The Galois group

**Definition 1.** Let $K/F$ be a field extension. The Galois group of $K$ over $F$ is defined as

$\text{Gal}(K/F) = \{ \phi : K \to K \mid \phi \text{ is an automorphism and } \phi(a) = a \text{ for all } a \in F \}$

We have one useful result for finding the size of the Galois group.

**Proposition 1.** Suppose that $K = F(\alpha)$ and let $f(x)$ be the minimal polynomial of $\alpha$. Then the size of $\text{Gal}(K/F)$ equals the number of roots of $f(x)$ which lie in $K$.

1. Consider the field extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$. Find the Galois group of this extension.
2. Consider the field extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$, where $\zeta_5 = e^{2\pi i/5}$. Find the Galois group of this extension.
3. Consider the field extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$. Find the Galois group of this extension.
   (We discussed this in class on Wednesday.)

Intermediate fields and subgroups

**Definition 2.** Let $H$ be a subgroup of $\text{Gal}(K/F)$. The fixed field of $H$, denoted $\text{Inv}(H)$ or $\hat{I}(H)$, consists of all the elements of $K$ that are fixed by all the automorphisms in $H$. In other words,

$\hat{I}(H) = \{ \alpha \in K : \phi(\alpha) = \alpha \text{ for all } \phi \in H \}$

4. Show that $\hat{I}(H)$ is a field which contains $F$.
5. If $H_1 \leq H_2$ are subgroups of $\text{Gal}(K/F)$, how are $\hat{I}(H_1)$ and $\hat{I}(H_2)$ related?
6. List all the subgroups of $\text{Gal}(K/\mathbb{Q})$ for $K = \mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\zeta_5), \mathbb{Q}(\sqrt[2]{2}, \sqrt[2]{3})$ and find the corresponding fixed fields.

**Definition 3.** If $M$ is a field such that $F \subseteq M \subseteq K$, we call $M$ an intermediate field between $F$ and $K$. We denote $\text{Gal}(K/M)$ by $\hat{G}(M)$.

7. Show that $\hat{G}(M)$ is a subgroup of $\text{Gal}(K/F)$.
8. If $M_1 \subseteq M_2$ are intermediate fields, how are $\hat{G}(M_1)$ and $\hat{G}(M_2)$ related?
9. Find all the intermediate fields between $\mathbb{Q}$ and $K$ for $K = \mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\zeta_5), \mathbb{Q}(\sqrt[2]{2}, \sqrt[2]{3})$.
   For each intermediate field $M$, find $\hat{G}(M)$. 
The Galois correspondence

Note that we have defined two functions:

\[ \widehat{I} : \{ \text{subgroups of } \text{Gal}(K/F) \} \to \{ \text{intermediate fields between } F \text{ and } K \} \]
\[ \widehat{G} : \{ \text{intermediate fields between } F \text{ and } K \} \to \{ \text{subgroups of } \text{Gal}(K/F) \} . \]

10. For any intermediate field \( M \) between \( F \) and \( K \), how are \( M \) and \( \widehat{I}(\widehat{G}(M)) \) related? Find an example (among ones we’ve seen so far) where they are not equal.

11. For any subgroup \( H \) of \( \text{Gal}(K/F) \), how are \( H \) and \( \widehat{G}(\widehat{I}(H)) \) related?

12. Find some examples (among ones we’ve seen so far) where the functions \( \widehat{G} \) and \( \widehat{I} \) actually are inverses.

13. In class, we discussed \( K = \mathbb{Q}(\omega, \sqrt[3]{2}) \) where \( \omega = e^{2\pi i/3} \). We saw that \( \text{Gal}(K/\mathbb{Q}) = S_3 \). In this case the Galois correspondence is a bijection. Find the lattice of subgroups of \( S_3 \) and the corresponding intermediate fields of \( K/\mathbb{Q} \).