MAT 347
Subrings, ideals, and ring homomorphisms
November 27, 2015

A lot of the concepts we want to define for rings, and a lot of the theorems we want to prove for rings, are entirely the same as their counterparts for groups. Your task today is to think of all the group theory you know and find out what generalizes and how.

Subrings

1. Let A be a ring. Let $R \subseteq A$ be a subset. We say that R is a *subring* of A, and we write it $R \leq A$ when R is also a ring with the same operations as R. Write down the criterion for quickly checking when a subset of a ring is a subring (analogous to the criterion for quickly checking when a subset of a group is a subgroup).

2. Find all the subrings of $\mathbb{Z}$.

   Hint: You already know what all the subgroups of $(\mathbb{Z}, +)$ are.

Ideals

3. Let $(A, +, \cdot)$ be a ring and let $I \leq A$ be a subring. We want to define the quotient A/I. For every $a \in A$ we define

   $$a + I =: \{a + r \mid r \in I\}$$

   We also define the quotient set

   $$A/I := \{a + I \mid a \in A\}$$

   What is this relation between this quotient set and the quotient construction for groups that we learned earlier?

4. We want to define operations on the quotient A/I. Specifically, we want to define

   $$(a + I) + (b + I) := (a + b) + I$$
   $$(a + I)(b + I) := ab + I$$

   We say that I is an *ideal* when these operations are well-defined. We write $I \trianglelefteq A$. (Think of ideals as "normal subrings"). Find a simple algebraic characterization of ideal, just like the characterization of normal subgroups as "closed under conjugation".

5. Assume that I is an ideal. What else do we need to require for A/I to be a ring? If A is abelian/has an identity, does it follow that A/I is abelian/has an identity?
6. Find all the ideals of $\mathbb{Z}$.
7. Find all the ideals of $\mathbb{Q}$. In which other rings would you get the same result as on $\mathbb{Q}$?
8. Given two rings A and B, their cartesian product $A \times B$ is a ring with operations defined componentwise. Find an example of a subring of $A \times B$ (for particular rings A and B) which is not an ideal.

Ring homomorphisms

9. Define ring homomorphism and ring isomorphism.
10. Define kernel of a ring homomorphism.
11. The kernel of a group homomorphism is always a normal subgroup. State and prove the corresponding result for rings.
12. A group homomorphism is injective if and only if its kernel is trivial. State and prove the corresponding result for rings.
13. State and prove the First Isomorphism Theorem for rings. (Imitate the corresponding theorem for groups.)

Extra: the other isomorphism theorems

14. State and prove the second isomorphism theorem for rings.

    Note: For groups, this required looking at intersection and product of subgroups, but "product" now means "sum". Or something.
15. State and prove the third and fourth isomorphism theorems for rings.