Corollary: For each $\lambda$, $H^0(\mathbb{F}_a^\lambda)$ is an irep of $S_n$.

Proof:

$$\mathcal{O}[S_n] = \text{End}(\mathbb{F}_a^\lambda) \otimes I(\mathcal{O})$$

$$\otimes \text{End}_\mathbb{F}(H^0(\mathbb{F}_a^\lambda))$$

Since we abstract finite objects $\text{End}(\mathbb{F}_a^\lambda) = \mathcal{O}$.

Thus, the map $\mathcal{O}[S_n] \to \text{End}_\mathbb{F}(H^0(\mathbb{F}_a^\lambda))$ is surjective $\iff$ irep.

Actually, we want to show that $H^0(\mathbb{F}_a^\lambda)$ is the irep associated to $\lambda$. Note since $\mathbb{F}_a^\lambda$ do two - factor using generality. So take...

We can also use sheaf theory to give an alternate proof of the iso. $H(\mathbb{F}_a^\lambda) \cong \mathcal{O}[S_n]$.

First consider $\mathcal{F} = \mathcal{O}(X/U) : X = M_{\ast}, V = \mathbb{F}_a, XU, \mathcal{O}V, V$.

We have $\mathcal{F} \to \mathbb{C}$, $U_{\mathcal{M}}$ disconnected with distinct elements.

So a $U_{\mathcal{M}}$ covering.

$f_U$ is seminormal, in fact normal, so $f^* \mathcal{F} = I_{\mathcal{M}_{\mathcal{F}}} L$.

where $L$ is just the dual system $[(\mathcal{F})^\lambda]$ coming from the $\nu!$ above.

Now $f^* \mathcal{F} = \mathbb{F}_a[\lambda] = I_{\mathcal{M}_{\mathcal{F}}} [\lambda]$ by our chase.

$\mathcal{O}[S_n] \to \mathcal{O}$. 

We have an action of $S_n$ on $f 	imes g$
and so on... $f_1 	imes g_1$

"Categorical" framework for Springer theory.

We have $C(S_n) \to \text{End}(f \times g)$.

We define a functor:

$$\text{Per}_{GL}(N) \to \text{Rep}(S_n)$$

category of
Perwitt sheaves
on $N$ constructible
not the sheaf
by GL-orbits

This functor is an equivalence
(\text{i.e., a module version as well})

Affine Grassmannian.

Now we will study

$$\text{Gr} = \left\{ \text{0-lattices in } K^n \right\} \quad K = C(\mathbb{A}), \ 0 = C(0)$$

$$\text{Gr} = \left\{ L \subseteq K^n : L = \text{span}_K (v_1, v_n) \right\}$$

$s.t. \ v_1, \ldots, v_n$ is a $K$-basis for $K^n$

Gr is analogous to \{\text{0-lattices in } \mathbb{R}^2\}

We have $GL(K) \acts \text{Gr}$ transitively
and $\text{Gr} = GL_n(K) / GL_n(0)$.

Gr has the structure of an \"ind-scheme\".
Let $\lambda = (\mu, \lambda) \in \mathbb{Z}^n$: get $L_\lambda = \text{span}_\mathbb{R}(Z^\mu \nu, Z^{\lambda \cdot \mathbb{R}_n})$

Let $G_{\mathbb{R}^n} = \bigcup_{\lambda \in \mathbb{Z}^n} L_\lambda$

**Prop.** $G_{\mathbb{R}^n} = \bigcup_{\lambda \in \mathbb{Z}^n} L_\lambda$

*Proof.* Row and column operations.

If $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 > 0$, then $L_\lambda \subset L_0 = 0^n$

and $Z^{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}$ is a nilpotent operator of Jordan type.

**Lemma.**

$G_{\mathbb{R}^n} = \bigcup_{\lambda \in \mathbb{Z}^n} L_\lambda$: $Z^{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}$ is a nilpotent operator.

