijection. This defines a map $\phi : G \to S_A$ by the equation $\phi(g) := \phi_g$. Show that $\phi$ is a group homomorphism.

3. Conversely, show that every group homomorphism $G \to S_A$ comes from an action of $G$ on $A$. In other words, there is a natural one-to-one correspondence between actions of $G$ on $A$ and group homomorphisms from $G$ to $S_A$. This is why some authors define an action as a group homomorphism $G \to S_A$ instead.
The Orbit-Stabilizer Lemma

Definitions. Let $G$ be a group acting on a set $A$.

- Given $g \in G$, we define the fixed set of $g$ as the set
  \[ \text{Fix}(g) := \{a \in A \mid g \cdot a = a\} \subseteq A \]

- Given $a \in A$, we define the stabilizer of $a$ as the set
  \[ \text{Stab}(a) := \{g \in G \mid g \cdot a = a\} \subseteq G \]

- Given $a \in A$ we define the orbit of $a$ as the set
  \[ \Omega_a := \{g \cdot a \mid g \in G\} \subseteq A \]

4. Say we want to count how many different necklaces we can build with 6 stones each, if we have stones of two different colours. Define a diagram to be any way of colouring each of the six vertices of a hexagon with black or white. Notice that $|A| = 64$. Show that $D_{12}$ acts on $A$, and that the number of orbits of this action equals the number of different necklaces.

Note: This shows that the problem of counting the number of orbits of an action is an interesting problem in combinatorics.

5. Regarding the previous question, consider the following diagrams:

For each one of them, compute the size of its orbit and the size of its stabilizer. Make a conjecture or a formula that relates these two numbers for an arbitrary element in an arbitrary action. Then prove it.