Now we have $H(Z) \cong \mathbb{C} S_n$

This leads to an action of S_n on $H_*(Fl_n^X)$ for any X.

$$\begin{array}{ccc} & \tilde{N} \times \tilde{N} & \\ \swarrow^{\pi_1} & & \downarrow^{\pi_2} \\ \tilde{N} & & \tilde{N} \end{array}$$

$\dim_{\mathbb{R}} \tilde{N} = 2n(n-1)$

Given $a \in H(Z)$, $b \in H_k(Fl_n^X)$

we define $a \cdot b = \pi_{2*}\underbrace{(\underset{2n(n-1)}{a} \cap \underset{k+2n(n-1)}{\pi_1^*(b)})}_{k} \in H_k(Fl_n^X)$

From this perspective, it is hard to see why $H_{top}(Fl_n^X)$ is an irreducible representation.

We will need sheaf theoretic means.

We study constructible sheaves of vector spaces in the complex topology (not Zariski)

X any reasonable top space.
$\underline{\mathbb{C}}_X$ sheaf with $\underline{\mathbb{C}}_X(U) = \mathbb{C}$ if U connected
$= \text{locally constant maps } U \to \mathbb{C}$

We can take $H^*(X, \underline{\mathbb{C}}_X) =$ Cech cohomology of X
$= H^*(X, \mathbb{C})$

More generally if $\pi : X \to Y$ is any map, we have

$R\pi_*(\underline{\mathbb{C}}_X)$ – this is a complex of sheaves on Y

$H_y^k(R\pi_* \underline{\mathbb{C}}_X) = H^k(\pi^{-1}(y))$

(π proper

For any X, we have a complex of sheaves "responsible for homology":

$$C_i^{BM}(U) = \{\sum a_j f_j \quad f_j : \Delta^i \to U\}$$

$$C_i^{BM}(U) \to C_i^{BM}(V) \qquad \text{for } V \subset U$$

$$C_i(U) \leftrightarrow C_i(V) \qquad d : C_i(U) \to C_{i-1}(U)$$

~~tet~~

$$C_i(U)^* \to C_i(V)^*$$

There is a complex of sheaves $\mathcal{F}^i$ with $\mathcal{F}^i = (C_i(U))^*$ $\quad \mathcal{F}^\bullet \simeq \mathbb{C}_X$
and D^i with $D^i = C_{-i}^{BB}(U)$

Note that $H^i(\mathbb{C}_X, U) = H_c^i(U, D^\bullet)^*$

For any $\mathcal{F} \in D(X)$, $\mathbb{D}(\mathcal{F})$ is a complex sheaf with $H^\bullet(U, \mathcal{F})^* = H_c^\bullet(U, \mathbb{D}(\mathcal{F}))$

In fact $\mathbb{D}(\mathcal{F}) = \mathrm{Hom}(\mathcal{F}, D^\bullet)$

If X is smooth of dim n, then $D^\bullet \simeq \mathbb{C}_X[n]$.

The existence of stalks in $D^\bullet$ is related to singularities of X.

A sheaf $\mathcal{F} \in D(X)$ is called <u>perverse</u> if

$$\dim\{x \in X : H_x^k(\mathcal{F}) \neq 0\} < k \qquad \dim\{x \in X : H_x^i(\mathbb{D}(\mathcal{F})) \neq 0\} < i$$

Last time

X any reasonable top space, have $\mathbb{C}_X$ - constant sheaf

$H^i(X, \mathbb{C}_X)$ = singular cohomology of X

and $\mathbb{C}_X \cong C^*_\bullet$ in $D(Sh(X))$

where $C^*_i(U) = \{\overset{\text{finite}}{\text{linear comb}}$ of maps of i-simplices into $U\}^*$

Theorem

There exists a $\overset{\text{(complex of)}}{\text{sheaf}}$ $\mathbb{D}_X$ on X s.t.
$(H^\bullet(U, \mathbb{C}))^* = H^\bullet_c(U, \mathbb{D}_X)$ for all U.

More gen. for any $\mathcal{F}$
$H^\bullet_c(X, \mathcal{F})^* \cong H^\bullet(X, \mathrm{Hom}(\mathcal{F}, \mathbb{D}_X))$

In fact $\mathbb{D}_X$ can be constructed as
$C_i^{BM}(U) = \{\overset{\text{locally finite}}{\text{linear comb}}\ \Delta_i \to X\}$

Theorem (Poincaré duality)

If X is an n-dimensional oriented manifold, then $\mathbb{D}_X \cong \mathbb{C}_X[n]$.

