MAT 347
Fields of fractions
December 4, 2015

The only thing missing from $\mathbb{Z}$ to be a field is that there are no multiplicative inverses. We can construct $\mathbb{Q}$ by "adding multiplicative inverses" for every non-zero element of $\mathbb{Q}$. We are going to imitate this construction and try to make every ring into a field.

The construction

Let R be a ring. We are going to need some extra properties on R. Instead of listing them now, we invite you to discover them. Try to do the process layed out below. At various points it will not work without adding extra properties to R. Keep track of those properies.

Note: Some of you may be tempted to think about each step for two seconds and to pretend you get it without checking it in detail. You do not like writing things in detail. Don't be lazy and check things in detail! You need to do so to realize which properties our ring needs to satisfy.

1. We define the set $A := R \times (R \setminus \{0\})$. A "fraction" in R is like an element of A, except that different ordered pairs represent the same fraction. Define an equivalence relation in A that will correspond to the idea that two ordered pairs are related if and only if they represent the same fraction. (Think about how you would define this for $R = \mathbb{Z}$.) Prove that this relation is actually an equivalence relation. We denote by $\frac{a}{b}$ the equivalence class of an element $(a, b) \in A$. We will denote the quotient set as F.

2. F is going to be our candidate for a field. First of all, we need to define operations $+$ and $\cdot$ in F. Define them. Before we even explore which properties these operations satisfy, there is something else to check. What is it? Actually check.

3. Is $(F, +, \cdot)$ a field?

4. We want to say that F contains R as a subring. Define what this means formally (after all, R is not a subset of F properly speaking). Then prove it.

Minimality and uniqueness

5. We have constructed a field F that contains R. There could be other fields that contain R (for example: all of $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ contain $\mathbb{Z}$ as subrings) but this one is special because it is the *smallest*. We can make the meaning of this precise in three different ways.

   (a) Every element of F is the quotient of two elements of R. Verify this is true.

   (b) The subfield of F generated by R is F itself. Define what this means and verify it is true.

   (c) If K is another field that "contains" R, then K also "contains" F. Formally state what this should mean and prove it.

6. The *field of fractions* of R is a field that satisfies the property in Question 5c. We have proven that such a field exist by explicitly constructing it. But even before we constructed it, we could have proven that it had to be unique. Prove, directly from the property in Question 5c, that if F and F' are both fields satisfying this same property, then they are isomorphic.

Further questions

7. Let K be a field. Prove that K has a single smallest subfield, and that is is isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}/\mathbb{Z}p$ for some prime p. This subfield is called the *prime subfield* of K.

   Hint: Start by building the subring of K generated by 1.

8. We want to generalize the main construction in this worksheet. Let R be a ring and let $D \subseteq R$ be a subset. (We will need some extra properties that you should keep track of.) This time, we want to construct the *smallest* ring Q such that

   - R is subring of Q, and
   - every element of D is a unit in Q.

   How do we do this construction and when is it possible?