1. Let $V, \langle , \rangle$ be an inner product space. Let $W \subset V$ be a subspace.

(a) Give the definition of $W^\perp$, the orthogonal complement of $W$.
(b) Suppose that $W^\perp = V$. Prove that $W = \{0\}$.

Solution:

(a) 
\[ W^\perp = \{v \in V : \langle w, v \rangle = 0 \text{ for all } w \in W \} \]

(b) Suppose that $w \in W$. Then, since $W^\perp = V$, we have $\langle v, w \rangle = 0$ for all $v \in V$. In particular $\langle w, w \rangle = 0$. Thus $w = 0$. 
2. Consider $\mathbb{R}^3$ with the usual inner product. Let $W$ be the span of $(1, 0, 0)$ and $(1, 1, 1)$.

(a) Perform the Gram-Schmidt process to these vectors to find an orthonormal basis for $W$.
(b) Find the orthogonal projection of $(0, 0, 1)$ onto $W$.

Solution:

(a) Let $w_1 = (1, 0, 0), w_2 = (1, 1, 1)$. Then since $w_1$ is already unit length, we set $e_1 = w_1$. Then we define

$$v_2 = w_2 - (e_1, w_2)e_1 = (1, 1, 1) - 1(1, 0, 0) = (0, 1, 1)$$

and we set $e_2 = \frac{v_2}{||v_2||} = \frac{1}{\sqrt{2}}(0, 1, 1)$. Thus $e_1, e_2$ is an orthonormal basis for $W$.

(b) We compute

$$v = (e_1, (0, 0, 1))e_1 + (e_2, (0, 0, 1))e_2 = 0 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}(0, 1, 1) = (0, \frac{1}{2}, \frac{1}{2})$$

Thus $v = (0, \frac{1}{2}, \frac{1}{2})$ is the projection of $(0, 0, 1)$ onto $W$. 

2
3. Let $V$ be a real inner product space.

(a) Give the definition of a self-adjoint linear operator on $V$.

(b) Suppose that a linear operator $T : V \to V$ is orthogonally diagonalizable (i.e. there exists an orthonormal basis for $V$ consisting of eigenvectors for $T$). Show that $T$ is self-adjoint.

Solution:

(a) A self-adjoint linear operator is a linear operator $T : V \to V$ where $T = T^*$.

(b) Choose an orthonormal basis $e_1, \ldots, e_n$ for $V$ consisting of eigenvectors for $T$. Consider the matrix

$$A = [T]_{e_1, \ldots, e_n}$$

of $T$ with respect to this basis. Since this is an orthonormal basis,

$$[T^*]_{e_1, \ldots, e_n} = [T]^*_{e_1, \ldots, e_n} = A^*.$$

Since this is a basis of eigenvectors, $A$ is a diagonal matrix (with real entries since we are working with a real vector space) and so $A^* = A$. Thus $[T^*]_{e_1, \ldots, e_n} = A$ and so $T^* = T$. Hence $T$ is self-adjoint.
4. Let $V$ be an inner product space.

(a) Give an example of a linear operator $T : V \to V$ such that $\text{null}(T) \neq \text{null}(T^*)$.

(b) Show that it is not possible to find an example when $T$ is normal.

(c) Show that for any linear operator $T : V \to V$, $\dim \text{null}(T) = \dim \text{null}(T^*)$.

Solution:

(a) Consider $V = \mathbb{R}^2$ and consider the linear operator

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then since the standard basis of $\mathbb{R}^2$ is an orthonormal basis,

$$T^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$ 

A simple computation shows that $\text{null}(T) = \text{span}(1, 0)$ and $\text{null}(T^*) = \text{span}(0, 1)$. Thus $\text{null}(T) \neq \text{null}(T^*)$.

(b) If $T$ is normal, then for all $v \in V$.

$$\langle Tv, Tv \rangle = \langle T^*Tv, v \rangle = \langle TT^*v, v \rangle = \langle T^*v, T^*v \rangle$$

and thus $||Tv|| = ||T^*v||$.
Thus $||Tv|| = 0$ if and only $||T^*v|| = 0$. Hence $v \in \text{null}(T)$ if and only if $v \in \text{null}(T^*)$. Hence $\text{null}(T) = \text{null}(T^*)$ for all normal operators $T$.

(c) For any linear operator $T$,

$$\dim V = \dim \text{null}(T) + \dim \text{range}(T).$$

Also we know that $\text{range}(T) = \text{null}(T^*)^\perp$. Hence

$$\dim V = \dim \text{range}(T) + \dim \text{null}(T^*).$$

Combining these two equations, we obtain that $\dim \text{null}(T) = \dim \text{null}(T^*)$. 

4
5. Let \( V, \langle \cdot, \cdot \rangle \) be an inner product space and let \( T : V \to V \) be a linear operator. Suppose that for all pairs of vectors \( v, w \in V \), \( \langle Tv, Tw \rangle = 0 \) if and only if \( \langle v, w \rangle = 0 \) (in other words, \( T \) preserves the property of orthogonality). Show that there exists some scalar \( a \) such that \( aT \) is an isometry.

**Solution:**
First, notice that \( T \) is injective, since if \( Tv = 0 \), then \( \langle Tv, Tv \rangle = 0 \), so \( \langle v, v \rangle = 0 \) by the hypothesis and hence \( v = 0 \).

Pick an orthonormal basis \( e_1, \ldots, e_n \) for \( V \). We want to show that there exists a scalar \( a \), such that \( aTe_1, \ldots, aTe_n \) is an orthonormal basis. By the hypothesis, we see that for all \( a \), and all \( i \neq j \),

\[
\langle aTe_i, aTe_j \rangle = 0.
\]

So it remains to show that we can pick \( a \) so that \( ||aTe_i|| = 1 \) for all \( i \).

Pick some \( i > 1 \) and consider \( e_1 - e_i \) and \( e_1 + e_i \). We have

\[
\langle e_1 - e_i, e_1 + e_i \rangle = \langle e_1, e_1 \rangle - \langle e_i, e_1 \rangle + \langle e_1, e_i \rangle - \langle e_i, e_i \rangle = 1 - 0 + 0 - 1 = 0
\]

Thus by the hypothesis, \( \langle T(e_1 - e_i), T(e_1 + e_i) \rangle = 0 \). Hence,

\[
0 = \langle Te_1 - Te_i, Te_1 + Te_i \rangle = \langle Te_1, Te_1 \rangle - \langle Te_i, Te_i \rangle
\]

since by the hypothesis \( \langle Te_1, Te_i \rangle = 0 \). So \( ||Te_1|| = ||Te_i|| \) for all \( i \).

Since \( T \) is injective, \( ||Te_1|| \neq 0 \). Let

\[
a = \frac{1}{||Te_1||}.
\]

Then, since \( ||Te_1|| = ||Te_i|| \), we see that \( ||aTe_i|| = 1 \) for all \( i \). Thus \( aTe_1, \ldots, aTe_n \) is an orthonormal basis and hence \( aT \) is an isometry.