*Proof.*

If $G_{\mathbb{R}^n}$ then $L_0 \subset L_0$ and get Jordan type. Since $G_{\mathbb{R}^n}$ has an $z^{\mu \cdot \mathbb{R}_n} L_0 \subset L_0$

If $z_{\lambda \mu} L_0$ has type $\lambda$, then $L_{\mu \cdot \mathbb{R}_n}$ are similar.

Since $L_{\mu \cdot \mathbb{R}_n} \in L_0$, $\lambda$ for all $i$. $\Box$

Note that if $L_{\mu \cdot \mathbb{R}_n}$ then $L_0 \supset L_0 = Z^{\lambda_1 \cdot \lambda_0}$

(since for all $v \in \mathbb{R}_n$, $Z^{\lambda_1 \cdot \lambda_0}$).

So $L$ can be thought of as a point is a Grassmannian.$\Box$

For any $\lambda$

$G_{\mathbb{R}^n} = \bigcup_{\mu \cdot \mathbb{R}_n} G_\mu$ where $\lambda \mu$ means $\lambda - \mu \in \mathbb{Z}^{n \cdot \mathbb{R}_n}$, $\mathbb{R}_n$.

$G_{\mathbb{R}^n}$ is a finite dimensional projective variety.

$G_{\mathbb{R}^n}$ has connected components labelled by integers.

For each $k \in \mathbb{Z}$

$G_k = \bigcup_{\lambda \cdot \mathbb{R}_n} G_{\lambda \mu}$ where $\lambda \cdot \mathbb{R}_n$
Eq
\[G^{1,2}, \quad k=0\]

\[G^1_{(\omega,0)} = p^+\]
\[G^1_{\alpha^{-1}} = \text{weighted } IP^2\]
\[G^1_{(\omega,2)} = 4 - \dim\]

Prop.
\[\dim \overline{G^1} = (n-1)\lambda_1 + (n-3)\lambda_2 + \cdots + (1-n)\lambda_n\]

Proof.
\[G^1 \cong \mathbb{C}^2 \times \	ext{up}, \text{and we have a map } G^1 \to G^1_{\alpha^2/p}, \text{ which is a vector bundle.}\]

\[\lambda = (2, 0, 0)\]

\[L \subseteq \mathbb{C}^6 \text{ has Jordan type } (2, 0, 0)?\]

\[L \subseteq C^6 = L_0 / \mathbb{Z}^2 L_0\]
\[\dim L = 4\]

\[\mathbb{Z}^2 L_0 \text{ s.t. } 2 \text{ line has Jordan type } (2, 0, 0)\]

\[\mathbb{C}^2 \text{ fiber is } \mathbb{C}^2\]

\[\langle f_1 + a_2 + b_3 \rangle\]
Recall $G_\mathbf{r} = O_{r(n+r, r)}$ in $k^r$.

Loci $L$ have $L_\lambda = \langle z^{\lambda_1}_{1}, \ldots, z^{\lambda_r}_{r} \rangle$.

$G_{r} = G(l_{r}(r)) \ L_\lambda \ \lambda \in \Lambda^r_+.$

If $\mu \geq 0$, $G_{r} = \emptyset$, $L = L_{0}$: $G_{0}$ has Jacobian type $\lambda^\vee$.

So get $G_{r}$ locally closed in some Grassmannian.

For example, $\lambda = (6, 0)$, $L_{0} = (0, 1, 0, 0)$ then $G_{r} = G(k, n)$ and is closed.

We also see that $G_{r} = U / G_{r}$.

Recall $G_{r} = \Sigma_{r} \mu_{r} = \lambda_{r}$.

In fact $G_{r}$ breaks into connected components $G_{r}(\mu) = U / G_{r}$.

$G(r, \mu) = \{ L : \dim L \cap L_{0} = m \} = \{ L : \ker \psi(L) = m \}$

All connected components are isomorphic $L \mapsto \langle z^{\lambda}_{\lambda} \rangle_{L}$.

Proof:

$\dim G_{r} = \sum \lambda_{i} + \ldots + (1 - n) \lambda_{n}$

Proof:

We have a map $G_{r} \rightarrow L_{n_{r}(r)}$ to partial flag variety.

