MAT 347
Constructions with straight-edge and compass
February 12, 2016

General constructions

For centuries, mathematicians searched for methods to solve geometric constructions using only ruler and compass. It never ocurred to the ancient Greeks that some of these constructions were impossible.

Definition 1. *Let P_0 be a set of points in the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$. The two basic constructions are*

Operation 1 (ruler)*: draw a line through any two points of P_0.*

Operation 2 (compass)*: draw a circle centered at any point of P_0 and with radius equal to the distance between a pair of points in P_0.*

The points of intersection of any lines and circles drawn using Operations 1 and 2 are said to be constructible from P_0 in one step*. Given $r \in \mathbb{R}^2$, we say that r is* constructible from P_0 *if there exist points $r_1, r_2, \ldots, r_n = r$ such that r_i is constructible in one step from*

$$P_0 \cup \{r_1, \ldots, r_{i-1}\}$$

for $i = 1, \ldots, n$.

1. Show that using Operations 1 and 2 we can do the following constructions:

   (a) Bisect a given segment.

   (b) Draw the bisector of a given angle.

   (c) Given a line l and a point P, draw a line through P perpendicular to l

   (d) Given a line l and a point P, draw a line through P parallel to l.

2. Let $z \in \mathbb{C}$. Show that the following are equivalent:

   (a) z can be constructed from $\{0, 1\}$.

   (b) We can construct segments with length $\Re z$ and $\Im z$ starting from a single segment of length 1 by using operations 1 and 2.

3. Here are three classical construction problems. Translate each one of them into a statement of the form "the point z can be constructible from a set of points P_0".

(a) Squaring the circle: given a circle, build a square that has the same area.
(b) Doubling the cube: given a cube, build a cube with volume twice as large.
(c) Trisecting an arbitrary angle.

Definition 2. *We denote by* $\mathbb{K}$ *the set of all* $z \in \mathbb{C}$ *that can be cosntructed from* $\{0, 1\}$.

We will now show that $\mathbb{K}$ is a field.

4. Show that if $\mathbb{K}$ is an subgroup (under addition) of $\mathbb{C}$.
5. Show that if $\alpha, \beta \in \mathbb{K}$, then $\alpha\beta \in \mathbb{K}$.
6. Show that if $\alpha \in \mathbb{K}$, then $\frac{1}{\alpha} \in \mathbb{K}$. Conclude that $\mathbb{K}$ is a field.

Now, we have to prove that $\mathbb{K}$ is not all of $\mathbb{C}$, or at least not all of $\overline{\mathbb{Q}}$. The way we will do this is by examining what happens when we do one construction.

7. Let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Let $z \in \mathbb{C}$. Assume that z can be be build from a set of points in F by using operation 1 or 2 just once. Prove that $[F(z) : F] = 1$ or 2.
8. Let $z \in \mathbb{C}$. Assume that $z \in \mathbb{K}$. Then there exist fields $Q = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n$ such that $z \in F_n$ and $[F_i : F_{i-1}] = 2$ for every i.
9. Let $z \in \mathbb{C}$. If $z \in \mathbb{K}$ then $[\mathbb{Q}(z) : \mathbb{Q}] = \deg m_{z,\mathbb{Q}}(X)$ is a power of 2.
10. Prove the following results.
    (a) It is impossible to square the circle.
    (b) It is impossible to double the cube.
    (c) It is impossible to trisect an arbitrary angle.