Symmetric bilinear forms

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1 Symmetric bilinear forms

We will now assume that the characteristic of our field is not 2 (so $1 + 1 \neq 0$).

1.1 Quadratic forms

Let H be a symmetric bilinear form on a vector space V. Then H gives us a function $Q : V \to \mathbb{F}$ defined by $Q(v) = H(v, v)$. Q is called a quadratic form. We can recover H from Q via the equation

$$H(v, w) = \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$$

Quadratic forms are actually quite familiar objects.

Proposition 1.1. *Let $V = \mathbb{F}^n$. Let Q be a quadratic form on $\mathbb{F}^n$. Then $Q(x_1, \ldots, x_n)$ is a polynomial in n variables where each term has degree 2. Conversely, every such polynomial is a quadratic form.*

Proof. Let Q be a quadratic form. Then

$$Q(x_1, \ldots, x_n) = [x_1 \cdots x_n] A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

for some symmetric matrix A.

Expanding this out, we see that

$$Q(x_1, \ldots, x_n) = \sum_{1 \leq i,j \leq n} A_{ij} x_i x_j$$

and so it is a polynomial with each term of degree 2. Conversely, any polynomial of degree 2 can be written in this form. $\square$

Example 1.2. Consider the polynomial $x^2 + 4xy + 3y^2$. This the quadratic form coming from the bilinear form H_A defined by the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

We can use this knowledge to understand the graph of solutions to $x^2 + 4xy + 3y^2 = 1$. Note that H_A has a diagonal matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with respect to the basis $(1, 0), (-2, 1)$. This shows that $Q(a(1, 0) + b(-2, 1)) = a^2 - b^2$. Thus the solutions of $x^2+4xy+3y^2 = 1$ are obtained from the solutions to $a^2-b^2 = 1$ by a linear transformation. Thus the graph is a hyperbola.

1.2 Diagonalization

As we saw before, the bilinear form is symmetric if and only if it is represented by a symmetric matrix. We now will consider the problem of finding a basis for which the matrix is diagonal. We say that a bilinear form is *diagonalizable* if there exists a basis for V for which H is represented by a diagonal matrix.

Lemma 1.3. *Let H be a non-trivial bilinear form on a vector space V. Then there exists $v \in V$ such that $H(v, v) \neq 0$.*

Proof. There exist $u, w \in V$ such that $H(u, w) \neq 0$. If $H(u, u) \neq 0$ or $H(w, w) \neq 0$, then we are done. So we assume that both u, w are isotropic. Let $v = u + w$. Then $H(v, v) = 2H(u, w) \neq 0$. □

Theorem 1.4. *Let H be a symmetric bilinear form on a vector space V. Then H is diagonalizable.*

This means that there exists a basis $v_1, \ldots, v_n$ for V for which $[H]_{v_1,\ldots,v_n}$ is diagonal, or equivalently that $H(v_i, v_j) = 0$ if $i \neq j$.

Proof. We proceed by induction on the dimension of the vector space V. The base case is $\dim V = 0$, which is immediate. Assume the result holds for all bilinear forms on vector spaces of dimension $n - 1$ and let V be a vector space of dimension n.

If $H = 0$, then we are already done. Assume $H \neq 0$, then by the Lemma we get $v \in V$ such that $H(v, v) \neq 0$.

Let $W = \text{span}(v)^{\perp}$. Since v is not isotropic, $W \oplus \text{span}(v) = V$. Since $\dim W = n - 1$, the result holds for W. So pick a basis $v_1, \ldots, v_{n-1}$ for W for which H_W is diagonal and then extend to a basis $v_1, \ldots, v_{n-1}, v$ for V. Since $v_i \in W$, $H(v, v_i) = 0$ for $i = 1, \ldots, n-1$. Thus the matrix for H is diagonal. □

1.3 Diagonalization in the real case

For this section we will mostly work with real vector spaces. Recall that a symmetric bilinear form H on a real vector space V is called *positive definite* if $H(v, v) > 0$ for all $v \in V$, $v \neq 0$. A postive-definite symmetric bilinear form is the same thing as an inner product on V.

Theorem 1.5. *Let H be a symmetric bilinear form on a real vector space V. There exists a basis $v_1, \ldots, v_n$ for V such that $[H]_{v_1,\ldots,v_n}$ is diagonal and all the entries are $1, -1,$ or 0.*

We have already seen a special case of this theorem. Recall that if H is an inner product, then there is an orthonormal basis for H. This is the same as a basis for which the matrix for H consists of just 1s on the diagonal.

