MAT 247, Winter 2014
Assignment 4
Due Feb 4

1. Let T be a linear operator on a complex vector space V. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of T.

   (a) Let $v \in V$. Prove that there exist unique vectors $v_1, \dots, v_k$ such that $v = v_1 + \cdots + v_k$ and $v_i \in K_{\lambda_i}$ for all i.

   (b) Define a linear operator $D : V \to V$ by $D(v) = \lambda_1 v_1 + \cdots + \lambda_k v_k$ and let $N = T - D$. Prove that D is diagonalizable, N is nilpotent, and $DN = ND$.

   (c) Pick a basis β for V for which $[T]_\beta$ is a Jordan form matrix. Describe $[D]_\beta$ and $[N]_\beta$.

2. Let A be an $n \times n$ matrix with real entries. Prove that the minimal polynomial of A when considered as a real matrix is the same as the minimal polynomial of A when considered as a complex matrix.

3. (a) Using the property $\det(AB) = \det(A)\det(B)$ prove that the determinant of a matrix is unchanged if we add a multiple of one row to another row.

   (b) Find the determinant of the following matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 3 & 7 \end{bmatrix} \tag{1}$$

4. Let P_n denote the vector space of polynomials (with complex coefficients) of degree at most $n-1$. Let $c_1, \dots, c_n$ denote n complex numbers. Define a linear map $T : P_n \to \mathbb{C}^n$ by $T(f) = (f(c_1), \dots, f(c_n))$.

(a) Let $\alpha = \{1, x, \ldots, x^{n-1}\}$ be the usual basis for P_n and let β be the standard basis for $\mathbb{C}^n$. Prove that

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & c_1 & \ldots & c_1^{n-1} \\ 1 & c_2 & \ldots & c_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & c_n & \ldots & c_n^{n-1} \end{bmatrix}$$

(b) Prove that the determinant of the above matrix is

$$\prod_{1 \leq i < j \leq n} c_j - c_i.$$

[Hint: first use column operations to make the first row [1 0 ... 0] and then evaluate the determinant by expanding along the first row.]

(c) Prove that T is invertible if and only if $c_1, \ldots, c_n$ are distinct.