MAT 347
Classification of finitely generated abelian groups
November 13, 2015

Recall that a group G is *generated* by a set $S \subset G$, if S is not contained in any proper subgroup of G. Equivalently, G is generated by S if the map $F(S) \to G$ is surjective. If G is abelian and S has size n, then G is generated by S, if the map $\mathbb{Z}^n \to G$ is surjective.

A group G is called *finitely generated* if is generated by a finite subset $S \subset G$. Of course, every finite group G is finitely generated. In this worksheet, we will classify finitely generated abelian groups.

Throughout the worksheet we will work with abelian group and use additive notation. Also if $k \in \mathbb{Z}$ and $g \in G$, then we will write $kg = g + \cdots + g$ (k times), if $k \geq 0$ with obvious modification if $k < 0$.

Some examples

1. Prove that $\mathbb{Z}/2 \times \mathbb{Z}/3 \cong \mathbb{Z}/6$.
2. Prove that $\mathbb{Z}/2 \times \mathbb{Z}/2$ is not isomorphic to $\mathbb{Z}/4$.
3. Give an example of an abelian group which is not finitely generated.

The classification

Theorem 1. *Suppose that G is a finitely generated abelian group. Then there exists unique integers $r, k_1, \ldots, k_n$ with $r \geq 0, k_i \geq 2$ and $k_i | k_{i+1}$ for $i = 1, \ldots, n-1$ such that*

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/k_1 \times \cdots \times \mathbb{Z}/k_n$$

4. Use the classificaiton to find all abelian groups (up to isomorphism) of order 120.
5. *[Putnam 2009 - A5]* Is there a finite abelian group such that the product of the orders of all its elements is 2^{2009}?

Integer matrices

Suppose that G is a finitely generated abelian group and that we choose n generators for G. Thus we get a surjective homomorphism $\mathbb{Z}^n \to G$ with kernel K. Then by the first isomorphism theorem $G \cong \mathbb{Z}^n/K$. We will use the following result, which we will not prove.

Theorem 2. *Let $K \subset \mathbb{Z}^n$ be any subgroup. Then $K \cong \mathbb{Z}^m$ for some $m \leq n$.*

If particular, this means that we can choose a set of n generators for K.

6. Consider the group $G = \mathbb{Z}/5$. Let us pick two generators $\bar{2}, \bar{3}$ for G. Let K be the kernel of $\mathbb{Z}^2 \to \mathbb{Z}/5$. Draw a picture of K. Find two generators of K. Explain how this gives rise to a presentation of G with two generators and two relations.

7. Let G be a finitely generated abelian group with n generators. Explain how we get a presentation of G with n generators and n relations.

8. Explain how to encode this presentation using an $n \times n$ integer matrix L. Find this matrix for the above presentation of $\mathbb{Z}/5$.

9. Conversely, suppose you are given an square integer matrix L. Explain how this leads to a finitely generated abelian group.

We will now explore how the matrix L depends on the various choices we made.

10. Return to the example of $G = \mathbb{Z}/5$. Choose different generators for the kernel K. How does this affect the resulting matrix L? Choose different generators for G. How does this affect the resulting matrices?

11. Let G be an abelian group with generators $g_1, \ldots, g_n$ and relations $r_1, \ldots, r_n$ (generators for the kernel K), resulting in an integer matrix L. How does L change if we choose another set of generators $g'_1, \ldots, g'_n$ for G or another set of generators $r'_1, \ldots, r'_n$ for the kernel? What about if n changes?

Smith Normal Form

An integer $n \times n$ matrix S is said to be in *Smith Normal Form* if it is diagonal with diagonal entries $d_1, \ldots, d_n$ satisfying $d_1 | d_2 | \cdots | d_n$, $d_i \geq 0$.

Let $GL_n\mathbb{Z}$ denote the group of integer matrices of determinant ± 1 (this is equivalent to saying that their inverse is still an integer matrix).

Theorem 3. *Let L be an integer $n \times n$ matrix. Then there exists a unique Smith Normal Form matrix S and matrices $A, B \in GL_n\mathbb{Z}$ such that $ALB = S$.*

12. Prove the existence part of this theorem by describing an algorithm to reduce any matrix to Smith Normal Form using elementary row and column operations. (Compared to usual linear algebra, which row and column operations are allowed?)

13. Prove the uniqueness part of this theorem.

14. Use this theorem to prove the classification of finitely generated abelian groups.

15. Reprove that $\mathbb{Z}/2 \times \mathbb{Z}/3 \cong \mathbb{Z}/6$ by considering the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

16. Consider the abelian group defined by the presentation:

$$G = \langle a, b, c : a + 3b + c = 0, 3a + b + 3c = 0, a + 3b + 5c = 0 \rangle$$

Express G as a product of cyclic groups.