The fibres are $L_{0} c_{L_{0}} L_{0}$.

Vector bundles $Z_{e_{0}} \rightarrow L_{0}$ are $c_{L_{0}} c_{L_{0}} L_{0}$ and $\mathbb{P}^{2}$.

$\mathbb{P}^{2} \rightarrow L_{0} \rightarrow L_{0} C^{2} \rightarrow L_{0}$.

$E_{0} c_{Z_{e_{0}}} Z_{e_{0}}$.
\[ \chi_0(h) = B \]
\[ \chi_0 \rightarrow X \]
\[ \chi \leftrightarrow F_n \]

Analog: \[ Gr \leftrightarrow X \]

In fact, we can consider

Proposition
\[ \{ L < L_0 : [e_1, e_2] \text{ forms } \} \leftrightarrow X \]

Proof
\[ [x_{L_0}] \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \]

\[ \text{Span}(Z_{2,1} + X_{1,1}, \ldots) \leftarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

Then \[ [e_1] - [2e_1] = [X(e_1)] \]

Moreover: \[ Gr \cap \{ L < L_0 \} = \emptyset \]

We define \( d \): \( Gr \times Gr \rightarrow X \)

by \[ d((L_1, L_2)) = X \text{ if } L_1 \text{ and } L_2 \text{ have type } \lambda \]

\[ d((L_1, L_2)) = d((L_1, L_1)) - d((L_1, 1)) \]

Relative position of two lattices

Analog of Bott-Samelson

\[ T^*F_n \]

Given a sequence \( \lambda_1, \ldots, \lambda_m \rightarrow \lambda_+ \) we define

\[ Gr \leftarrow \{ (L_1 \cup L_{1-1} \cup \cdots) : d(L_1, L_{1}) = \lambda_1 \} \]

Often we will use \( \lambda_1 = 0 \).

We have \( \omega : Gr \rightarrow Gr \)
\[ m = 2 \quad \Delta = (\omega_2, \omega_4, \omega_6, \omega_8) \]

Get
\[ \text{span} \left< l_0, l_1, \cdots, l_4 \right>, \quad \text{dim} \left< l_1 / \langle l_0 / l_1 \rangle \right> = 1 \] \rightarrow \mathbb{C}

\[ l_0 \rightarrow l_4 \]

So,
\[ m_2^{-1}(\mathcal{L}) = \left\{ l_0 \geq l_1 \geq \cdots \geq l_4 = \lambda : \pm l_i < \lambda \right\} \]

= Springer fiber for \( \mathcal{L} = l_0 / l_1 \)

\[ \text{eg} \quad \mathcal{L} = 2 \cdot l_0 = l_{n,2} \]

So \( m_2^{-1}(\mathcal{L}_{n,2}) = (2,2) \) Springer fibre

Prop
More generally, if \( \Delta = (\omega_1, \cdots, \omega_m) \)

Then we have
\[ m_2^{-1}(\text{span} \left< l_0, l_2, \cdots, l_m \right>) \rightarrow \mathbb{C} \]

and for any \( \mu = (\mu_1, \mu_m) \)
\[ m_2^{-1}(\mathcal{L}_\mu) = \mathcal{P}_\mu \] when \( \chi \) is a nilpotent of type \( \mu \)

if \( \Delta = (\omega_1, \cdots, \omega_{km}) \)

then \( m_2^{-1}(\mathcal{L}_\mu) = \left\{ 0 \leq V_1 \leq \cdots \leq V_m = \mathbb{C}^N : \text{dim} \lambda V_i = \text{dim} V_{i-1} + \text{dim} V_{i+1} \right\} \)

where \( \chi \) is a nilpotent of type \( \mu \)

\[ \chi = (\omega_1, \cdots, \omega_1) \]

