Final exam study suggestions

April 4, 2011

Review the definitions of the following notions.
1. inner product
2. adjoint of a linear operator (in the presence of an inner product)
3. orthogonal complement
4. normal, self-adjoint, isometry, and positive operators
5. singular values
6. bilinear form
7. quadratic form
8. isotropic vector and subspace
9. non-degenerate bilinear form
10. signature of a symmetric bilinear form
11. radical of symmetric bilinear form
12. symplectic form
13. Lagrangian subspace
14. dual vector space
15. dual basis
16. multilinear forms (also known as k-forms)
17. tensor products

Review all of your homework problems and also the following questions.
1. Axler 6.13, 6.17, 6.21, 7.3, 7.4, 7.15, 7.21, 7.23, 7.24, 7.31, 7.32
2. Let \( p(x, y) = x^2 + 3xy + 2y^2 \). Find a symmetric bilinear form on \( \mathbb{R}^2 \) for which this polynomial is the associated quadratic form. Find the signature of this bilinear form.
3. Let $H$ be a symmetric bilinear form on a vector space $V$. Let $W$ be a subspace of $V$ which is complementary to $\text{rad}(H)$ (i.e. $\text{rad}(H) \oplus W = V$). Prove that $H$ restricts to a non-degenerate bilinear form on $W$.

4. Let $\Omega$ be a symplectic form on a vector space $V$. Let $W \subset V$ be a subspace. Prove that $\Omega|_W$ is non-degenerate if and only if $W \oplus W^\perp = V$.

5. Let $H$ be a bilinear form on a vector space $V$. Assume the characteristic of the field is not 2. Prove that $H$ is skew-symmetric if and only if $H(v, v) = 0$ for all $v \in V$.

6. Consider the symmetric bilinear form on $\mathbb{C}^2$ given by the matrix

$$
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
$$

Find a basis for $\mathbb{C}^2$ for which this bilinear form is represented by the identity matrix.

7. Let $H$ be a non-degenerate symmetric bilinear form on a complex vector space $V$. Show that we can find a basis for $V$ for which $H$ is represented by the identity matrix.

8. Let $W$ be a vector space and let $U \subset W$ be a subspace. Recall that $V = W \oplus W^*$ carries a symplectic form $\Omega$. Let

$$
U' = \{ \alpha \in W^* : \alpha(u) = 0 \text{ for all } u \in U \}.
$$

Prove that $U \oplus U'$ is a Lagrangian subspace of $V$.

9. Let $V, W$ be vector spaces. Prove that $L(V^*, W)$, the space of linear maps from $V^*$ to $W$, is a tensor product for $V, W$.

10. Let $V$ be a vector space and let $v_1, \ldots, v_k \in V$. Prove that $H(v_1, \ldots, v_k)$ is non-degenerate if and only if $v_1, \ldots, v_k$ are linearly dependent.

11. Let $V, W$ be vector spaces. Let $v_1, \ldots, v_k \in V$ be linearly independent. Suppose that there exists $w_1, \ldots, w_k \in W$ such that

$$
\sum_{i=1}^k v_i \otimes w_i = 0
$$

in $V \otimes W$. Show that $w_i = 0$ for all $i$.

12. Let $V, W$ be two vector spaces. Let $v_1, v_2 \in V$ be linearly independent and let $w_1, w_2 \in W$ be linearly independent. Show that there do not exist $v \in V$ and $w \in W$ such that

$$
v_1 \otimes w_1 + v_2 \otimes w_2 = v \otimes w
$$