MAT 247, Winter 2014
Assignment 10
Due April 1

1. Let $V = \mathbb{F}_2^2$ (here $\mathbb{F}_2$ is the field with two elements). Find a non-zero bilinear form H on V for which every vector is null.

2. An integer n is called square-free if it is not divisible by the square of any integer.

   (a) Let $\mathbb{F} = \mathbb{Q}$, the field of rational numbers. Let H be a symmetric bilinear form on a vector space V. Prove that there exists a basis for V for which the matrix representing H is diagonal with square-free integers on the diagonal.

   (b) Give an example of H, V as above which shows that the resulting diagonal matrix is not unique (even up to permutation).

3. Let Q be a non-degenerate quadratic form on $\mathbb{R}^3$. Given any real number c, the set of solutions to the equation $Q(x, y, z) = c$ is called a non-degenerate quadric surface.

   (a) Up to linear transformation, how many different non-degenerate quadric surfaces are there? Draw a picture of each of them.

   (b) Which quadric surface is defined by the equation

   $$x^2 + 2xz + y^2 + z^2 + 6yz = 10$$

4. Let H be a symmetric bilinear form on a n-dimensional vector space V. Let β be a basis for V. Let A be the matrix of H with respect to the basis β.

(a) Prove that $A = [\tilde{H}]_{\beta}^{\beta^*}$. (Here β^* is the dual basis for V^* constructed from β.)

(b) Use (a) to prove that if γ is another basis for V, then the matrix of H with respect to γ is $P^t AP$ where P is the change of basis matrix from β to γ.

(c) Use (a) to prove that $rank(A) + \dim rad(H) = n$, where

$$rad(H) = \{v \in V : H(v, w) = 0 \text{ for all } w \in V\}.$$