Proof. By the previous theorem, we can find a basis $w_1, \ldots, w_n$ for V such that $H(w_i, w_j) = 0$ for $i \neq j$. Let $a_i = H(w_i, w_i)$ for $i = 1, \ldots, n$. Define

$$v_i = \begin{cases} \frac{1}{\sqrt{a_i}} w_i, \text{ if } a_i > 0 \\ \frac{1}{\sqrt{-a_i}} w_i, \text{ if } a_i < 0 \\ w_i, \text{ if } a_i = 0 \end{cases} \tag{1}$$

Then $H(v_i, v_i)$ is either $1, -1$, or 0 depending on the three cases above. Also $H(v_i, v_j) = 0$ for $i \neq j$ and so we have found the desired basis. □

Corollary 1.6. *Let Q be a quadratic form on a vector space V. There exists a basis $v_1, \ldots, v_n$ for V such that the quadratic form is given by*

$$Q(x_1 v_1 + \cdots + x_n v_n) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

Proof. Let H be a associated bilinear form. Pick a basis $v_1, \ldots, v_n$ as in the theorem, ordered so that the diagonal entries in the matrix are 1s then -1s, then 0s. The result follows. □

Given a symmetric bilinear form H on a real vector space V, pick a basis $v_1, \ldots, v_n$ for V as above. Let p be the number of 1s and q be the number of -1s in the diagonal entries of the matrix. The following result is known (for some reason) as "Sylveter's Law of Inertia".

Theorem 1.7. *The numbers p, q depend only on the bilinear form. (They do not depend on the choice of basis $v_1, \ldots, v_n$.)*

To prove this result, we will begin with the following discussion which applies to symmetric bilinear forms over any field. Given a symmetric bilinear form H, we define its radical (sometimes also called kernel) to be

$$\text{rad}(H) = \{w \in V : H(v, w) = 0 \text{ for all } v \in V\}$$

In other words, $\text{rad}(H) = V^\perp$. Another way of thinking about this is to say that $\text{rad}(H) = \text{null}(H^\#)$.

Lemma 1.8. *Let H be a symmetric bilinear form on a vector space V. Let $v_1, \ldots, v_n$ be a basis for V and let $A = [H]_{v_1, \ldots, v_n}$. Then*

$$\dim \text{rad}(H) = \dim V - \text{rank}(A)$$

Proof. Recall that A is actually the matrix for the linear map $H^\#$. Hence $\text{rank}(A) = \text{rank}(H^\#)$. So the result follows by the rank-nullity theorem for $H^\#$. □

Proof of Theorem 1.7. The lemma shows us that $p+q$ is an invariant of H. So it suffices to show that p is independent of the basis.

Let

$$\tilde{p} = \max\left(\dim W : W \text{ is a subspace of } V \text{ and } H|_W \text{ is positive definite}\right)$$

Clearly, $\tilde{p}$ is independent of the basis. We claim that $p = \tilde{p}$.

Assume that our basis $v_1, \ldots, v_n$ is ordered so that

$$\begin{aligned} &H(v_i, v_i) = 1 \text{ for } i = 1, \ldots p, \\ &H(v_i, v_i) = -1 \text{ for } i = p+1, \ldots, p+q, \text{ and} \\ &H(v_i, v_i) = 0 \text{ for } i = p+q+1, \ldots, n \end{aligned}$$

Let $W = \operatorname{span}(v_1, \ldots, v_p)$. Then $\dim W = p$ and so $p \leq \tilde{p}$.

To see that $\tilde{p} \leq p$, let $\tilde{W}$ be a subspace of V such that $H|_{\tilde{W}}$ is positive definite and $\dim \tilde{W} = \tilde{p}$.

We claim that $\tilde{W} \cap \operatorname{span}(v_{p+1}, \ldots, v_n) = 0$. Let $v \in \tilde{W} \cap \operatorname{span}(v_{p+1}, \ldots, v_n)$, $v \neq 0$. Then $H(v,v) > 0$ by the definition of $\tilde{W}$. On the other hand, if $v \in \operatorname{span}(v_{p+1}, \ldots, v_n)$, then

$$v = x_{p+1}v_{p+1} + \cdots + x_n v_n$$

and so $H(v,v) = -x_{p+1}^2 - \cdots - x_{p+q}^2 \leq 0$. We get a contradiction. Hence $\tilde{W} \cap \operatorname{span}(v_{p+1}, \ldots, v_n) = 0$.

This implies that

$$\dim \tilde{W} + \dim \operatorname{span}(v_{p+1}, \ldots, v_n) \leq n$$

and so $\tilde{p} \leq n - (n-p) = p$ as desired. □

The pair (p,q) is called the signature of the bilinear form H. (Some authors use $p-q$ for the signature.)

Example 1.9. Consider the binear form on $\mathbb{R}^2$ given by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It has signature $(1,-1)$.

Example 1.10. In special relativity, symmetric bilinear forms of signature $(3,1)$ are used.

In the complex case, the theory simplifies considerably.

Theorem 1.11. *Let H be a symmetric bilinear form on a complex vector space V. Then there exists a basis $v_1, \ldots, v_n$ for V for which $[H]_{v_1,\ldots,v_n}$ is a diagonal matrix with only 1s or 0s on the diagonal. The number of 0s is the dimension of the radical of H.*

Proof. We follow the proof of Theorem 1.5. We start with a basis $w_1, \ldots, w_n$ for which the matrix of H is diagonal. Then for each i with $H(w_i, w_i) \neq 0$, we choose a_i such that $a_i^2 = \frac{1}{H(w_i, w_i)}$. Such a_i exists, since we are working with complex numbers. Then we set $v_i = a_i w_i$ as before. □