"Proof"

Locally $X \cong \mathbb{R}^n$, so we just need to prove this for $\mathbb{R}^n$,
and then it is just $H_k^{BM}(\mathbb{R}^n) = \begin{cases} \mathbb{C} & k=n \\ 0 & \text{otherwise} \end{cases}$

Or you can think that k-cochains on X are just codim k cycles in X.

D.

Cor

$H^k(X, \mathbb{C})^* = H^k_c(X, \mathbb{C}_X[n]) = H^{n-k}_c(X)$

For any sheaf $\mathcal{F}$ and $x \in X$ we can consider the stalk $H^i_x(\mathcal{F})$.

Note that $H^i_x(\mathbb{C}_x) = 0$ if $i \neq 0$
and $H^i_x(\mathbb{C}_x[n]) = 0$ if $i \neq -n$.

The existence of stalks in $\mathbb{D}_x$ is thus a measure of the singularity of X.

$$H^i_x(\mathbb{P}_x) = H^{BM}_i(U) \text{ where } U \text{ is a small nbd of } x.$$

A sheaf $\mathcal{F} \in D(Sh(X))$ is called <u>perverse</u> if $\mathcal{F}$ is constructible with respect to a stratification and

$\dim_{\mathbb{R}}\{x : H^j_x(\mathcal{F}) \neq 0\} \leq 2j$

$\dim_{\mathbb{R}}\{x : H^{-j}_x(D(\mathcal{F})) \neq 0\} \leq 2j$

<u>eg</u> $\mathcal{F} = \mathbb{C}_X[\frac{n}{2}]$ and X is a manifold, so $D(\mathcal{F}) = \mathcal{H}om(\mathbb{C}_X[\frac{n}{2}], \mathbb{D}_X)$
$= \mathbb{C}_X[\frac{n}{2}]$

Then $H^i_x(\mathcal{F}) = 0$ unless $i = \frac{n}{2}$
$\dim\{x : H^{\frac{n}{2}}_x(\mathcal{F}) \neq 0\} = n \leq 2 \cdot \frac{n}{2} = n$.

<u>Theorem</u>
The set of perverse sheaves forms an abelian category (heart of a t-structure on $D(Sh(X))$)

For any X, $\exists$ an IC sheaf IC_X which is defined by

$$IC^i_X(U) = \text{locally finite IH-chains in } X.$$

X has a stratification $X_n \supset X_{n-1} \supset \cdots$ $X_i \setminus X_{i-1}$ is smooth of dim i

An IC i-chain is map $\Delta^i \to X$ that is transverse
$\dim \Delta^i \cap X_j \leq i-(n-j)$

X smooth $IC_X = \mathbb{C}_X[n]$
codim n

$IH(X) = H^\bullet(X, IC_X)$ is the intersection homology.

We have $D(IC_X) = IC_X$ and thus $IH(X)$ satisfies P.D.

Moreover every simple perverse sheaf in $P(X)$ is of the form $IC(Z, L)$

where $Z \subset X$ subvariety

and L is a local system on an open subset of Y.

Theorem.

Let $f: Y \to X$ be a proper map of complex alg varieties

Then $f_* \mathbb{C}_Y$ is a direct sum of shifted simple perverse sheaves.

$$f_* \mathbb{C}_Y = \bigoplus_i IC(Z_i, L_i)[n_i]$$

so $H^\bullet(Y) = \bigoplus IH^\bullet(Z_i, L_i)[n_i]$.

