MAT 347
The Galois Correspondence
February 26, 2016

The Galois group

Definition 1. *Let K/F be a field extension. The Galois group of K over F is defined as*

$$\mathrm{Gal}(K/F) = \{\phi : K \to K \mid \phi \text{ is an automorphism and } \phi(a) = a \text{ for all } a \in F\}$$

We have one useful result for finding the size of the Galois group.

Proposition 1. *Suppose that $K = F(\alpha)$ and let $f(x)$ be the minimal polynomial of α. Then the size of $\mathrm{Gal}(K/F)$ equals the number of roots of $f(x)$ which lie in K.*

1. Consider the field extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$. Find the Galois group of this extension.

2. Consider the field extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$, where $\zeta_5 = e^{2\pi i/5}$. Find the Galois group of this extension.

3. Consider the field extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$. Find the Galois group of this extension. (We discussed this in class on Wednesday.)

Intermediate fields and subgroups

Definition 2. *Let H be a subgroup of $\mathrm{Gal}(K/F)$. The* fixed field *of H, denoted $\mathrm{Inv}(H)$ or $\widehat{I}(H)$, consists of all the elements of K that are fixed by all the automorphisms in H. In other words,*

$$\widehat{I}(H) = \{\alpha \in K : \phi(\alpha) = \alpha \text{ for all } \phi \in H\}.$$

4. Show that $\widehat{I}(H)$ is a field which contains F.

5. If $H_1 \le H_2$ are subgroups of $\mathrm{Gal}(K/F)$, how are $\widehat{I}(H_1)$ and $\widehat{I}(H_2)$ related?

6. List all the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$ for $K = \mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\zeta_5), \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and find the corresponding fixed fields.

Definition 3. *If M is a field such that $F \subseteq M \subseteq K$, we call M an* intermediate *field between F and K. We denote $\mathrm{Gal}(K/M)$ by $\widehat{G}(M)$.*

7. Show that $\widehat{G}(M)$ is a subgroup of $\mathrm{Gal}(K/F)$.

8. If $M_1 \subseteq M_2$ are intermediate fields, how are $\widehat{G}(M_1)$ and $\widehat{G}(M_2)$ related?

9. Find all the intermediate fields between $\mathbb{Q}$ and K for $K = \mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\zeta_5), \mathbb{Q}(\sqrt{2}, \sqrt{3})$ For each intermediate field M, find $\widehat{G}(M)$.

The Galois correspondence

Note that we have defined two functions:

$$\widehat{I} : \{\text{subgroups of } \operatorname{Gal}(K/F)\} \longrightarrow \{\text{intermediate fields between } F \text{ and } K\}$$
$$\widehat{G} : \{\text{intermediate fields between } F \text{ and } K\} \longrightarrow \{\text{subgroups of } \operatorname{Gal}(K/F)\}.$$

10. For any intermediate field M between F and K, how are M and $\widehat{I}(\widehat{G}(M))$ related? Find an example (among ones we've seen so far) where they are not equal.

11. For any subgroup H of $\operatorname{Gal}(K/F)$, how are H and $\widehat{G}(\widehat{I}(H))$ related?

12. Find some examples (among ones we've seen so far) where the functions $\widehat{G}$ and $\widehat{I}$ actually *are* inverses.

13. In class, we discussed $K = \mathbb{Q}(\omega, \sqrt[3]{2})$ where $\omega = e^{2\pi i/3}$. We saw that $\operatorname{Gal}(K/\mathbb{Q}) = S_3$. In this case the Galois correspondence is a bijection. Find the lattice of subgroups of S_3 and the corresponding intermediate fields of $K/\mathbb{Q}$.