MAT 247 midterm

Name:

February 15, 2011

1. Let $V, \langle , \rangle$ be an inner product space. Let $W \subset V$ be a subspace.
   (a) Give the definition of $W^\perp$, the orthogonal complement of W.
   (b) Suppose that $W^\perp = V$. Prove that $W = \{0\}$.

Solution:

(a)
$$W^\perp = \{v \in V : \langle w, v\rangle = 0 \text{ for all } w \in W\}$$

(b) Suppose that $w \in W$. Then, since $W^\perp = V$, we have $\langle v, w\rangle = 0$ for all $v \in V$. In particular $\langle w, w\rangle = 0$. Thus $w = 0$.

2. Consider $\mathbb{R}^3$ with the usual inner product. Let W be the span of $(1,0,0)$ and $(1,1,1)$.

   (a) Perform the Gram-Schmidt process to these vectors to find an orthonormal basis for W.

   (b) Find the orthogonal projection of $(0,0,1)$ onto W.

Solution:

(a) Let $w_1 = (1,0,0), w_2 = (1,1,1)$. Then since w_1 is already unit length, we set $e_1 = w_1$. Then we define

$$v_2 = w_2 - \langle e_1, w_2 \rangle e_1 = (1,1,1) - 1(1,0,0) = (0,1,1)$$

and we set $e_2 = \frac{v_2}{||v_2||} = \frac{1}{\sqrt{2}}(0,1,1)$. Thus e_1, e_2 is an orthonormal basis for W.

(b) We compute

$$v = \langle e_1, (0,0,1) \rangle e_1 + \langle e_2, (0,0,1) \rangle e_2 = 0 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}(0,1,1) = (0, \frac{1}{2}, \frac{1}{2})$$

Thus $v = (0, \frac{1}{2}, \frac{1}{2})$ is the projection of $(0,0,1)$ onto W.

3. Let V be a real inner product space.

   (a) Give the definition of a self-adjoint linear operator on V.

   (b) Suppose that a linear operator $T : V \to V$ is orthogonally diagonalizable (i.e. there exists an orthonormal basis for V consisting of eigenvectors for T). Show that T is self-adjoint.

Solution:

(a) A self-adjoint linear operator is a linear operator $T : V \to V$ where $T = T^*$.

(b) Choose an orthonormal basis $e_1, \ldots, e_n$ for V consisting of eigenvectors for T. Consider the matrix

$$A = \big[T\big]_{e_1,\ldots,e_n}$$

of T with respect to this basis. Since this is an orthonormal basis,

$$\big[T^*\big]_{e_1,\ldots,e_n} = \big[T\big]^*_{e_1,\ldots,e_n} = A^*.$$

Since this is a basis of eigenvectors, A is a diagonal matrix (with real entries since we are working with a real vector space) and so $A^* = A$. Thus $\big[T^*\big]_{e_1,\ldots,e_n} = A$ and so $T^* = T$. Hence T is self-adjoint.

4. Let V be an inner product space.

   (a) Give an example of a linear operator $T : V \to V$ such that $\operatorname{null}(T) \neq \operatorname{null}(T^*)$.

   (b) Show that it is not possible to find an example when T is normal.

   (c) Show that for any linear operator $T : V \to V$, $\dim \operatorname{null}(T) = \dim \operatorname{null}(T^*)$.

Solution:

(a) Consider $V = \mathbb{R}^2$ and consider the linear operator

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then since the standard basis of $\mathbb{R}^2$ is an orthonormal basis,

$$T^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

A simple computation shows that $\operatorname{null}(T) = \operatorname{span}(1, 0)$ and $\operatorname{null}(T^*) = \operatorname{span}(0, 1)$. Thus $\operatorname{null}(T) \neq \operatorname{null}(T^*)$.

(b) If T is normal, then for all $v \in V$.

$$\langle Tv, Tv\rangle = \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle = \langle T^*v, T^*v\rangle$$

and thus $||Tv|| = ||T^*v||$.
Thus $||Tv|| = 0$ if and only $||T^*v|| = 0$. Hence $v \in \operatorname{null}(T)$ if and only if $v \in \operatorname{null}(T^*)$. Hence $\operatorname{null}(T) = \operatorname{null}(T^*)$ for all normal operators T.

(c) For any linear operator T,

$$\dim V = \dim \operatorname{null}(T) + \dim \operatorname{range}(T).$$

Also we know that $\operatorname{range}(T) = \operatorname{null}(T^*)^\perp$. Hence

$$\dim V = \dim \operatorname{range}(T) + \dim \operatorname{null}(T^*).$$

Combining these two equations, we obtain that $\dim \operatorname{null}(T) = \dim \operatorname{null}(T^*)$.

5. Let $V, \langle , \rangle$ be an inner product space and let $T : V \to V$ be a linear operator. Suppose that for all pairs of vectors $v, w \in V$, $\langle Tv, Tw\rangle = 0$ if and only if $\langle v, w\rangle = 0$ (in other words, T preserves the property of orthogonality). Show that there exists some scalar a such that aT is an isometry.

Solution:

First, notice that T is injective, since if $Tv = 0$, then $\langle Tv, Tv\rangle = 0$, so $\langle v, v\rangle = 0$ by the hypothesis and hence $v = 0$.

Pick an orthonormal basis $e_1, \ldots, e_n$ for V. We want to show that there exists a scalar a, such that $aTe_1, \ldots, aTe_n$ is an orthonormal basis. By the hypothesis, we see that for all a, and all $i \neq j$,

$$\langle aTe_i, aTe_j\rangle = 0.$$

So it remains to show that we can pick a so that $||aTe_i|| = 1$ for all i.

Pick some $i > 1$ and consider $e_1 - e_i$ and $e_1 + e_i$. We have

$$\langle e_1 - e_i, e_1 + e_i\rangle = \langle e_1, e_1\rangle - \langle e_i, e_1\rangle + \langle e_1, e_i\rangle - \langle e_i, e_i\rangle = 1 - 0 + 0 - 1 = 0$$

Thus by the hypothesis, $\langle T(e_1 - e_i), T(e_1 + e_i)\rangle = 0$. Hence,

$$0 = \langle Te_1 - Te_i, Te_1 + Te_i\rangle = \langle Te_1, Te_1\rangle - \langle Te_i, Te_i\rangle$$

since by the hypothesis $\langle Te_1, Te_i\rangle = 0$. So $||Te_1, Te_1|| = ||Te_i Te_i||$ for all i.

Since T is injective, $||Te_1|| \neq 0$. Let

$$a = \frac{1}{||Te_1||}.$$

Then, since $||Te_1|| = ||Te_i||$, we see that $||aTe_i|| = 1$ for all i. Thus $aTe_1, \ldots, aTe_n$ is an orthonormal basis and hence aT is an isometry.