Order

Let $G$ be a group. Let $a \in G$. We want to compare the powers of $a$. In other words, when do we have $a^n = a^m$?

First, we define the order of $a$ as the smallest positive integer $n$ such that $a^n = 1$, if there is such a thing. Otherwise we define the order of $a$ to be infinity. We denote the order of $a$ by $|a|$.

1. Let $G$ be a group, let $a \in G$, and let $r = |a|$. Complete the following statements and prove them:

   (a) $a^n = 1 \iff \ldots$ (something about $n$ and $r$)
   (b) $a^n = a^m \iff \ldots$ (something about $n$, $m$, and $r$)

2. Find the order of every element in $\mathbb{Z}/18\mathbb{Z}$.

3. Find an example of a group $G$ that contains one element of order $n$ for every positive integer $n$ and which also contains an element of order infinity.

4. Find the order of every element of the Dihedral group $D_{10}$.

Order in symmetric groups

The cycle notation for symmetric groups is well-adapted to finding the order of elements.

1. What is the order of a $k$-cycle?

2. What is the order of the following elements of $S_6$?

   (a) $(123)(45)$
   (b) $(12)(34)(56)$
   (c) $(123)(456)$

3. Given a permutation expressed as a product of disjoint cycles, explain how you would compute its order.

4. What is the maximal order of an element in $S_7$?