(1) Let $V, \langle \cdot, \cdot \rangle$ be a real inner product space. Let $T$ be a self-adjoint operator on $V$. On the previous assignment, we saw that $T$ defines a symmetric bilinear form $H$ by the formula $H(v, w) = \langle Tv, w \rangle$.

(a) Show that $\text{null}(T) = \text{rad}(H)$.

(b) Find the signature of $H$ in terms of information about the eigenvalues of $T$.

(2) Let $V, W$ be two real vector spaces of the same dimension and let $H_V, H_W$ be symmetric bilinear forms on $V, W$ respectively. We say that an invertible linear map $T : V \to W$ is an orthogonal isomorphism if

$$H_V(v_1, v_2) = H_W(Tv_1, Tv_2), \text{ for all } v_1, v_2 \in V.$$ 

Prove that there exists an orthogonal isomorphism $T : V \to W$ if and only if the signature of $H_V$ is the same as the signature of $H_W$.

(3) Let $V = \mathbb{R}^2$. Define a bilinear form $H_A$ on $V$ using the matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$.

What is the signature of $H$?