A map $f: Y \to X$ is called semismall, if Y smooth

$\dim\{x \in X : \dim f^{-1}(x) \geq k\} \leq n - 2k$ fibre dim $= k$ codim of strata $= n - d$ $k \leq \frac{1}{2}(n-d)$ $\Rightarrow$ $d \leq n - 2k$

$\Rightarrow$ generically finite $\dim\{x \in X : \dim f^{-1}(x) \geq 1\} \leq n-2$

$\exists$ a strati. $X = \bigcup X_i$ s.t. f is a fibre bundle on each X_i with $\dim f^{-1}(x_i) \leq \frac{1}{2} \operatorname{codim} X_i$

Prop

If f is semismall, then $f_* \mathbb{C}_Y[n]$ is perverse
$n = \dim Y$

Proof

$\{x \in X : H^k(f^{-1}(x)) \neq 0\} \subseteq \{x \in X : \dim f^{-1}(x) \geq \frac{k}{2}\}$

So $\dim_{\mathbb{C}}\{x \in X : H^k(f^{-1}(x)) \neq 0\} \leq n - k$

Thus $\dim_{\mathbb{C}}\{x \in X : H_x^{-k}(f_* \mathbb{C}_Y[n]) \neq 0\} \leq k$.

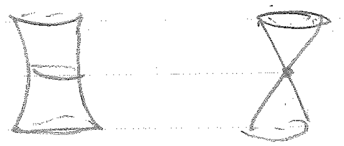
since $H_x^{-k}(f_* \mathbb{C}_Y[n]) = H^{n-k}(f^{-1}(x))$

[Recall F is perverse means
$\dim_{\mathbb{C}}\{x : H_x^{-j}(F) \neq 0\} \leq j \quad \forall j$
$\dim_{\mathbb{C}}\{x : H_x^{-j}(D(F)) \neq 0\} \leq j$]

Now if proper maps
$Df_* = f_* D$ - example push to a point
so $D f_* \mathbb{C}_Y[n] = f_* D(\mathbb{C}_Y[n])$
$= f_* \mathbb{C}_Y[n]$.

Eg

① $T^*\mathbb{P}^1 \longrightarrow \mathcal{N}_{\mathfrak{sl}_2}$ is semismall



only fibre is $\mathbb{P}^1$
and codim of pt is 2

Blowup origin $\longrightarrow \{(x,y,z) : x^2+y^2+z^2=0\}$

$\|$

$\{(v, L) : v \in L$, L an isotropic line in $\mathbb{C}^3\}$

$\downarrow$

isotropic lines in $\mathbb{C}^3 = \{[x,y,z] : x^2+y^2+z^2=0\}$

$= \{[u,v,w] : uv = w^2\} = \mathbb{P}^1$

$f : \{(\overset{\in \mathbb{C}^n}{v}, \overset{\in \mathbb{P}^{n-1}}{L}) : v \in L$, L isotropic line$\} \longrightarrow \{(x_1, \ldots, x_n) : x_1^2 + \ldots + x_n^2 = 0\}$

$\to$ isotropic lines in $\mathbb{C}^n$.

Then $\dim f^{-1}(0) = n-2 \not\leq \frac{1}{2}(n-1)$

② More generally

Thm $T^*Fl_n \to \mathcal{N}$ is semismall.

Proof
We have $\mathcal{N} = \bigsqcup \mathcal{O}_\lambda$
and $\dim \mathcal{O}_\lambda = n^2 - \sum (\lambda_i')^2$

and for each $X \in \mathcal{O}_\lambda$

$\dim Fl_n^X = \sum \lambda_i (i-1)$

We checked before that

$\cancel{2 \sum_i \lambda_i (i-1) + \underbrace{n^2 - \sum (\lambda_i')^2}_{\dim \mathcal{O}_\lambda} =}$

$2 \dim Fl_n^X + \dim \mathcal{O}_\lambda = \dim T^* Fl = \dim \mathcal{N}$

so $\dim Fl_n^X = \frac{1}{2}(\dim \mathcal{N} - \dim \mathcal{O}_\lambda)$ $\square$

In fact $f : Y \to X$ is semismall iff $\dim Y \times_X Y \leq \dim Y$

Theorem

Suppose that $f : Y \to X$ is semismall and let $X = \bigcup X_i$ s.t. $f^{*} : f^{-1}(X_i) \to X_i$ is a fibre bundle for each i.

Then

$$f_*(\mathbb{C}_Y[n]) = \bigoplus_{i, \chi} H^{2d_i}(f^{-1}(x_i))_\chi \otimes IC_{\bar{X}_i, \chi}$$

where
- $x_i \in X_i$
- $d_i = \frac{1}{2} \operatorname{codim} X_i$
- χ irreducible local system on X_i (i.e. a irrep of $\pi_1(X_i)$)
- $IC_{\bar{X}_i, \chi}$ resulting IC sheaf (some authors write $IC_{X_i, \chi}$)
- $H^{2d_i}(f^{-1}(x_i))_\chi = \operatorname{Hom}(V_\chi, H^{2d_i}(f^{-1}(x_i)))$

Proof

By decomposition theorem, $f_*(\mathbb{C}_Y[n])$ is a direct sum of shifted simple perverse sheaves

Since $f_*(\mathbb{C}_Y[n])$ is perverse, there are no shifts

Since f is a fibre bundle over each X_i, the perverse sheaves must all be $IC_{\overline{X}_i,\chi}$.

