MAT 247, Winter 2014
Assignment 8
Due March 18

1. The purpose of this exercise is to establish the properties of the sign of a permutation. Recall that S_n denotes the set of all permutations of $\{1, \dots, n\}$. If σ, τ are two permutations, then $\sigma\tau$ denotes their composition, so $(\sigma\tau)(i) = \sigma(\tau(i))$. We will write id for the identity permutation. In this exercise, do not use the determinant of matrices.

   (a) A simple transposition is a permutation $\sigma \in S_n$ which exchanges a pair of neighbouring elements. More precisely, there exists i with

   $$\begin{aligned} \sigma(i+1) &= i \\ \sigma(i) &= i+1 \\ \sigma(j) &= j \text{ otherwise.} \end{aligned}$$

   Prove that if $\sigma_1, \dots, \sigma_k$ are all simple transpositions and $\sigma_1\sigma_2 \cdots \sigma_k = id$, then k is even. Hint: one way to do this is to consider the length function $\ell(\sigma) = |\{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}|$.

   (b) Use (a) to prove that there exists a unique function $sign : S_n \to \{1, -1\}$ such that $sign(\sigma\tau) = sign(\sigma)sign(\tau)$ for any two permutations and $sign(\sigma) = -1$ whenever σ is a simple transposition.

   (c) Prove that $sign(\sigma) = -1$ whenever σ is any transposition.

2. Let V, W be two vector spaces. Let $B(V, W)$ denote the set of all bilinear pairings $B : V \times W \to \mathbb{F}$.

   (a) Construct a natural linear map $V^* \otimes W^* \to B(V, W)$.

(b) Prove that this map is an isomorphism when V, W are finite-dimensional.

(c) Suppose that V, W are finite-dimensional. Construct a natural isomorphism $V \otimes W \to B(V^*, W^*)$. (This is the definition of $V \otimes W$ from *Linear Algebra Done Wrong*.)

3. Let $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$ the field with two elements. In this field $1 + 1 = 0$.

   Let $V = \mathbb{F}^2$. Consider the linear map $\tau : V \otimes V \to V \otimes V$, defined by $\tau(v \otimes w) = w \otimes v$. Find a basis β for $V \otimes V$ such that $[\tau]_\beta$ is a Jordan form matrix.

4. Let V be a vector space with basis $\{x_1, \ldots, x_n\}$. Then every element of $V \otimes V$ can be written uniquely as $y = \sum_{1 \leq i,j \leq n} c_{ij} x_i \otimes x_j$. So every element of $V \otimes V$ can be represented by a matrix $C = (c_{ij})$. Prove that $y \in Sym^2 V$ if and only if C is symmetric and that $y \in \Lambda^2 V$ if and only if C is skew-symmetric.