Prop
We have
\[ m_2^{-1}\left( \left\{ l_0 \geq l_1 \cdot \cdots \cdot l_n / l_1 \right\} \right) = T^* P_n \]

\[ \downarrow \]

\[ \mathcal{L} \cong \chi \]
Theorem

\[ G_r \cong \{ (E, \varphi) : E \text{ is a vector bundle on } \mathbb{P}^1, \varphi : E \mid_{\mathbb{P}^1 \setminus 0} \to E_0 \mid_{\mathbb{P}^1 \setminus 0} \text{ is trivial away from } 0 \} \]

Proof

Given \((E, \varphi)\), we let \( L = \{ s \in \Gamma(\mathbb{P}^1 \setminus 0; E) : \varphi^*(s) \text{ extends to a section} \} \)

\[ \text{i.e. we get } L = \Gamma(\mathbb{P}^1 \setminus 0; E) \text{ which is embedded into } \Gamma(\mathbb{P}^1; E_0). \]

Then \(L\) is a \(\mathbb{C}[z]\) lattice in \(\mathbb{C}(\mathbb{Z}, \mathbb{Z})^n\).

\[ \text{From this perspective, } L = \bigoplus_{i=0}^{m} \mathcal{O}(x_i^a) \text{ with canonical trivialization} \]

where \(\mathcal{O}(x_i^a) = \text{rational functions allowed to have a pole of order } n \text{ at } 0\).

\[ \text{deg} = \{ (E, \varphi) : \text{deg}(E) = -m \} \]

\[ \text{let } \{ E_1, E_2 \} \text{ is a basis } \Rightarrow \bigcup_{r} \{ (E, \varphi) : E = \mathcal{O}(-r) \} \]

\[ \bigcup_{r} \{ (E, \varphi) : \text{deg}(E) = n \text{ and the trivialization } E \to E_0 \text{ extends to an inclusion of coherent sheaves} \} \]

\[ \text{let } \{ E_1, E_2 \} \text{ is a basis } \Rightarrow \bigcup_{r} \{ (E, \varphi) : E = \mathcal{O}(-r) \} \]

\[ \bigcup_{r} \{ (E, \varphi) : \text{deg}(E) = n \text{ and the trivialization } E \to E_0 \text{ extends to an inclusion of coherent sheaves} \} \]
Last time I described $Gr \to$ partial flag variety.

A $\mathbb{Z}$-flag in $\mathbb{C}^n$ is a sequence $0 \leq V_j \leq V_{j+1} \leq \cdots \leq \mathbb{C}^n$.

$$\text{Fl}_\mathbb{Z}(\mathbb{C}^n) = \bigsqcup_{\lambda \in \mathbb{Z}} \text{Fl}_\lambda(\mathbb{C}^n)$$

Where $\text{Fl}_\lambda(\mathbb{C}^n) = \{V_0 \leq V_1 \leq \cdots \leq V_r = \mathbb{C}^n \mid r = \dim V_j - \dim V_{j-1} = \lambda \}$

E.g. $\text{Fl}_0(\mathbb{C}^n) = \{0 = V_1 \leq V_2 = \mathbb{C}^n \}$

$$\text{Fl}_{(3,3,1)}(\mathbb{C}^3) = \{0 = V_0 \leq V_1 \leq V_2 \leq V_3 = \mathbb{C}^3 \}$$

So $\text{Fl}_\lambda(\mathbb{C}^n)$ is a partial flag variety.

Define $\Pi : Gr \to \text{Fl}_\mathbb{Z}(\mathbb{C}^n)$

by $L \mapsto \{V_j = \{v \in \mathbb{C}^n : \exists v \in L\}\}$

($\Pi$ is not a morphism).

We get $\Pi : Gr \to \text{Fl}_\mathbb{Z}(\mathbb{C}^n)$ a vector bundle.

We have $i : \text{Fl}_\mathbb{Z}(\mathbb{C}^n) \to Gr$

by $V_j \mapsto \bigoplus_{k \in \mathbb{Z}} V_k z^k$

In fact $\mathbb{C}^n \cap Gr$ by "loop rotation" and $i(\text{Fl}_\mathbb{Z}(\mathbb{C}^n))$ is the set of itself.