So we get $f_* \mathbb{C}_X[n] \cong \bigoplus IC_{\overline{X}_i,\chi} \otimes M(i,\chi)$

for some multiplicity space $M(i,\chi)$

Now take $H^{2d_i-n}_{x_i}$ of both sides. $2d_i - n = -\dim X_i$

we get $H^{2d_i}(f^{-1}(x)) \cong \bigoplus\limits_{j,\chi} H^{-\dim X_i}_{x_i}(IC_{\overline{X}_j,\chi}) \otimes M(i,\chi)$

So we get $H^{-\dim X_i}_{x_i}(IC_{\overline{X}_i,\chi}) = V_\chi$ rep of $\pi_1(X_i)$

and $H^{-\dim X_i}_{x_i}(IC_{\overline{X}_j,\chi}) = 0$ for all $j \neq i$

since $IC_{\overline{X}_j}$ is an IC-sheaf so it has no $H^{-\dim X_i}_{x_i}$ stalks along X

Thus $H^{2d_i}(f^{-1}(x)) = \bigoplus\limits_{\chi} V_\chi \otimes M(i,\chi)$

So $M(i,\chi) = \mathrm{Hom}(V_\chi, H^{2d_i}(f^{-1}(x_i)))$ $\square$

Let us apply this to the Springer resolution

<u>Lemma</u>
For any λ, $\mathcal{O}_\lambda$ is simply-connected.

<u>Proof</u>
We have $\pi_1(\mathcal{O}_\lambda) = \pi_0(\mathrm{stab}_{GL_n}(X))$ since $\mathcal{O}_\lambda = SL_n/\mathrm{stab}_{GL_n}(X)$

for some $X \in \mathcal{O}_\lambda$.

$\mathrm{stab}_{GL_n}(X) < \{Y \in M_n : XY = YX\}$ connected

$\pi_1(\mathcal{O}_\lambda) \to \pi_0(\mathrm{stab}) \to \pi_0(GL_n) \to \pi_0(\mathcal{O}_\lambda)$

$\det Y \neq 0$ complement of a linear hypersurface in a vector space

Corollary

$$f_* \mathbb{C}_{T^*Fl_n}[n(n-1)] = \bigoplus_\lambda IC_{\mathcal{O}_\lambda} \otimes H^{2d_\lambda}(Fl_n^X) \qquad d_\lambda = \dim Fl_n^\lambda$$

Earlier we showed $H(Z) = \mathbb{C}[S_n]$

Now we want to related $H(Z)$ to this decomposition.

Theorem

Let $f: Y \to X$ be any proper map, $Z = Y \times_X Y$

Then there is a graded alg iso

$$H_*(Z) \cong \mathrm{Ext}^\bullet_{D(Sh(X))}(f_*\mathbb{C}_Y, f_*\mathbb{C}_Y)$$

alg by convolution

$$H_k(Z) \cong \mathrm{Ext}^{2n-k}(\quad) \qquad n = \dim Y$$

So $H_{2n}(Z) \cong \mathrm{End}_{D(Sh(X))}(f_*\mathbb{C}_Y)$

H

Corollary

$$\mathrm{End}(f_*\mathbb{C}_{T^*Fl_n}[n(n-1)]) \cong H(Z) \cong \mathbb{C}[S_n]$$

$$\cong \bigoplus_\lambda \mathrm{End}(H^{2d_\lambda}(Fl_n^X))$$

Note that since $IC_{\mathcal{O}_\lambda}$ are distinct simple objects there are no morphisms between them and their endomorphisms are just multiples of the identity.

Can use $H(Z) = \mathrm{End}(f_*\mathbb{C}_{T^*Fl})$ to give a alternated proof of $H(Z) \cong \mathbb{C}[S_n]$ using $\tilde{g} \to g$ and Fourier-transform.