# Final exam study suggestions

April 4, 2011

Review the definitions of the following notions.

1. inner product
2. adjoint of a linear operator (in the presence of an inner product)
3. orthogonal complement
4. normal, self-adjoint, isometry, and positive operators
5. singular values
6. bilinear form
7. quadratic form
8. isotropic vector and subspace
9. non-degenerate bilinear form
10. signature of a symmetric bilinear form
11. radical of symmetric bilinear form
12. symplectic form
13. Lagrangian subspace
14. dual vector space
15. dual basis
16. multilinear forms (also known as $k$-forms)
17. tensor products

Review all of your homework problems and also the following questions.

1. Axler 6.13, 6.17, 6.21, 7.3, 7.4, 7.15, 7.21, 7.23, 7.24, 7.31, 7.32
2. Let $p(x, y) = x^2 + 3xy + 2y^2$. Find a symmetric bilinear form on $\mathbb{R}^2$ for which this polynomial is the associated quadratic form. Find the signature of this bilinear form.

3. Let $H$ be a symmetric bilinear form on a vector space $V$. Let $W$ be a subspace of $V$ which is complementary to $rad(H)$ (i.e. $rad(H) \oplus W = V$). Prove that $H$ restricts to a non-degenerate bilinear form on $W$.

4. Let $\Omega$ be a symplectic form on a vector space $V$. Let $W \subset V$ be a subspace. Prove that $\Omega|_W$ is non-degenerate if and only if $W \oplus W^\perp = V$.

5. Let $H$ be a bilinear form on a vector space $V$. Assume the characteristic of the field is not 2. Prove that $H$ is skew-symmetric if and only if $H(v, v) = 0$ for all $v \in V$.

6. Consider the symmetric bilinear form on $\mathbb{C}^2$ given by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find a basis for $\mathbb{C}^2$ for which this bilinear form is represented by the identity matrix.

7. Let $H$ be a non-degenerate symmetric bilinear form on a complex vector space $V$. Show that we can find a basis for $V$ for which $H$ is represented by the identity matrix.

8. Let $W$ be a vector space and let $U \subset W$ be a subspace. Recall that $V = W \oplus W^*$ carries a symplectic form $\Omega$. Let

$$U' = \{\alpha \in W^* : \alpha(u) = 0 \text{ for all } u \in U\}.$$

Prove that $U \oplus U'$ is a Lagrangian subspace of $V$.

9. Let $V, W$ be vector spaces. Prove that $L(V^*, W)$, the space of linear maps from $V^*$ to $W$, is a tensor product for $V, W$.

10. Let $V$ be a vector space and let $v_1, \ldots, v_k \in V$. Prove that $H(v_1, \ldots, v_k) = 0$ for all skew-symmetric $k$-forms $H$ if and only if $v_1, \ldots, v_k$ are linearly dependent.

11. Let $V, W$ be vector spaces. Let $v_1, \ldots, v_k \in V$ be linearly independent. Suppose that there exists $w_1, \ldots, w_k \in W$ such that

$$\sum_{i=1}^{k} v_i \otimes w_i = 0$$

in $V \otimes W$. Show that $w_i = 0$ for all $i$.

12. Let $V, W$ be two vector spaces. Let $v_1, v_2 \in V$ be linearly independent and let $w_1, w_2 \in W$ be linearly independent. Show that there do not exist $v \in V$ and $w \in W$ such that

$$v_1 \otimes w_1 + v_2 \otimes w_2 = v \otimes w$$